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Infrared behavior of non-Abelian gauge theories*

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Infrared singularities in non-Abelian gauge (Yang-Mills) theories are studied in the leading-logarithm approximation. In addition to the usual infrared limit in which the infrared cutoff μ approaches zero for fixed on-shell momenta, we consider high-energy wide-angle scattering and form factors at large momentum transfer t in which infrared-induced powers of $g^2 \ln^2 t$ (g= coupling constant) typical of vector field theories appear in perturbation expansions. Our techniques include asymptotic estimates of Feynman integrals (to sixth order) and a nonperturbative approach based on a conjectured formula for soft-meson emission. Remarkably, the logarithms sum up into exponential factors much as in QED. Unlike QED, cross sections for production of nonsinglet particles, even including an indefinite number of soft gauge quanta, vanish as $\mu \to 0$, a phenomenon we interpret as evidence of particle confinement. We discuss exclusive hadronic processes in the context of the non-Abelian quark-gluon theory based on color SU(3). We find justification for the scaling laws at large momentum transfers ("quark counting rules") in that the infrared logarithms cancel in the scaling pieces of hadronic amplitudes while providing a fast damping of the "pinch" contributions associated with Landshoff graphs.

I. INTRODUCTION

Non-Abelian (Yang-Mills[1]) gauge theories, as the only renormalizable field theories which are asymptotically free,[2] seem uniquely qualified to explain approximate Bjorken scaling.[3] An attractive Yang-Mills model of the hadronic world is the quark-gluon model[4] with an SU(3) of "color" as the gauge group. Since quarks do not, apparently, exist as real free particles, one supposes that the color symmetry is exact and that all non-color-singlet particles are somehow dynamically suppressed so that one has a theory of quarks without either real quarks or real gluons.

Another aspect of Yang-Mills theories with unbroken symmetry is that their direct physical interpretation is clouded by severe infrared divergences in matrix elements of what are ordinarily presumed to be physical observables. Does this infrared "instability" provide the dynamical mechanism which miraculously suppresses all colored quanta? This question motivated much of the work in this paper. We report evidence of an affirmative answer and a specific mechanism for "particle confinement."

The hadronic constituent picture has been recently invoked for exclusive high-energy wide-angle hadron scattering where scaling laws[5] of the form $d\sigma/dt \sim s^{2-N}$ have provided a good fit to existing data, N being the total number of quarks in the participating hadrons (quark counting rule). In nonvector renormalizable field theories, amplitudes for scattering of elementary quanta obey the same renormalization-group equations (RGE) as in the Euclidean regime[6]; much less is known, however, about the scattering of composite particles.[7] Moreover, in vector field theories (such as the non-Abelian gluon model) the proof of RGE for on- or near-mass-shell amplitudes fails in general because of the presence, in perturbation theory, of powers of $\ln^2(t/\mu^2)$, where t is a typical large momentum transfer and μ is the vector mass (or an infrared cutoff). In this paper we study the fixed-angle limit in perturbation theory for non-Abelian gauge theories. We find that, remarkably, when leading logarithms are summed to all orders they provide damping exponential factors[8]—one for each nonsinglet near-mass-shell particle. This suppression of "colored" amplitudes at large momentum transfers indeed helps us understand the origin of the scaling laws.

The following outline of the framework and scope of our work is intended as a preliminary orientation for this long paper.

Our perturbative calculations are based on the Yang-Mills Lagrangian density

$$-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}+\bar{q}(i\not\partial - M - ig\not B^j t^j)q \qquad (1.1)$$

which couples a set of vector gauge fields B^i_μ ("gluon fields") to a multiplet of fermion fields q ("quark fields"). The matrices t^j are the group generators in quark-field space and $F^i_{\mu\nu}$ is the gauge-covariant curl of the gauge field:

$$F^i_{\mu\nu}=\partial_\mu B^i_\nu - \partial_\nu B^i_\mu + g c_{ijk} B^j_\mu B^k_\nu .$$

The matrices t^j are normalized according to $[t^i, t^j] = i c_{ijk} t^k$, where c_{ijk} are the structure constants of the group. The gauge group is assumed to be semisimple, the physically interesting case being SU(3) of color. The Lagrangian can accommodate several quark multiplets corresponding to the basis states of ordinary SU(3) or SU(4): $u, d, s, c, \ldots$ quarks.

Just as in an Abelian gauge theory such as quantum electrodynamics (QED), infrared singularities appear in perturbation expansions of on-shell amplitudes as powers of the logarithm of an infrared-cutoff mass μ. Our work is primarily based on extracting the leading such power of $\ln\mu$ in each order of perturbation theory (Sec. II). When these leading logarithms are summed to all orders, we find formulas displaying an unexpectedly simple pattern of infrared behavior not unlike that of the Abelian theory[9] in many respects (exponentiation and factorization). The simplicity of these results is difficult to understand in the context of perturbation theory where remarkable cancellations take place between graphs of different topology. There is, however, a way to derive these results from a low-energy theorem which summarizes all our perturbative estimates in a simple differential equation (Sec. III).

We find it convenient to work in the Feynman gauge. As an infrared cutoff we introduce a mass term "by hand" in the vector propagator: $\delta^{ab} \times g_{\alpha\beta}(k^2 - \mu^2)^{-1}$. Violations of unitarity, etc., thereby introduced[10] are of order μ and do not contribute to leading powers of $\ln\mu$.

An alternative, "legitimate," way to introduce an infrared cutoff into the theory is to give a common mass μ to the gauge mesons via the Higgs-Kibble mechanism[11] by adding appropriate multiplets of scalar fields. (In that case, however, the $\mu=0$ theory cannot be asymptotically free[2].)

Still another way to have an effective infrared cutoff is to leave the gauge mesons massless but keep the external momenta off their mass-shell values by an amount proportional to μ, say.

We use mostly the first method (i.e., "by hand" inclusion of a vector mass) because it expedites the differential formulation of Sec. III. However, we obtained qualitatively similar results by the off-shell method—at least in leading-logarithm approximations. Actually, the off-shell method seems potentially more versatile since it would allow one to probe the infrared behavior even when various momenta approach their respective mass-shell values at *different* rates.

Our work deals primarily with three distinct asymptotic regimes:

(1) *The infrared regime*. We shall take the quark masses (as defined in terms of the quark propagator singularities in perturbation theory) to be nonzero. By "infrared regime" we shall mean the $\mu \to 0$ limit of S-matrix elements and form factors under the following conditions:

(a) All external three-momenta are fixed.

(b) External gauge vector quanta (gluons) are given a mass μ and are transversely polarized.

(c) No momentum transfer vanishes.

Thus, all invariant squared energies and momentum transfers $s, t, \ldots$ as well as $M^2>0$ (M being a typical quark mass) are much larger than μ^2 in the infrared regime. Whenever condition (c) does not hold, we shall speak of the near-forward regime (see below). As the most important feature of the infrared regime we find that *cross sections of nonforward processes involving non-neutral (i.e., non-group singlet) particles, whether or not an indefinite number of soft gauge mesons are included, vanish in the limit* $\mu \to 0$ (see Sec. III). Of course this is quite unlike the Abelian case, where the emission of soft real (bremsstrahlung) photons prevents the vanishing of the cross section[9] ("cancellation between soft real and virtual photons"). The difference stems from the fact that all gauge mesons in a Yang-Mills theory with a semisimple gauge group are non-neutral, while photons are neutral: Namely, they do not carry the charge to which they couple.

On the other hand, *processes involving only neutral (i.e., color-singlet) particles have nonvanishing cross sections as* $\mu \to 0$ (the Abelian analog is light-by-light scattering in QED). As neutral particles one may envisage singlet bound states or couple group-singlet fields in the Lagrangian; for example, one may couple the (color-singlet) electromagnetic field to the quarks. In any case, the differential formulation of Sec. III does not distinguish between bound states and fundamental field quanta and on that basis the above statements follow quite generally, with the possible exception of forward processes (more specifically, processes in which one or several nonsinglet particles retain their momentum). But even this exception does not interfere with the main conclusion: *In a collision between color-singlet particles (hadrons) the probability for producing any colored particle or colored bound state vanishes as* $\mu \to 0$. The reader will hardly need prompting to recognize the significance of these results as evidence for particle confinement, *if only the sum of leading logarithms reflects more or less correctly the infrared behavior of the theory.*

(2) *The fixed-angle regime*. All invariant squared energies and momentum transfers $(s, t, \ldots)$ are much larger than both μ^2 and M^2 while ratios s/t, etc. are of order 1 (for example, for the quark-quark scattering amplitude this limit corresponds to high-energy wide-angle scattering). In this regime the leading-logarithmic approximation essentially consists of powers of $g^2\ln^2 t$, where t is a

typical momentum transfer—obviously no distinction need be made between $s, t, \ldots$ at this level. Summation of these leading logarithms results in a remarkably simple situation: Asymptotic form factors are found by exponentiating the one-loop approximation, just as in the Abelian case[12]; fixed-angle scattering amplitudes are found by multiplying the tree approximation by $\Pi F_i^{1/2}(t)$, where F_i is the asymptotic form factor for the ith external on-shell particle coupled to a group-singlet current.[13] Accordingly, these amplitudes vanish faster than any power as $t\to\infty$, as determined by the exponential factor

$$\exp\left[-\frac{g^2}{32\pi^2}\left(\sum_j c_j\right)\ln^2\frac{t}{m^2}\right], \tag{1.2}$$

where m is an arbitrary mass scale. To simplify the discussion, we will always set $M=0$ in the fixed-angle regime, in which case $m=\mu$. In (1.2) c_j is the eigenvalue of the quadratic Casimir operator for the group representation to which the jth particle belongs; for example, for a fundamental fermion we have $t^a t^a = c_F I$. The summation in Eq. (1.2) runs over all external on-shell particles. Thus if all on-shell particles are "neutral" (i.e., $c_j=0$) the exponential is absent; in other words, such amplitudes have no leading logarithms of the type $g^2\ln^2(t/\mu^2)$. In fact, such amplitudes[14] most probably obey RGE in this asymptotic limit just as in nonvector theories.[6] The Abelian analog is light-by-light scattering which lacks infrared-type logarithms in the fixed-angle regime.

An interesting variation of this situation occurs if we consider "composite" particle scattering: Suppose that, for a given process, the external particles are divided into several clusters and define the fixed-angle limit by letting all invariants $p_i p_j$, where particles i and j belong to *different* clusters, approach infinity at the same rate, all other invariants being kept fixed. In this case we find that the behavior of *connected* amplitudes is given by Eq. (1.2) where now $\sum c_i$ stands for the sum of Casimir values of the *clusters*, i.e., only the coherent sum of the "group charges" of each cluster enters and not that of the individual constituents—just as an infrared photon responds coherently to the total charge of a spatially bound system. Note again that the leading logarithmic powers of $g^2\ln^2 t$ are absent from connected amplitudes when all clusters are group singlets.

Hadron amplitudes are constructed by convoluting irreducible cluster amplitudes with appropriate bound-state wave functions. In Sec. V we discuss the implications of our work for high-energy fixed-angle hadron scattering in the framework of the non-Abelian gluon model. In particular, we argue that the accumulation of "infrared" logarithms of the type $g^2\ln^2(t/\mu^2)$ is responsible for suppressing the contribution of the disconnected Landshoff graphs.

(3) *The near-forward regime.* Whenever one or more momentum transfers vanish (or are of order μ) the singularity structure is more severe and must be examined separately. This is the regime in which eikonal-type approximations are appropriate in the Abelian case (in the $\mu\to 0$ limit). The corresponding formulas for the non-Abelian case are briefly discussed in Sec. III B.

In Sec. III we propose a simple differential equation for the dependence on μ of a general on-shell amplitude T_r (r standing for the collection of particle group indices). This equation summarizes, in compact form, all our leading-logarithm results; it has the form

$$\mu\frac{\partial}{\partial\mu}T_r=\sum_i \Gamma_{rs}^{(i)}T_s\,. \tag{1.3}$$

The quantities $\Gamma_{rs}^{(i)}$—one for each particle i—are essentially given by one-loop integrals. In the fixed-angle regime $\sum_i \Gamma_{rs}^{(i)}\sim g^2(\sum c_i)\,\delta_{rs}\ln(t/\mu^2)$ and Eq. (1.2) follows.

Equation (1.3) bears an intriguing resemblance to a renormalization-group equation; in Sec. IV we pursue this connection. The $\Gamma^{(i)}$ are momentum-dependent quantities which generalize the notion of anomalous dimensions to the infrared-singular domain of vector theories; to $O(g^2)$ they behave like a single power of $\ln(t/\mu^2)$. We argue that they are closely related to the large-N behavior of the conventional anomalous dimensions γ_N of the operators in the Wilson expansion of the product of two group-singlet currents. To $O(g^2)$ γ_N behaves like $\ln N$ when $N\to\infty$; we show how to recover the fixed-angle Γ^i from γ_N and vice versa. For theories without vector mesons there are no dominant infrared singularities[6] and γ_N vanishes as $N\to\infty$.[15]

We speculate further in Sec. IV that nonleading infrared singularities as well as ultraviolet singularities may be incorporated in Eq. (1.3) by adding a $\beta(\partial T/\partial g)$ term to the left-hand side and by including in $\Gamma^{(i)}$ effects other than leading infrared singularities. The consequences for asymptotically free theories are indicated.

In Sec. V we discuss various aspects of the physical interpretation of our results: particle confinement, constituent models of wide-angle hadron scattering, hadron binding, etc.

This is a long paper and the reader may wish to skip, on first reading, the material on perturbation theory in Sec. II, the Appendixes A and B giving a detailed treatment, in fourth-order, of the low-energy formula in the fixed-angle regime from which the basic equation (1.3) follows and Appendix C containing a heuristic proof of the "external-line rule" which characterizes the leading graphs of all orders in the infrared limit.

II. PERTURBATION THEORY

In this section we briefly outline the findings of graphical "experiments" carried out to sixth order for the quark group-singlet form factor and for the elastic quark-quark amplitude in both the infrared and the fixed-angle regime. We also discuss in some generality one-loop corrections to various lowest-order processes, since we argue in Sec. III that these one-loop quantities essentially determine the leading logarithms to all orders.

Most technical details will be omitted; they will be the subject of a separate paper. However, some points of methodology should be mentioned here. Our calculations are made with a vector propagator $-ig^{\alpha\beta}(k^2-\mu^2)^{-1}$. We express the contribution of each graph in Feynman-parameter form, using well-known topological rules[16] as extended to cases with spin- and momentum-dependent numerators.[17] The asymptotic behavior of the Feynman-parametric integral is extracted according to standard scaling techniques.[18]

In the fixed-angle regime of scattering processes or form factors at large momentum transfer t ($t \gg M^2, \mu^2$) the leading logarithms are of the form $g^{2N}\ln^{2N}(t/\mu^2)$ for an N-loop graph just as in the Abelian case,[12] and these leading contributions always come from regions of integration where the gauge meson momenta are small compared to $\sqrt{t}$. The coefficients of the leading logarithms have no ultraviolet divergences.

In the infrared regime, for processes with only (massive) quarks on-shell the leading logarithms appear as powers of $g^2\ln(\mu^2)$ (one for each loop) as in the Abelian case.[9] However, for amplitudes involving on-shell gauge mesons as well, the leading logarithms are powers of $g^2\ln^2(\mu^2)$—the infrared singularities are stronger because the non-Abelian gauge mesons are both massless *and* charged, unlike the photon in QED.

In general, we find that in both the infrared and the fixed-angle regimes leading asymptotic terms come from "end-point" contributions of the Feynman-parametric integrals, i.e., regions of integration where some set of Feynman parameters are vanishingly small. The so-called "pinch" contributions are asymptotically smaller than the leading ones by some power of μ or the momentum transfer, depending on the regime. Pinch contributions are associated with regions of integration where the internal momenta of certain reduced graphs take approximately the values of an actual nontrivial space-time process, the vertices of the graph representing local collision-production events. Pinch effects are discussed in Sec. V in connection with bound-state scattering (i.e., hadronic scattering).

Finally, there are certain classes of graphs giving no leading contributions, in the fixed-angle regime. These are graphs with closed loops of ordinary scalars, Higgs scalars, ghosts, and quarks. Also four-vector Yang-Mills couplings are unimportant in higher-order corrections.

A. Fixed-angle regime: Form factors and scattering amplitudes to sixth order

The simplest and most general results are achieved in this regime, which we therefore study first. Two processes have been completely calculated through sixth order: the group-singlet fermion form factor and the elastic fermion-fermion amplitude. Other processes which have been calculated to fourth order only will be discussed in Sec. IIB.

There are two asymptotic regions of interest: the mass-shell region, where all fermions are on-shell and the vector is given a mass μ (in the Feynman gauge), and the Sudakov[19] region, for the form factor, in which the fermions are off-shell so that $t \gg p^2$, $p'^2 \neq M^2$. In this region it is not necessary to give the vector a mass. The form factor exponentiates in both regions, the exponent being the one-loop term of Fig. 1(a), aside from a Dirac matrix $\gamma^\mu, 1, \gamma_5, \ldots$ depending on the spin and parity of the group-singlet current. For brevity we restrict ourselves to the form factor for a group-singlet scalar current (carrying momentum $p-p'$), although exponentiation of the one-loop graph of Fig. 1(a) also happens for the Yang-Mills current form factor (the one-loop graph with two vectors coupled to the Yang-Mills current is nonleading). In a way familiar from Abelian calculations[12,9] we drop k in the numerator of Fig. 1(a), and we find for the one-loop graph $[t=(p-p')^2,\ M^2=0]$

$$F^{(a)} = c_F\tilde{B}(t) = -c_F\frac{g^2}{16\pi^2}\ln^2\left(\frac{-t}{\mu^2}\right) \text{ (mass shell)}$$

$$= c_F S(t) = -c_F\frac{g^2}{8\pi^2}\ln\left(\frac{-t}{p^2}\right)\ln\left(\frac{-t}{p'^2}\right) \text{ (Sudakov).} \tag{2.1}$$

We use the notation $\tilde{B}(t)$ to emphasize that $\tilde{B}$ is the large-t limit of a function $B(t)$ which enters in the infrared regime [see Eq. (2.15)]. In (2.1), c_F is the eigenvalue of the quadratic Casimir operator of the quark representation. For the Yang-Mills current form factor, replace c_F by $c_F-\frac{1}{2}c_A$, where

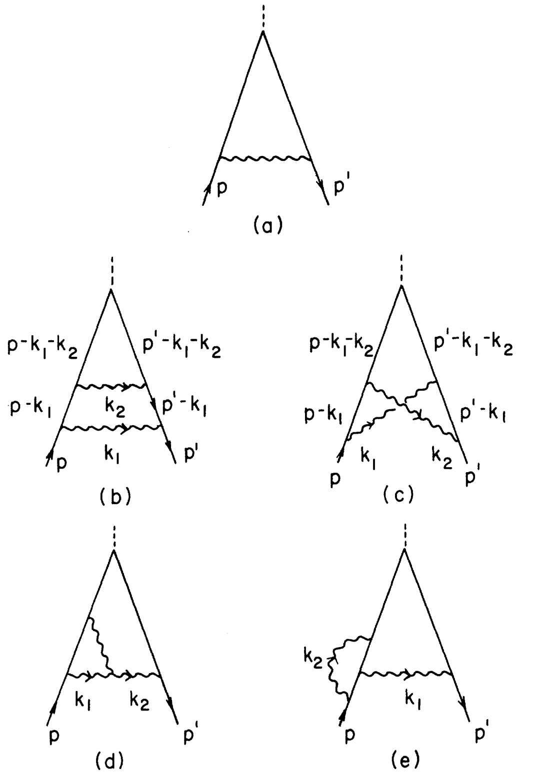

FIG. 1. Second- and fourth-order fermion form-factor graphs.

$\delta_{ab}c_A = \sum c_{aij}c_{bij}$ and the c_{aij} are structure constants. c_A is the Casimir operator for the vector particles.

Write the fourth-order contributions of Fig. 1 as

$$F^{(j)} = \tfrac{1}{2}\tilde{B}^2(t)X^{(j)} \quad \text{(mass shell)}$$

$$= \tfrac{1}{2}S^2(t)\,Y^{(j)} \quad \text{(Sudakov)},$$

$j = b, c, d, e$. Then, for the group-singlet form factor,

$$X^{(b)} = \tfrac{1}{3}c_F{}^2, \quad X^{(c)} = \tfrac{2}{3}c_F(c_F - \tfrac{1}{2}c_A), \quad X^{(d)} = \tfrac{1}{3}c_Fc_A,$$

$$Y^{(b)} = \tfrac{1}{2}c_F{}^2, \quad Y^{(c)} = \tfrac{1}{2}c_F(c_F - \tfrac{1}{2}c_A), \quad Y^{(d)} = \tfrac{1}{4}c_Fc_A,$$

$$X^{(e)} = Y^{(e)} = 0. \tag{2.2}$$

When (2.2) is added to (2.1) and the Born term, exponentiation occurs to fourth order.

The sixth-order graphs and their values (for the group-singlet form factor) are given in Fig. 2 for the Sudakov and mass-shell regions. These values agree with the Sudakov-region values of Ref. 20 except for the last graph and the graph with three Yang-Mills vertices. Evidently exponentiation still holds; through sixth order,

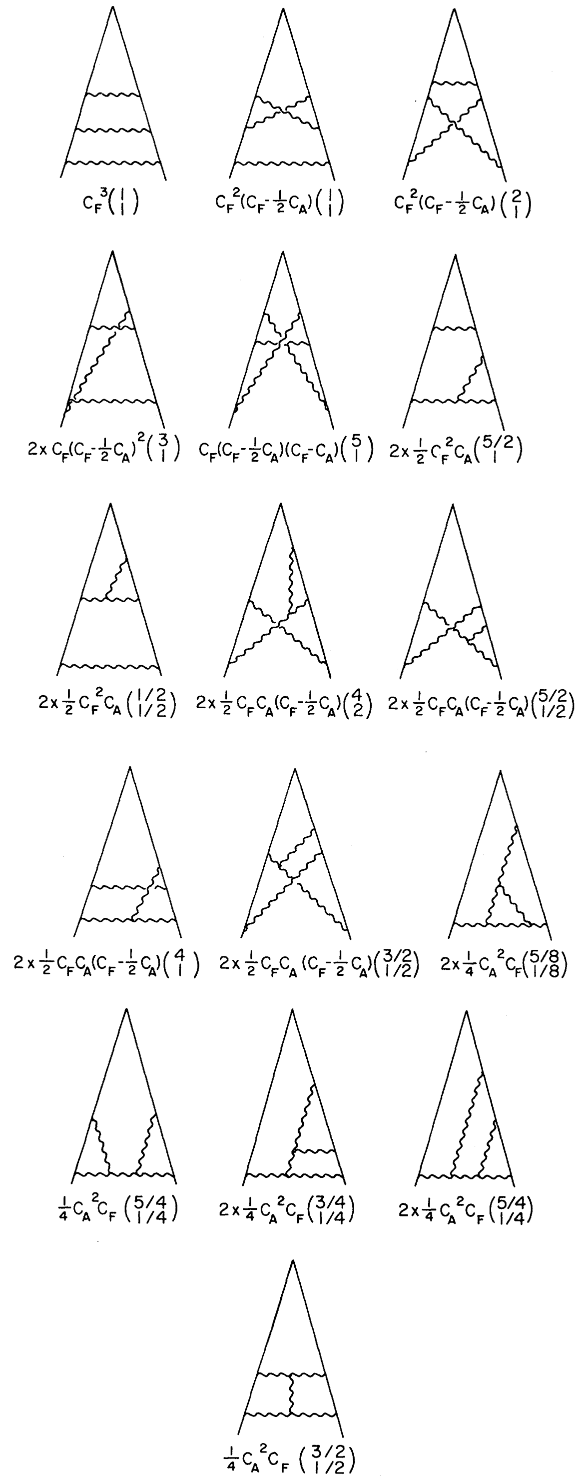

FIG. 2. Sixth-order fermion form-factor graphs. Graphs not shown are nonleading. To find the value of any graph (plus its mirror image if it is asymmetric) in the mass-shell region, multiply the weight given by the upper number by $\frac{1}{90}\tilde{B}^3$. To find the value in the Sudakov region, use the lower number and multiply by $\frac{1}{36}S^3$.

$$F(t)=\exp[c_F\tilde{B}(t)] \quad \text{(mass shell)}$$
$$=\exp[c_F S(t)] \quad \text{(Sudakov)}. \tag{2.3}$$

We have calculated numerous other form factors to fourth order, and they all exponentiate. Examples are: quark-quark-vector with one quark off-shell, vector-vector-vector with one vector off-shell, vector–vector–group-singlet. This last is the vector analog of (2.3). Denote it by $G^{\alpha\beta}$; then

$$G^{\alpha\beta}(t)=G^{B\alpha\beta}\exp[c_A\tilde{B}(t)] \tag{2.4}$$

in the mass-shell region (analogously in the Sudakov region), where $G^{B\alpha\beta}$ is the Born approximation.

Fixed-angle scattering amplitudes are expressible in terms of the group-singlet form factors (2.3) and (2.4) and the Born approximation T_B. For fermion-fermion elastic scattering the second- and fourth-order graphs are shown in Fig. 3, while the sixth-order graphs are shown in Fig. 4. It is not difficult to see that the leading terms in these graphs come when the large momentum transfer is routed through only one vector line, while all the other vector lines are soft (i.e., vector momenta k can be neglected compared to fermion momenta).[21] Removing the vector line carrying large momentum "pinches" the graph into form-factor-like expressions which are evaluated using the techniques already discussed for the form factor. A truly miraculous cancellation of complicated group-theoretic coefficients, in which the Jacobi identity plays a crucial role in sixth order, allows the graphs to be grouped into terms with the simple group properties of the Born term [Fig. 3(a)]. As with the form factor, graphs involving ghosts, closed loops of scalars and fermions, four-vector couplings, etc., are nonleading, but so are some graphs which have none of these features. Thus Fig. 3(d) is nonleading (as already mentioned in connection with the form factors). Figures 3(b), 3(c), and 3(e) add up to

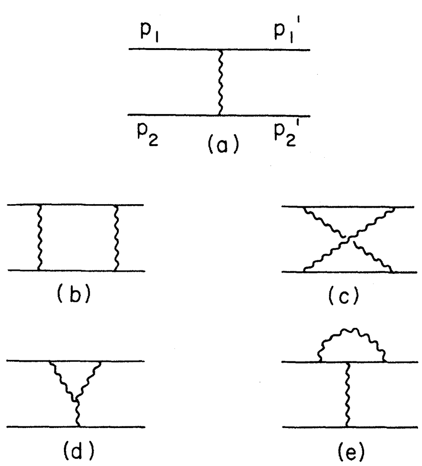

FIG. 3. Second- and fourth-order fermion-fermion elastic scattering amplitudes.

$$T^{(bce)}=2T_B\tilde{B}\,c_F\,. \tag{2.5}$$

We need not specify whether the argument of $\tilde{B}$ is s, t, or u, since, e.g., $\tilde{B}(s)-\tilde{B}(t)$ is nonleading in the fixed-angle regime. The weights for the sixth-order graphs (apart from a common factor of T_B) are shown in Fig. 4. Adding everything up, we have

$$T \xrightarrow[\text{angle}]{\text{fixed}} T_B(1+2\tilde{B}\,c_F+2\tilde{B}^2 c_F{}^2)\sim T_B F^2 \tag{2.6}$$

in terms of the fermion form factor F of (2.3). It is already tempting to speculate that, in the fixed-angle regime, on-shell processes are described by multiplying the Born approximation by $F^{1/2}$ for every fermion line and by $G^{1/2}$ for every vector line. Below, we show that this is true for one-loop corrections to any tree graph, and in Sec. III we offer arguments that it is true in general.

B. Fixed-angle regime: General one-loop processes

Consider the S-matrix element, in the *tree* approximation, for an arbitrary on-shell process:

$$T_{\alpha\beta}=\langle\alpha\text{ in}|\beta\text{ out}\rangle$$
$$=\langle\lambda_1\cdots\lambda_N|\lambda_{N+1}\cdots\lambda_M\rangle, \tag{2.7}$$

where particles of momentum p_i and group index λ_i are involved; it is unnecessary to specify the spin of the particles. The fixed-angle regime for such a process has $p_i^2=m_i^2$, $|p_i\cdot p_j|\gg m^2$ for $i\neq j$, and the ratios of all invariants $p_i\cdot p_j$ are $O(1)$. We further specify that all internal propagators are far off-shell: the momentum q of any internal line is such that $q^2=O(p_i\cdot p_j)$. (If for some internal line $q^2\approx m^2$, the graph describes the compounding of two S-matrix elements, each of which can be treated separately.)

The leading one-loop singularities are generated by adding one vector propagator of momentum k in all possible ways, and saving only the most singular terms for small k. Because all internal lines are far off-shell, it is readily seen that the vector need only be attached to *external* lines, and that on each such line one may use a simple convective vertex (i.e., spin zero) no matter what the spin of the external particle is. The vector propagator is described by the emission and reabsorption of a vector momentum k. According to the above discussion the emission process leads to a modified S-matrix element

$$\sum_{i=1}^{M}\frac{gp_\alpha^i}{p_i\cdot k}\,t^{c(i)}_{\lambda_i\lambda_i'}\,\langle\lambda_1\cdots\lambda_N|\lambda_{N+1}\cdots\lambda_M\rangle'\eta_i\,, \tag{2.8}$$

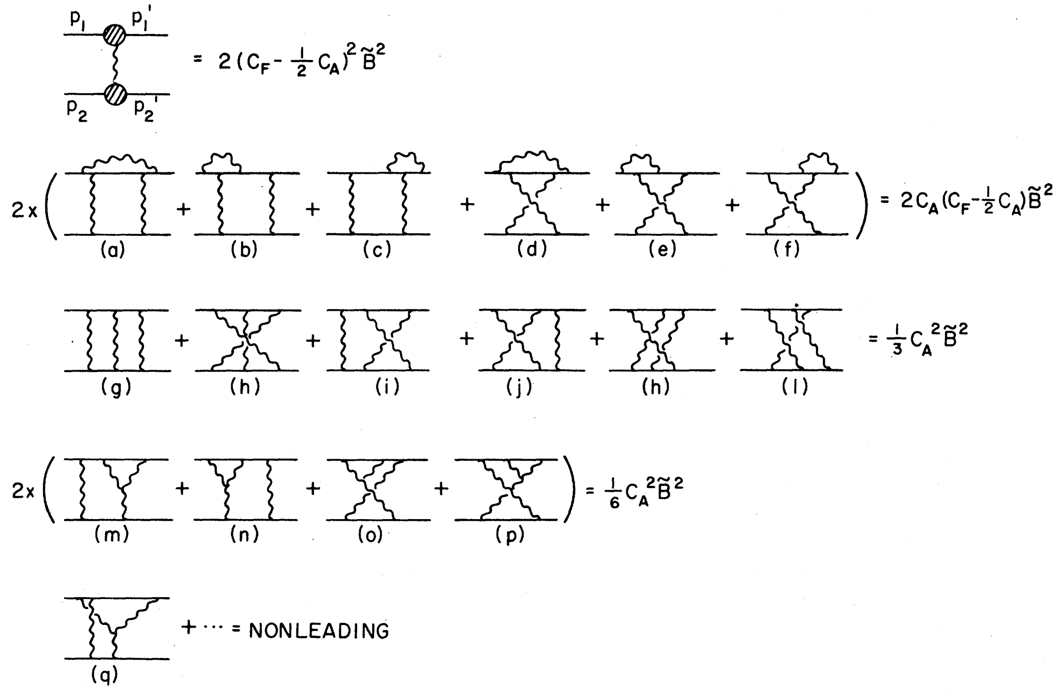

FIG. 4. Sixth-order fermion-fermion elastic scattering amplitudes. In the fixed-angle regime multiply the weights shown by the Born approximation.

where the emitted vector has momentum k_α and group index c, and the $t^{c(i)}_{\lambda\lambda'}$ are generators of the group representation for the various external particles. (Use the transpose of t^c for outgoing particles.) It is important to note that the k_α line need never be associated with a four-vector vertex; these are nonleading. In (2.8), $\eta_i = +1$ for an ingoing particle, $\eta_i = -1$ for an outgoing particle; the notation $\langle\,|\,\rangle'$ means that λ_i is replaced by $\lambda_{i'}$ and a sum over the possible values of $\lambda_{i'}$ is performed.

Observe that if the factors $p^i_\alpha/p^i\cdot k$ were replaced by 1 in (2.8), the resulting sum vanishes identically, or, in other words, k_α times (2.8) gives zero. This represents the conservation of the Yang-Mills charges $Q^c = \int d^3x\, J^c_0$, which is manifestly true in the tree approximation. Depending on whether Q^c is considered to act to the right or to the left in (2.9), two equivalent expressions are generated for

$$\langle \lambda_1 \cdots \lambda_N | Q^c | \lambda_{N+1} \cdots \lambda_M \rangle = \sum_{i=1}^{M} t^{c(i)}_{\lambda_i\lambda_{i'}} T' = \sum_{i=N+1}^{M} t^{c(i)}_{\lambda_i\lambda_{i'}} T' \qquad (2.9)$$

which establishes the vanishing of (2.8) when $p^i_\mu/p^i\cdot k \to 1$.

The reabsorption of the vector meson is described by another application of (2.8),

$$-\tfrac{1}{2}\sum_{i,j} g^2 \frac{p^i_\alpha p^j_\beta}{p^i\cdot k\, p^j\cdot k}\, t^{c(i)}_{\lambda_j\lambda_{j'}} t^{c(j)}_{\lambda_i\lambda_{i'}} \times \langle \lambda_1 \cdots \lambda_N | \lambda_{N+1} \cdots \lambda_M \rangle'' \eta_i\eta_j\,, \qquad (2.10)$$

in an obvious extension of the notation in (2.8). The one-loop description is completed by multiplying (2.10) by the vector propagator and integrating over k. Observe that those terms in (2.10) for which $i=j$ are nonleading; they correspond to emission and reabsorption of the vector on the same external line. These nonleading effects are to be absorbed into renormalization constants. Also note that[22]

$$\frac{ig^2}{(2\pi)^4}\int \frac{d^4k\, p^i\cdot p^j}{(k^2-\mu^2)\,p^i\cdot k\, p^j\cdot k} = -\tilde{B}(p^i\cdot p^j) \qquad (2.11)$$

and that we need not distinguish the various arguments of the $\tilde{B}$. Then we find

$$T(\text{1-loop}) = -\tfrac{1}{2}\tilde{B}\sum_{i\neq j} t^{c(i)}_{\lambda_j\lambda_{j'}} t^{c(j)}_{\lambda_i\lambda_{i'}} T''\eta_i\eta_j\,. \qquad (2.12)$$

If the terms with $i=j$ were included in (2.12), the sum would be zero, by virtue of (2.9). We thus find (setting $i=j$)

$$T(\text{1-loop}) = \tfrac{1}{2}\tilde{B}\sum_i (t^c)_{\lambda_i\lambda_{i'}}{}^2\, T' = \tfrac{1}{2} C_i \tilde{B} T\,, \qquad (2.13)$$

where C_i is the eigenvalue of the quadratic Casimir operator $(t^c)^2$ for the ith particle. Adding this to the tree approximation,

$$T=\left(1+\tfrac{1}{2}\tilde{B}\sum C_i\right)T(\text{tree})$$
$$\simeq\prod_i\left[\exp(\tfrac{1}{2}C_i\tilde{B})\right]T(\text{tree})\,. \qquad (2.14)$$

We conjecture (and try to establish more generally in Sec. III) *that the fixed-angle limit of any process is found by multiplying the Born approximation by* $\Pi^i F_i^{1/2}(t)$, *where* $F_i(t)=\exp(C_i\tilde{B})$ *is the asymptotic form factor for the ith particle.*

So far our results refer only to processes involving elementary quanta. We expect (and we present evidence in Sec. III C) that in a non-Abelian color gluon theory no colored particle will appear in asymptotic states. The question then remains: What happens to *composite* color singlets (i.e., hadrons) in the fixed-angle regime? We can only give a partial answer now, because we do not yet know how to handle the dynamics of binding of elementary constituents into hadronic singlets. In the spirit of various parton-model calculations of such processes we think of the constituents as essentially free in the fixed-angle regime and ignore the effects of binding. Thus we are led to a "composite" hadron state as introduced in Sec. I: a group-singlet cluster of particles whose momenta are not far off-shell. A typical *connected* Born term for the scattering of such composites is shown in Fig. 5. In the fixed-angle limit all invariants $p_i\cdot p_j$, where p_i and p_j belong to *different* clusters, approach infinity at the same rate; all other invariants are kept fixed. Note that here three "hard" vector-meson lines must be present rather than one [as for the elementary-particle process of Fig. 3(a)]. The leading one-loop corrections to the Born term come only from the exchange of a soft virtual gauge meson between external lines belonging to *different* composites.

Repeating the steps which lead from Eq. (2.8) to

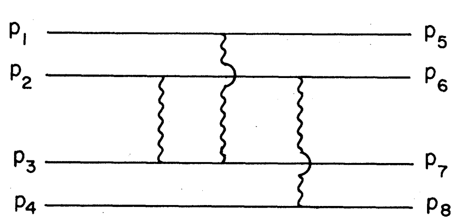

FIG. 5. An example of a connected Born term with three hard gauge mesons for the scattering of "composite" states.

Eq. (2.13) we easily find that Eq. (2.13) continues to hold except that C_i is the quadratic Casimir operator eigenvalue for the group representation to which the ith cluster belongs as a whole and not the individual constituents. In the Abelian case, this will be recognized as the statement that a long-wavelength photon probes only the total charge of a bound state, not the individual constituent charges. We refer to this as *coherent addition* of group charges in the non-Abelian case.

The rule given above for fixed-angle processes in leading-logarithm approximation in terms of a product of form factors and the Born term continues to hold. Its most important consequence is: For connected scattering amplitudes and form factors of group-singlet composites there are no leading logarithmic powers of the type $g^2\ln^2(t)$ in the fixed-angle regime. The implications of this fact for hadron scattering are discussed in Sec. V in connection with the "quark-counting" rule.

C. Infrared regime

We consider first amplitudes having only massive quanta (i.e., fermions) on the mass shell. For such amplitudes it turns out that the infrared regime (as defined in Sec. I) is, in a sense, simpler than the fixed-angle regime, at least when only leading logarithms are considered, because a rather limited class of graphs is leading; in fact, if we ignore vertex and self-energy insertions it is possible to extract and sum up the leading logarithmic terms to all orders of perturbation theory. The result is not always simple exponentiation as in the fixed-angle case (since we cannot ignore the distinction between various invariants $p_i\cdot p_j$, the group-algebraic details are more complicated) but they can still be simply stated in terms of an equation involving a derivative with respect to the infrared cutoff μ (see Sec. III).

Let us begin by considering once more the group-singlet quark form factor. In the regime $\mu^2\ll t=(p'-p)^2$ the one-loop contribution of Fig. 1(a) is correctly given by the integral (M = quark mass)

$$F^{(2)}(t)=c_F B(t,M^2,\mu^2)\,,$$
$$B(t,M^2,\mu^2)=-\frac{g^2}{16\pi^2}\int_0^1 d\beta\frac{2M^2-t}{M^2-\beta(1-\beta)t}\times\ln\left[\frac{M^2+\mu^2-\beta(1-\beta)t}{\mu^2}\right]. \qquad (2.15)$$

The asymptotic limit of this integral for $t\gg M^2,\mu^2$ yields the fixed-angle result (2.1) (which, as mentioned above, is equivalent to setting $M=0$). Being now interested in the infrared limit $\mu\to 0$ at *fixed* $t,M^2\neq 0$, we have

$$B \to \frac{g^2}{8\pi^2} H\left(\frac{t}{M^2}\right) \ln(\mu^2)\,, \tag{2.16}$$

where

$$H\left(\frac{t}{M^2}\right) = \frac{1}{2}\int_0^1 d\beta \frac{2M^2 - t}{M^2 - \beta(1-\beta)t}\,. \tag{2.17}$$

The fourth-order graphs, except for self-energy insertions to the fermion and gluon propagators, are shown in Fig. 1 [the mirror graphs of 1(d) and 1(e) must also be included]. A detailed calculation (which we shall report in a subsequent publication) shows that the contributions from graphs 1(b), 1(d), 1(e) and the self-energy corrections to 1(a) all behave like $[g^2\ln(\mu^2)]^2$ whereas 1(c) behaves like $g^4\ln(\mu^2)$, i.e., it is nonleading [this is far different from the fixed-angle regime where the leading graphs are 1(b), 1(c), and 1(d)]. The following *heuristic* argument shows why: Let k_1 and k_2 be the virtual meson momenta as shown in Fig. 1 and observe that in all graphs but 1(c) there is a logarithmic divergence in the k_1 integration (for small k_1) no matter what the value of k_2 is. The situation is well illustrated by the simplified integral

$$\int_0^1 \frac{dk_1}{k_1+\mu}\int_0^1 \frac{dk_2}{k_1+k_2+\mu} \underset{\mu\to 0}{\sim} \frac{1}{2}\int_0^1 \frac{dk_1}{k_1+\mu}\int_0^1 \frac{dk_2}{k_2+\mu} \sim \frac{1}{2}\ln^2\mu\,. \tag{2.18}$$

Accordingly, to get the leading behavior of Fig. 1(b) simply drop k_1 in the upper fermion lines and divide by two. The result then is obviously half the square of the one-loop contribution:

$$\frac{1}{2}\left[\frac{c_F g^2}{8\pi^2} H\left(\frac{t}{M^2}\right)\ln(\mu^2)\right]. \tag{2.19}$$

The "crossed ladder" graph 1(c) is nonleading and behaves only like $g^4\ln(\mu^2)$ because its infrared divergence comes from a region of integration where k_1 and k_2 become small together. More precisely, after performing the k_2 integration one obtains the integral

$$\int \frac{d^4k_1}{k_1^2-\mu^2}\frac{1}{p\cdot k_1} f(k_1)\,,$$

where, for small k_1, the function f behaves like $1/k_1$. This contrasts with the analogous function for graph 1(b) which behaves like $(\ln k_1)/k_1$.

If we now ignore for the moment the vertex correction graphs 1(d) and 1(e) as well as the self-energy insertions to propagators (not shown), we find that the second- and fourth-order results exponentiate according to

$$F \sim \exp\left[\frac{c_F g^2}{8\pi^2} H\left(\frac{t}{M^2}\right)\ln(\mu^2)\right]. \tag{2.20}$$

If we ignore vertex and self-energy insertions, i.e., if we limit ourselves to *skeleton* graphs, it turns out that, according to an argument presented in Appendix C, only the straight ladder graphs give leading contributions to the form factor. For the ladder graph of order $2n$ the generalization of (2.18) reads

$$\int_0^1 \frac{dk_1}{k_1+\mu}\int_0 \frac{dk_2}{k_1+k_2+\mu}\cdots\int_0 \frac{dk_n}{k_1+k_2+\cdots+k_n+\mu} \sim \frac{1}{n!}\left(\int_0 \frac{dk}{k+\mu}\right)^n \sim \frac{1}{n!}(\ln\mu)^n\,, \tag{2.21}$$

which completes the exponential in (2.20). Note that the leading behavior of the integral in (2.21) is unaffected when the domain of integration is restricted by

$$\frac{k_{i+1}}{k_i} > \text{const},\quad i=0,1,2,\ldots,n,\quad k_0=\mu\,,$$

i.e., there is a hierarchy of size for the virtual-gluon momenta, the lowest one being associated with the outermost rung of the ladder.

How is this result modified when vertex and self-energy insertions are included? Going back to fourth order we find that the vertex correction graphs Fig. 1(d) and 1(e) as well as the self-energy insertions to Fig. 1(a) all behave like $[g^2\ln\mu^2]^2$, i.e., they are leading contributions. This reflects the infrared structure of the fermion-fermion-gluon vertex $\Gamma_\nu^a(p,p+k)$ (where p and $p+k$ are the fermion momenta with $p^2=M^2$), the fermion propagator $S_F(p+k)$ ($p^2=M^2$), and the gluon propagator $D_{F\mu\nu}(k)$ for $k\to 0$ ($k\gg\mu$). For all three of these quantities the leading singularities come as powers of $g^2\ln k$ (one for each loop) which give rise to powers of $g^2\ln\mu$ when inserted as corrections to skeleton graphs. Thus the leading singularities for the form factor come from the straight ladder graphs plus their vertex and self-energy corrections. Accordingly, Eq. (2.20) is modified—the exponent is not just the one-loop triangle graph, but all its vertex and propagator corrections must be included.

We have not completed the study of Γ_ν^a, S_F, and $D_{F\mu\nu}$ in a general gauge, so in this paper we limit ourselves to skeleton graphs in our discussion of the infrared limit of amplitudes with only massive quanta on the mass shell. In QED the validity of naive Ward identities ensures that the infrared logarithms cancel exactly between Γ_ν and S_F; also $D_{F\mu\nu}$ is nonsingular in QED. Thus in the Abelian case exponentiation of the one-loop result [Eq. (2.20) with e^2 replacing $c_F g^2$] is the exact answer. In the non-Abelian theory this cancellation does *not* occur in general, as the one-loop calculation for

Γ^a_ν, S_F, and $D_{F\mu\nu}$ shows. It is worth noting that in the ghost-free gauges (such as the light-cone or axial gauge) $Z_1 = Z_2$, a consequence of which is that the fermion vertex and self-energy singularities cancel. However, there remains a correction to the gluon propagator, which amounts to an effective modification of the $1/r^2$ character of the long-range force between "colored" quanta. This should be intimately related to the confinement problem (confinement folklore holds that the long-range "potential" between a quark and an antiquark grows linearly with the distance between them).

The generalization to arbitrary amplitudes (with only massive quanta on the mass shell) is given by the following "external-line rule" (see Appendix C): *To obtain the entire set of leading N-loop skeleton graphs, connect by a virtual gluon line a pair of external mass-shell fermions of each N-loop skeleton graph in all possible ways.*

We illustrate this rule for quark-quark scattering in Fig. 4 where graphs 4(a), 4(d), and 4(g)–4(l) are leading, 4(b), 4(c), 4(e), 4(f), and 4(m)–4(p) are leading but not skeleton graphs, and 4(q) is a skeleton graph but not leading.

It is obvious that no three-vector or four-vector coupling vertices appear in the leading skeleton graphs so that *topologically* they are the same set of graphs as in the Abelian theory. To avoid possible misunderstanding we emphasize again at least two important features which are not true in the Abelian case: (i) vertex and propagator corrections (which will be ignored in this paper) may modify crucially the long-range behavior of the theory, (ii) even for the skeleton graphs the group matrix structure of the bare fermion-fermion-vector vertices prevents an immediate summation of the leading logarithms. This second difficulty is resolved in Sec. III by noting that the sum of leading terms satisfies a linear differential equation. A similar differential equation may also provide an extension of these results to the near-forward regime (see Sec. III B).

We turn now to the infrared behavior of amplitudes with one or more gluons on the mass shell. As specified in Sec. I, these gluons are given a mass equal to the infrared cutoff μ, they are transversely polarized, and their three-momenta (as well as the three-momenta of the mass-shell fermions) are held fixed at nonzero values as $\mu \to 0$. Two- and three-loop calculations show that the leading infrared singularities involve *two* powers of $\ln\mu$ for each loop. Furthermore, the leading graphs are not just the ones given by the "external-line rule" described above for quark amplitudes; rather, the situation here resembles the fixed-angle regime where there is a great topological variety of leading graphs. Graphical calculations (up to three-loop graphs) show again that the leading logarithms sum up to exponential factors—one for each mass-shell gluon. The results are summarized in Sec. III.

Finally, we should emphasize an important fact which is obvious from perturbation theory: An amplitude with only *neutral* (i.e., color-singlet) particles on the mass shell is nonsingular in the infrared regime (this is analogous to the absence of infrared singularities in light-by-light scattering in QED). Such neutral particles may, of course, be introduced as the quanta of appropriate color-singlet fields coupled to the quarks. More importantly, in the colored quark-gluon model, hadronic amplitudes should have no infrared singularities since hadrons are color-singlet bound states—just as the amplitude for the scattering of hydrogen atoms has no infrared singularities.

As will be discussed in the next section, our results indicate that the analogy with QED breaks down when we look at the production of charged particles from a collision of neutral bound states (or neutral particles in general): In QED, as $\mu \to 0$ the cross sections for production of a given number of charged particles and photons ("exclusive" cross sections) vanish; however, cross sections for production of a given number of charged particles and an *indefinite* number of photons ("inclusive" cross sections) do not vanish as $\mu \to 0$. In Yang-Mills theories, as $\mu \to 0$ both exclusive and inclusive cross sections for non-neutral particles vanish—only neutral particles, i.e., hadrons, can be produced.

III. SUMMING PERTURBATION THEORY

A. Differential equation

Guided by the simplicity of the results found in the graphical experiments of Sec. II, we look for principles which allow us to generalize these results to all orders of perturbation theory. In this section we introduce a differential equation which summarizes all the leading-logarithm calculations reported in Sec. II in both the fixed-angle and the infrared regime. Our hope is that this equation actually incorporates *all* powers of $\ln\mu$ and not just the leading ones; this is certainly true in the Abelian theory,[9] for which all the remarks we make in this section can be fully justified—in fact, our differential relation seems to constitute a direct and comprehensive approach to the infrared singularities in QED.

The differential equation is based on a certain formula for the emission of soft gluons. It can be proven to all orders in QED using conventional eikonal techniques, as we indicate below. This soft-meson formula has only been verified to fourth order in the fixed-angle limit for non-Abelian gauge theories (see Appendix A), but in the infrared regime it correctly reproduces the leading singularities of all skeleton graphs since at that level it is equivalent to the external line rule (see Sec. III C and Appendix C). At present, the soft-meson formula should be treated as a conjecture, which in fact will have to be modified in the infrared regime (in a straightforward way) when nonskeleton graphs are considered. But this conjecture is much more far-reaching than the simple-minded conjecture that form factors exponentiate, based on the sixth-order results. We emphasize our belief that the soft-meson formula (and its generalization) is the key to further progress in understanding the infrared singularities of gauge theories.

Consider first the fermion form factor $F(p,p')$ for a scalar group-singlet current $J(x)$. It is constructed in the Feynman gauge with vector propagator $-g_{\alpha\beta}(k^2-\mu^2)^{-1}$. The derivative of F with respect to $\ln\mu$ (reminiscent of a mass-insertion term in the Callan-Symanzik equation) can be expressed as an integral over an amplitude $T^{ab}_{\alpha\beta}(p,p';k,q)$ with two extra gauge-meson lines:

$$\mu\frac{\partial}{\partial\mu}F(p,p')=\frac{-i\mu^2}{(2\pi)^4}\int\frac{d^4k}{(k^2-\mu^2)^2}\,T^{aa}_{\alpha\alpha}(p,p',k,-k)\,, \tag{3.1}$$

where repeated group indices (a) and Lorentz indices (α) are summed over. The amplitude T is given (aside from kinematical factors) by

$$T^{ab}_{\alpha\beta}(p,p';k,q)=\int d^4x\,d^4y\,e^{ikx+iqy}\langle p'|\,TJ^a_\alpha(x)J^b_\beta(y)J(0)\,|p\rangle\,. \tag{3.2}$$

The full (covariant) amplitude is gotten by adding to T graphs where the currents J^a_α and J^b_β meet at the same point and it obeys a Ward identity. These additional graphs do not contribute in Eq. (3.1) since they are always antisymmetric in the group labels a,b or else they involve nonleading four-meson seagull vertices. Also nonleading are all ghost-line contributions, which otherwise would have to be included in Eq. (3.1) if the mass μ is generated by a Higgs mechanism since the ghost lines have mass μ in the Feynman gauge. Again, these complications in the Ward identities[23] may be neglected since we shall see that in Eq. (3.1) the momenta k, $q=-k$ are essentially on-shell and for on-shell gauge mesons the Ward identities express current conservation. Thus we have the Ward identities:

$$k_\alpha T^{aa}_{\alpha\beta}=0\,,\quad q_\beta T^{aa}_{\alpha\beta}=0\,, \tag{3.3}$$

valid when the components of k and q are of order μ.

The point of singling out one internal gauge line in the form factor $F(p,p')$ by differentiating it as in Eq. (3.1) is that the leading contributions to the k integral in this formula come from values of the components of k of order μ, as would be readily apparent if the integral were Euclidean [note that T is $O(k^{-2})$ for small k as inferred from the tree approximation]. This argument may not be made for the momenta of the undifferentiated lines, since F is homogeneous of degree zero in the momenta of these lines.

If all we required was the behavior of T for $k,q\ll\mu$, the obvious pole graphs[24] would supply the answer: Attach the soft gauge mesons to the *external* lines of F in all possible ways. This yields

$$T^{ab}_{\alpha\beta}(p,p';k,q)\simeq g^2\left(\frac{p_\alpha}{p\cdot k+i\epsilon}+\frac{p'_\alpha}{-p'\cdot k+i\epsilon}\right)t^aF(p,p')\times t^b\left(\frac{p_\beta}{p\cdot q+i\epsilon}+\frac{p'_\beta}{-p'\cdot q+i\epsilon}\right). \tag{3.4}$$

Note that the kinematical structure of Eq. (3.4) is of order $1/kq$ and obeys the Ward identities of Eq. (3.3).

We now propose that Eq. (3.4) is a valid approximation not only when $k,q\ll\mu$ but also in the regime where k and q are of the same order of magnitude as μ and μ is small compared to p and p'. In QED, this soft meson formula can be proved, using familiar eikonal techniques, to all order of perturbation theory, the remainder being smaller than the leading term [i.e., the right-hand side of Eq. (3.4)] by a whole power of μ and not just a power of $\ln\mu$ (see the discussion in Appendix A). Unfortunately, these eikonal techniques do not work so simply in the non-Abelian case because of the noncommutative group structure of the vertices. Appendix A gives an explicit demonstration of Eq. (3.4) in the fixed-angle limit, to fourth order, for the non-Abelian theory, where a kind of eikonal structure is seen to emerge for three-vector vertices. Possibly, the proof could be extended to all orders by means of combinatorial methods analogous to those used by 't Hooft[10] for proving Ward identities. Note that in the infrared limit using Eq. (3.4) in Eq. (3.1) is equivalent to the "external-line" rule discussed in Sec. II C and Appendix C.

Inserting $T^{ab}_{\alpha\beta}$ from Eq. (3.4) into Eq. (3.1) yields

$$\mu\frac{\partial}{\partial\mu}F(p,p')=c_F\,F(p,p')\,\mu\frac{\partial}{\partial\mu}[B(p,p')-B(p,p)]\,, \tag{3.5}$$

where

$$B(p,p')=\frac{-ig^2}{(2\pi)^4}\int\frac{d^4k\,p'\cdot p}{(k^2-\mu^2)(k\cdot p')(k\cdot p)}\,. \qquad (3.6)$$

The general solution to Eq. (3.5) is

$$F(p,p')=F_0(p,p')\exp\{c_F[B(p,p')-B(p,p)]\}\,, \qquad (3.7)$$

where F_0 is a μ-independent amplitude. Note that the integral for $B(p,p')$ in Eq. (3.6) is initially defined in terms of an appropriate ultraviolet cutoff which, of course, does not appear in $B(p,p') - B(p,p)$. We thus identify the μ dependence of the vertex renormalization constant Z_1 as $Z_1 = \hat{Z}_1\exp[c_F B(p,p)]$, where $\hat{Z}_1$ is independent of μ. Evidently, Eq. (3.7) yields the correct fixed angle limit ($t\to\infty$) [Eq. (2.3)] and infrared limit [Eq. (2.20)] of F since

$$B(p,p')\sim\ln(\mu^2)\frac{g^2}{8\pi^2}H(t)\quad(p,p'\text{ fixed},\ \mu\to 0)$$
$$\sim -\frac{g^2}{16\pi^2}\ln^2\left(\frac{t}{\mu^2}\right)\quad(t\gg M^2,\mu^2)\,.$$

The generalization to other processes is straightforward in view of the one-loop discussion of Sec. II [see, in particular, Eq. (2.10)]. Consider an amplitude $T(\ldots;p_i,\lambda_i;\ldots)$ involving a number of particles with (on-shell) momenta p_i and internal group indices λ_i, as well as an unspecified set of off-shell external lines. The differential relation reads

$$\mu\frac{\partial}{\partial\mu}T(p_1,\lambda_1;\ldots) = -\frac{i\mu^2}{(2\pi)^4}\int\frac{d^4k}{(k^2-\mu^2)^2}\,T^{aa}_{\alpha\alpha}(p_1,\lambda_1;\ldots;k,-k)\,, \qquad (3.8)$$

where $T^{ab}_{\alpha\beta}(p_1,\lambda_1;\ldots;k,q)$ is the amplitude with two extra gauge-vector lines of momenta k,q, Lorentz indices α,β, and group indices a,b. The soft-meson formula for $T^{ab}_{\alpha\beta}$ reads

$$T^{ab}_{\alpha\beta}(p_1,\lambda_1,\ldots;k,q) \simeq g^2\sum_{i,j}\frac{p_{i\alpha}p_{j\beta}}{(\eta_i p_i\cdot k-i\epsilon)(\eta_j p_j\cdot q-i\epsilon)} \times t^{(i)a}_{\lambda_i\lambda_i'}\,t^{(j)b}_{\lambda_j\lambda_j'}\,T(\ldots;p_i,\lambda_i';\ldots;p_j,\lambda_j';\ldots)\,, \qquad (3.9)$$

where $D^{(i)a}$ is the ath group generator matrix for the ith particle or its transpose and $\eta_i=+1$ or -1 depending on whether the ith particle is incoming or outgoing. This form of the soft-meson formula is adequate for *nonforward amplitudes*, i.e., amplitudes with no vanishing momentum transfer. In the case of one or several vanishing momentum transfers, neglecting the dependence of T on k and q on the right-hand side of Eq. (3.9) is not justified. We comment on this case later in this section.

From Eqs. (3.8) and (3.9) we obtain

$$\mu\frac{\partial}{\partial\mu}T(p_1,\lambda_1;\ldots) = -\tfrac{1}{2}\sum_{i,j}\eta_i\eta_j\left(\mu\frac{\partial}{\partial\mu}B_{ij}\right)t^{(i)a}_{\lambda_i\lambda_i'}\,t^{(j)a}_{\lambda_j\lambda_j'} \times T(\cdots\lambda_i'\cdots\lambda_j'\cdots)\,, \qquad (3.10)$$

where

$$B_{ij}=\frac{-ig^2}{(2\pi)^4}\int\frac{d^4k}{k^2-\mu^2}\,\frac{\eta_i\eta_j(p_i\cdot p_j)}{(\eta_i p_i\cdot k-i\epsilon)(-\eta_j p_j\cdot k-i\epsilon)} \qquad (3.11)$$

Note that B_{ij} depends on the invariant $(\eta_i p_i+\eta_j p_j)^2\equiv t_{ij}$ and the squared masses $p_i^2\equiv\frac{1}{4}t_{ii}$, $p_j^2\equiv\frac{1}{4}t_{jj}$.

On the basis of Eqs. (3.10) and (3.11) we proceed to discuss the fixed-angle and infrared regimes separately.

Fixed-angle limit. We consider the general situation in which the particles of a given connected amplitude T are divided into several clusters $l_1, l_2,\ldots$ (a cluster may consist of just one particle) such that

$$t_{ij}=\text{fixed if } i,j \text{ belong to the same cluster}\,,$$
$$t_{ij}=t\rho_{ij},\quad \rho_{ij}\text{ fixed}\neq 0,\quad t\to\infty\text{ otherwise}\,. \qquad (3.12)$$

Consider, first, the case where all the momenta are *strictly* on-shell and a "vector mass" cutoff μ is used. We are encouraged to use the $\mu(\partial/\partial\mu)$ trick in this case because we know from our experience with graphs that the leading logarithmic terms never depend on the fixed t_{ij} invariants, so that we may replace $\mu(\partial/\partial\mu)$ by $-2t(\partial/\partial t)$ in Eq. (3.10). As $t\to\infty$ those B_{ij}'s for which i and j belong to *different* clusters dominate in Eq. (3.10) having the same leading asymptotic term

$$B_{ij}\simeq -\frac{g^2}{16\pi^2}\ln^2\left(\frac{t}{\mu^2}\right)+O(\ln t)\,.$$

Retaining only the leading $\ln^2(t/\mu^2)$ terms in Eq. (3.10) we have

$$t\frac{\partial}{\partial t}T=\frac{1}{2}\,t\frac{\partial}{\partial t}\left[\frac{g^2}{16\pi^2}\ln^2\left(\frac{t}{\mu^2}\right)\right] \times\left[\sum_{i,j}\eta_i\eta_j t^{i(a)}t^{j(a)} - \sum_{\nu}\sum_{j\in l_\nu}\eta_i\eta_j t^{i(a)}t^{j(a)}\right]T\,, \qquad (3.13)$$

which, in view of the group-charge conservation equation $\sum\eta_i t^{(i)a}T=0$, becomes

$$t\frac{\partial}{\partial t}T=-\frac{1}{2}\left(\sum_\nu c_\nu\right)T\,t\frac{\partial}{\partial t}\left[\frac{g^2}{16\pi^2}\ln^2\left(\frac{t}{\mu^2}\right)\right], \quad (3.14)$$

where

$$c_\nu=\left(\sum_{i\in l_\nu}\eta_i t^{(i)a}\right)\left(\sum_{j\in l_\nu}\eta_j t^{(j)a}\right).$$

In other words, c_ν is the value of the quadratic Casimir operator for the group representation to which the cluster l_ν belongs (in the Abelian theory c_ν is the total charge of the cluster squared in units of the fundamental fermion charge).

Equation (3.14) leads to the exponential form

$$T\propto\exp\left[-\frac{g^2}{32\pi^2}\left(\sum_\nu c_\nu\right)\ln^2\left(\frac{t}{\mu^2}\right)\right], \quad (3.15)$$

in agreement with our findings in perturbation theory (Sec. II).

Equation (3.15) is easily generalized to the case where a set of external momenta are *far off-shell*, i.e., their components grow like $\sqrt{t}$. The sums over i and j in Eqs. (3.10) and (3.13) do not sum over off-shell particles, and instead of Eq. (3.15) we find

$$T\propto\exp\left[-\frac{g^2}{32\pi^2}\left(\sum_\nu c_\nu-c\right)\ln^2\left(\frac{t}{\mu^2}\right)\right]. \quad (3.16)$$

Here c stands for the value of quadratic Casimir operator for the *coherent* set of all on-shell particles (which equals, of course, that of the coherent set of all off-shell particles[25]).

We would also like to discuss, in the fixed-angle-limit, the scattering of "composites" as introduced in Sec. I; in other words, to relax the strict mass-shell requirement for the momenta while, of course, retaining the cluster conditions of Eq. (3.12). In constructing such off-shell amplitudes no vector-mass cutoff is needed and the $\mu(\partial/\partial\mu)$ equation is not applicable. Nevertheless, our perturbation calculations suggest that the answer for $t\to\infty$ is obtained by exponentiating the one-loop result:

$$T\propto\exp\left[-\frac{g^2}{16\pi^2}\left(\sum_\nu c_\nu\right)\ln^2\frac{t}{m^2}\right]. \quad (3.15')$$

Here m^2 is a typical fixed invariant. A special case of Eq. (3.17) verified to sixth order in g is the off-shell quark form factor in the Sudakov limit (see Sec. II).

Infrared limit. Recall that the infrared limit ($\mu\to0$, momenta fixed) is distinct from the (high-energy) fixed-angle limit in a theory with massive fundamental fermions or quarks—the mass of a quark being defined in terms of the singularity of the fermion propagator. Of course, even in a theory of massless quarks the infrared limit is distinct from the fixed-angle limit for processes involving massive bound states.

Looking at Eqs. (3.10) and (3.11) we note that as $\mu\to0$ with the external momenta held fixed we have

$$\begin{aligned}B_{ij}&=-\frac{g^2}{16\pi^2}\ln^2(\mu^2)+O(\ln\mu^2)\quad\text{if } i,j \text{ are both massless}\\&=-\frac{g^2}{32\pi^2}\ln^2(\mu^2)+O(\ln\mu^2)\quad\text{if } i \text{ or } j \text{ is massless [see Eq. (2.25)]}\\&=O(\ln(\mu^2))\quad\text{if } i,j \text{ are both massive}.\end{aligned}$$

This if the amplitude in question has *at least one massless external particle which is nonsinglet* (e.g., gauge meson), the leading behavior comes from the $g^2\ln^2(\mu^2)$ terms and if we only retain those leading terms, we obtain

$$\mu\frac{\partial}{\partial\mu}T=-\frac{g^2}{32\pi^2}\ln^2\left(\frac{t}{\mu^2}\right)\times\left(\sum_{M_i=0}c_i+\tfrac{1}{2}c_M-\tfrac{1}{2}c_0-\tfrac{1}{2}c\right)T, \quad (3.16')$$

where the c_i refer to on-shell particles of zero mass and c_M, c_0, c are the coherent Casimir eigenvalues of the set of massive particles, the set of massless particles, and the set of off-shell lines, respectively. Note that when all external particles are on-shell $c_M-c_0-c=0$ and we conclude that nonforward on-shell amplitudes involving at least one massless nonsinglet particle vanish in the infrared limit faster than any power of the infrared cutoff μ.

We now come to processes involving no massless nonsinglet particles; for example, quark-quark scattering in a theory with massive quarks. For such amplitudes none of the B_{ij}'s in Eq. (3.10) has a $\ln^2(\mu^2)$ leading behavior. Instead, we have

$$B_{ij}=\frac{g^2}{16\pi^2}\ln(\mu^2)\times\int_0^1 d\beta\frac{M_i^2+M_j^2-t_{ij}}{\beta M_i^2+(1-\beta)M_j^2-\beta(1-\beta)t_{ij}}+O(\mu^0). \quad (3.17)$$

Since the B_{ij}'s are not, in general, asymptotically equal, the right-hand side of Eq. (3.10) does not become a simple multiple of T asymptotically; rather, we obtain a system of homogeneous differential equations with constant (i.e., μ independent) coefficients. In compact notation

$$\mu\frac{\partial}{\partial\mu}T_r=h_{rs}T_s\,, \tag{3.18}$$

where, in writing T_r, the index r stands for the collection of group indices of the external particles. The general solution of Eq. (3.18) is given in terms of the eigenvalues $\lambda^{(\rho)}$ and eigenvectors $\theta^{(\rho)}$ of the matrix h ($\rho=1,2,\ldots$):

$$T_r=\sum_\rho \theta_r^{(\rho)}e^{\lambda_\rho\ln(\mu^2)}\,. \tag{3.19}$$

We now show that $\mathrm{Re}\lambda_\rho>0$ for all nonforward amplitudes, i.e., nonforward amplitudes vanish in the infrared limit like some positive power of μ.

From Eq. (3.10) we see that if we write $B_{ij}=\mathrm{Re}\,B_{ij}+i\,\mathrm{Im}\,B_{ij}$, h_{rs} is decomposed accordingly as

$$h_{rs}=h_{rs}^{(1)}+i\,h_{rs}^{(2)}\,,$$

where $h^{(1)}$ and $h^{(2)}$ are *Hermitian* matrices. It suffices to show that $h^{(1)}$ is positive definite. Note that as $\mu\to 0$ we have

$$\mathrm{Re}\,B_{ij}\sim\frac{g^2}{(2\pi)^4}\frac{\pi}{4}\int\frac{d^3k}{(\vec{k}^2+\mu^2)^{1/2}}\left(\frac{p_{i\alpha}}{p_i\cdot k}\frac{p_{j\alpha}}{p_j\cdot k}\right)\bigg|_{k_0=(\vec{k}^2+\mu^2)^{1/2}}\,. \tag{3.20}$$

It follows that, for any eigenvector θ of h, we may write

$$\theta^*h^{(1)}\theta=\int\frac{d^3k}{(\vec{k}^2+\mu^2)^{1/2}}\,\xi_\alpha^{a*}\left(-g_{\alpha\beta}+\frac{k_\alpha k_\beta}{k^2}\right)\xi_\beta^a\bigg|_{k_0=(\vec{k}^2+\mu^2)^{1/2}}\geq 0\,, \tag{3.21}$$

where

$$\xi_\alpha^a=\sum_j\frac{\eta_jp_{j\alpha}}{p_j\cdot k}\,t_{\lambda_j\lambda_j'}^{(j)a}\,\theta(\ldots,\lambda_j',\ldots)\,. \tag{3.22}$$

Recall that, since $k_a\xi_\alpha^a=0$ by charge conservation, the insertion of the $k_\alpha k_\beta$ term in Eq. (3.21) is justified.

In Eq. (3.21) the equality sign occurs only if $\epsilon\cdot\xi^a=0$ for all lightlike k and all transverse polarization vectors $\epsilon(k)$. It is not difficult to show that this happens only if the nonsinglet external particles are divided into *group-singlet clusters*, the particles of each cluster being relatively at rest (i.e., having parallel four-momenta). Although such kinematical configurations generally correspond to zero volume in phase space, the particular case of *forward amplitudes* deserves special attention because their singularity structure as $\mu\to 0$ is more severe than the nonforward amplitudes and could compensate for the vanishing of the phase space.

A brief comment on the implications of infrared behavior for off-shell Green's functions is appropriate here. When all momenta are off-shell there are, of course, no infrared singularities in perturbation theory. A Green's function, however, becomes infrared singular if two or more momenta belonging to nonsinglet particles are on-shell. It is interesting to note that in certain cases of such partially on-shell amplitudes the infrared logarithms add up to an *exploding* exponential; an example is the quark-antiquark amplitude in the octet channel with, say, only the incoming two momenta on-shell. Can this be reconciled with a dispersion relation expressing the *finite* fully off-shell $q\bar{q}\to q\bar{q}$ amplitude in terms of its discontinuity which involves the *singular* half-off-shell amplitudes? Note that another indication of failure of dispersion relations for these off-shell quantities is the essential singularities appearing at two-particle thresholds since $B_{ij}\simeq i[t_{ij}-(m_i+m_j)^2]^{-1/2}$ near $t_{ij}=(m_i+m_j)^2$.

B. Near-forward regime

In implementing the basic differential equation (3.8) in the near-forward regime, where some invariant momentum transfer squared t vanishes or is of order μ^2, it becomes necessary to use a modified version of the soft-meson formula (3.9) in which k is not dropped as small compared to the relevant momentum transfer Δ ($\Delta^2=t$), but is dropped—according to the argument of Appendix A—as compared to the external momenta.

It is instructive to see how this works for near-forward fermion-fermion scattering in QED. Let p_1,p_1' and p_2,p_2' be the incoming and outgoing momenta for fermion 1 and fermion 2, respectively. The modified $\mu(\partial/\partial\mu)$ equation for the amplitude $T(\Delta,\ s=(p_1+p_2)^2)$ reads

$$\mu \frac{\partial}{\partial \mu} T(\Delta, s) = \mu \frac{\partial}{\partial \mu} T_B + \frac{2e^2}{(2\pi)^4 i} \int d^4k \frac{\mu^2}{(k^2-\mu^2)^2} \left(\frac{p_{1\alpha}}{p_1 \cdot k - i\epsilon} + \frac{p'_{1\alpha}}{-p'_1 \cdot k - i\epsilon}\right)\left(\frac{p_{2\alpha}}{-p_2 \cdot k - i\epsilon} + \frac{p'_{2\alpha}}{p'_2 \cdot k - i\epsilon}\right) T(\Delta - k, s)\,, \tag{3.23}$$

where T_B is the Born approximation. The two factors in parentheses correspond to attaching the two photons of the soft-meson formula to *different* fermions. The remaining terms where the two photons are attached to the *same* fermion (either both photons to the same external line or one photon to the ingoing and the other photon to the outgoing line of the fermion) cancel between themselves for $t \ll M^2$. Also, in the near-forward regime, setting $p'_i \simeq p_i$ reduces the factors in parentheses to

$$-(2\pi)^2 (p_1 \cdot p_2)\, \delta(p_1 \cdot k)\, \delta(p_2 \cdot k)$$

so that, in the center-of-mass frame, we have an integral over just the transverse components of k:

$$\mu \frac{\partial}{\partial \mu} T(\vec{\Delta}, s) = \mu \frac{\partial}{\partial \mu} T_B(\Delta) + \frac{1}{(2\pi)^2} \int d^2k\, T(\vec{\Delta} - \vec{k}, s)\, i\mu \frac{\partial}{\partial \mu} \left(\frac{e^2}{k^2 + \mu^2}\right). \tag{3.24}$$

This equation can be easily solved by first taking a two-dimensional Fourier transform with respect to $\vec{\Delta}$ which turns it into an ordinary linear differential equation. The solution is the eikonal formula

$$T = i(\bar{u}\gamma_\alpha u)(\bar{u}\gamma^\alpha u) \int d^2b\, e^{i\vec{\Delta}\cdot\vec{b}}(e^{i\chi(b)} - 1)\,, \tag{3.25}$$

where

$$\chi(b) = \frac{e^2}{(2\pi)^2} \int d^2q \frac{e^{i\vec{q}\cdot\vec{b}}}{\vec{q}^2 + \mu^2}\,. \tag{3.26}$$

Our purpose in recording here one more derivation of the well-known eikonal formula[26] is, firstly, to suggest that this formula may be profitably looked at as expressing an infrared rather than a high-energy limit, and, secondly, to make plausible the obvious generalization to the non-Abelian theory. In the non-Abelian theory the $\mu(\partial/\partial\mu)$ equation has a matrix form but the k integral is still a convolution; thus by Fourier transformation one obtains a set of ordinary linear differential equations with constant (i.e., μ-independent) coefficients. It would be interesting to test the results obtained this way with explicit calculations in perturbation theory. Also, a comparison with the high-energy, fixed-t calculations of Yao and Nieh[27] and, more recently, McCoy and Wu[28] should be instructive.

C. Cross sections in the infrared limit

We have found that cross sections for nonforward processes involving only massive particles (elementary or composite), some of which are not group-singlets, vanish like a positive power of μ in the infrared limit $\mu \to 0$. This is qualitatively not different from what happens in the Abelian case. The striking difference shows up when we look at processes involving on-shell gauge mesons as well.

Let T denote the amplitude for a process in which a fundamental fermion (or, more generally, any massive nonsinglet particle) of four-momentum p appears among the particles in the final state. Consider the amplitude for emitting, in addition, a gauge meson of momentum q. The lowest-order graph of Fig. 6(a) has a pole at $(p+q)^2 - M^2 = 2p\cdot q = 0$ and, of course, there is one such pole term for each nonsinglet external particle. Thus the cross section for emission of a (transverse) gauge meson with a three-momentum $\vec{q}$ smaller in magnitude than a given cutoff Q has, in lowest order, a logarithmic enhancement of the form

$$g^2 \int_{|\vec{q}|<Q} \frac{d^3q}{(\vec{q}^2+\mu^2)^{1/2}} \left[\frac{|T|^2}{(p\cdot q)^2}\right]_{q_0=(\vec{q}^2+\mu^2)^{1/2}} \simeq g^2 |T|^2 \ln\left(\frac{Q}{\mu}\right). \tag{3.27}$$

In the Abelian theory this logarithmic divergence is not altered by higher-order corrections—the *Abelian* full vertex Γ depicted in Fig. 6(b) is finite for $\mu \ll 2p\cdot q/M$. Similarly, N-fold pole terms make the N-photon emission cross section diverge like $(N!)^{-1}[g^2\ln(Q/\mu)]^N|T|^2$. When summed over N, these partial cross sections exponentiate to exactly cancel the vanishing infrared factor in $|T|^2$ (i.e., the μ dependence due to the soft virtual photons). Thus in the Abelian theory the cross section with emission of an indefinite number of soft photons approaches a finite, nonzero value as $\mu \to 0$.

In contrast, in the non-Abelian theory our leading-logarithm calculations indicate that higher-order corrections, due to exchanges of soft virtual gauge mesons between the on-shell meson of momentum q and all other nonneutral particles in the process, provide the exponential factor

$$\exp\left[-\frac{g^2}{32\pi^2} c_A \ln^2\left(\frac{|\vec{q}|}{\mu}\right)\right] \tag{3.28}$$

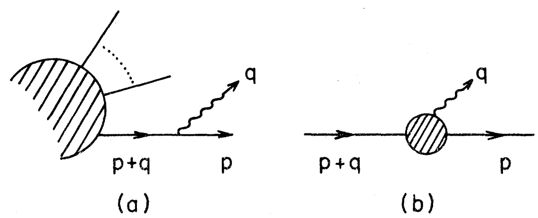

FIG. 6. Graphs for the emission of a soft real gauge meson form a fermion line in the infrared regime.

as long as $\mu \ll (p_j \cdot q)/M$ for all momenta p_j of the non-neutral particles. This follows immediately from the basic equations (3.10) and (3.11) and the fact that

$$\frac{-ig^2}{(2\pi)^4}\int \frac{d^4k}{k^2-\mu^2}\frac{(p_i\cdot q)}{(p_i\cdot k - i\epsilon)(-q\cdot k - i\epsilon)} \simeq -\frac{g^2}{32\pi^2}\ln^2\left(\frac{p_i\cdot q}{\mu M}\right)$$

for $\mu M \ll p_i\cdot q$. The exponential (3.28), included as a factor in the integrand of Eq. (3.27), provides an effective cutoff at $Q \simeq O(\mu)$ so that no logarithmic enhancement of the cross section is possible. To see this, note that

$$\int \frac{d^3q}{(\vec{q}^2+\mu^2)^{1/2}}\frac{1}{(p\cdot q)^2}\exp\left[-\frac{g^2}{16\pi^2}c_A\ln^2\left(\frac{|\vec{q}|}{\mu}\right)\right]_{\mu\to 0} = \text{finite}. \tag{3.29}$$

Observe also that the $\mu=0$ value of the integral in Eq. (3.29) behaves as $1/g$ for small g which means that it is impossible to recover the perturbation expansion *after* μ is set equal to zero.

For the emission of several gauge mesons with momenta $q_1, q_2, \ldots$ such that $q_i\cdot p_j \gg \mu M$, $q_i\cdot q_j \gg \mu^2$ we similarly obtain the exponential factor

$$\exp\left[-\frac{g^2}{32\pi^2}c_A\sum_j \ln^2\left(\frac{|\vec{q}_j|}{\mu}\right)\right],$$

which provides an effective cutoff for all gauge-meson three-momenta at $|\vec{q}_i| = Q_i \simeq O(\mu)$, thus eliminating the possibility of logarithmic enhancements from multiple particle poles. Note that configurations in which $q_i\cdot q_j = O(\mu^2)$ or, more generally, in which clusters consisting of two or more gauge mesons have total invariant mass of order μ, are appropriately suppressed by phase space.

If the above leading-logarithm calculations reflect more or less correctly the behavior of vector-emission cross sections we must conclude that in non-Abelian theories the N-vector-emission cross section σ_N is a *finite multiple* of the no-vector-emission cross section σ_0 in the infrared limit, i.e., $\sigma_N = c_N\sigma_0$. Thus, assuming that the sum over N does not introduce a new kind of divergence, the cross section with emission of an indefinite number of soft gauge mesons vanishes as $\mu\to 0$ just like the no-meson-emission cross section. The suppression of gluons with momenta $\{q_i\}$ satisfying $q_i\cdot p_j \gg \mu M$, $q_i\cdot q_j \gg \mu^2$ occurs only after summation to all orders of perturbation theory for each σ_N and therefore there is no violation of the Kinoshita-Lee-Nauenberg (KLN) theorem[29] which states that there are no "mass singularities" in the inclusive cross section *order-by-order* in perturbation theory. There are other theories in which the KLN theorem could be misleading. For example, in (2+1)-dimensional QED, the static Coulomb potential between charged fermions is proportional to the logarithm of the distance which suggests that the charged fermions are confined. The KLN theorem nonetheless predicts the complete cancellation of all photon mass singularities, order by order.

The most interesting aspect of this situation, from the physical point of view, emerges if we assume that there are group-singlet bound states in the theory representing ordinary hadrons. Our discussion then indicates that in a collision of hadrons no particles carrying the group charge (i.e., quarks, gluons or colored bound states) will ever be produced—only hadrons will appear in the final state. The analytic structure of hadronic amplitudes and color-singlet (e.g., electromagnetic) hadronic form factors cannot contain any trace of the elementary constituents in its poles and branch cuts. As a result, no *long-range* forces, analogous to the van der Waals forces between neutral atoms, exist between hadrons. In a world of such hadrons, quarks and gluons are permanently confined and "color" is a hidden degree of freedom.

IV. RELATION TO THE RENORMALIZATION GROUP

The techniques developed in Sec. III, although developed for reasons having nothing to do with the renormalization group (RG), yield results which call for interpretation in the language of the RG. This is especially clear in the fixed-angle regime where simple exponentiation and factorization occur. While there is certainly no bar in principle to extending this interpretation to the infrared regime (where simple exponentiation breaks down), we restrict ourselves for the present to the fixed-angle regime.

The properties of exponentiation and factorization developed in Sec. III suggest an RG interpretation with infrared singular anomalous dimensions (ISAD's). ISAD's differ from conventional (short-distance or ultraviolet-singular) anomalous dimensions in two respects: First, they are momentum dependent, and second, they add *coherently* for "composite" states. This property of coherency is to be expected in view of the long-range nature of the infrared singularities.

While we use the language of the RG in this section, in fact ISAD's come, in part, from mass-insertion terms in the inhomogeneous Callan-Symanzik[30] equations. These mass-insertion terms are the leading singularities near the mass shell for fixed-angle processes, dominating the ultraviolet singularities which give rise to only one logarithm for every power of g^2. (However, in the infrared regime long-distance singularities associated with massive particles yield one logarithm for every power of g^2.) The ultraviolet singularities are associated solely with coupling constant and wave-function renormalization (if the ultraviolet mass-insertion terms are nonleading, as we assume) and give rise to the Callan-Symanzik function $\beta(g)$ as well as the conventional (momentum-independent) anomalous dimensions γ_i. For the moment we ignore all such renormalization effects, but we consider their inclusion at the end of this section.

The prototype fixed-angle equation developed in Sec. III,

$$\mu\frac{\partial}{\partial\mu}S_{\alpha\beta}=\sum c_i\frac{g^2}{8\pi^2}\ln\left(\frac{-t}{\mu^2}\right)S_{\alpha\beta}\,, \tag{4.1}$$

looks very much like an RG equation with an ISAD,

$$\Gamma_i(t)=c_i\frac{g^2}{8\pi^2}\ln\left(\frac{-t}{\mu^2}\right), \tag{4.2}$$

assigned to each on-shell particle or to each "composite" state, with c_i the quadratic Casimir eigenvalue for the particle or for the "composite" state. A question arises which we cannot completely answer yet: Is μ to be considered as a vector-meson mass (as in Sec. III), so that (4.1) is in the spirit of the Callan-Symanzik equations, with the right-hand side considered as a mass insertion, or should μ enter as an off-shell renormalization parameter? In the latter case, we would focus our attention not on mass-shell amplitudes with massive vector mesons but on amplitudes with external momenta obeying $p_i^2=M_i^2-\mu^2$, and set the mass M_V of the vector mesons equal to zero. Then a simple calculation of the one-loop graph shows that the $\Gamma_i(t)$ in (4.2) should be multiplied by two (in effect, we enter the Sudakov regime). Until a better understanding of the confinement process is reached, it is not possible to draw a definite conclusion as to which is more convenient since the concept of the mass of a confined particle is rather elusive. In any event, the distinction will become significant only when nonleading effects are considered; then one will have to renormalize μ if it is treated as a mass.

There is an intimate connection between the ISAD's of (4.2) and the conventional anomalous dimensions γ_N occurring in the Wilson expansion[31] of $J(x)J(0)$, where $J(x)$ is the group-singlet scalar current whose form factor has been studied in Secs. II and III. For large N, the γ_N behave in leading order like $\ln N$ in vector theories,[32] and vanish for other theories.[15] We show that this $\ln N$ behavior comes from the same infrared singularities which produce the logarithmic momentum dependence of the ISAD's. In higher orders, there appears to be a mixture of infrared and ultraviolet effects which we defer to later work.

First, we recapitulate the well-known techniques[15,32] for finding the γ_N. The Wilson expansion is

$$T(J(x)J(0))\underset{x^2\sim 0}{\sim}\sum_N c_N(x^2-i\epsilon)O^N_{\alpha_1\cdots\alpha_N}(0)x^{\alpha_1}\cdots x^{\alpha_N}\,. \tag{4.3}$$

The forward matrix element of this reads

$$\langle p|T(J(x)J(0))|p\rangle=\sum c_N(x^2-i\epsilon)(x\cdot p)^NA_N\,, \tag{4.4}$$

$$\langle p|O^N_{\alpha_1\cdots\alpha_N}(0)|p\rangle=p_{\alpha_1}\cdots p_{\alpha_N}A_N+g_{\alpha\beta}\text{terms}\,, \tag{4.5}$$

where $|p\rangle$ is a one-quark state. Although we have not explicitly indicated it in (4.3)–(4.5), for each N there may be several operators, which leads to the problem of operator mixing in calculating the anomalous dimensions of the O_N.[33] But, according to the lowest-order calculations at least, in the limit of large N the off-diagonal mixing terms in the anomalous dimension matrix are nonleading. Furthermore, there should be no problems of mixing with ghost operators since these are not dominant in the infrared (large-N) region.

The short-distance behavior of the c_N in (4.4) is probed by expanding the Deser-Gilbert-Sudarshan (DGS) representation[34] for (4.4) in powers of $p\cdot x$:

$$iT(x,p)=\int d\lambda^2\int_{-1}^{1}d\beta\, e^{i\beta p\cdot x}h(\sigma,\beta)\Delta_\lambda(x)\,, \tag{4.6}$$

$$c_N(x^2)\sim\int d\lambda^2 d\beta\,\beta^N h(\sigma,\beta)\Delta_\lambda(x)\,, \tag{4.7}$$

where $\Delta_\lambda(x)$ is the free-field Feynman propagator for mass λ, and $\sigma=\lambda^2-\beta^2-p^2$. Note that h is an even function of β, so only even values of N are relevant. In (4.7), irrelevant x-independent factors are dropped. In practice, the Fourier transform of (4.7),

$$\tilde{c}_N(q^2) = \int dx\, e^{iq\cdot x} c_N(x^2)$$

$$= \int d\lambda^2 d\beta \frac{h(\sigma,\beta)\beta^N}{q^2-\lambda^2}, \qquad (4.8)$$

is approximated by finding the coefficient of ω^N ($\omega = -2p\cdot q/q^2$) in the expansion of the Fourier transform $\tilde{T}(q,p)$ of T for fixed q. This coefficient is also found as the $N+1$ moment of the imaginary part of iT (denoted by W below):

$$i\tilde{T}(q,p) = \sum \omega \int d\sigma \frac{d\beta\, h(\sigma,\beta)}{(q^2-\sigma)}\left(\frac{q^2\beta}{q^2-\sigma}\right)^N \qquad (4.9)$$

$$\equiv \sum \omega^N D_N(q^2), \qquad (4.10)$$

$$D_N(q^2) = \int_1^\infty d\omega\, \omega^{-N-1} W(\omega, q^2), \qquad (4.11)$$

$$W(\omega,q^2) = \int d\sigma\, d\beta\, h(\sigma,\beta)\delta(q^2(1-\beta\omega)-\sigma). \qquad (4.12)$$

For large q^2 at fixed N, $D_N \to \tilde{c}_N$; however, when N is also large, $|q^2|$ must grow at least as fast as N for this to be true.

The asymptotic behavior of the $\tilde{c}_N$ is expressed in terms of the anomalous dimensions γ_N of the operators O_N. If the effective coupling constant is fixed (i.e., $\beta\equiv 0$), then

$$q^2 c_N(q^2) \sim (-q^2/m^2)^{-\gamma_N/2}, \qquad (4.13)$$

but in the case of asymptotic freedom ($\beta = -bg^3$)

$$q^2\tilde{c}_N(q^2) \sim [\ln(-q^2/m^2)]^{-\gamma_N/2bg^2}. \qquad (4.14)$$

Only the first case is relevant for us at the moment. The mass scale m is arbitrary, and in the limit of large N, and to lowest order in g^2,

$$\gamma_N \to \frac{g^2}{2\pi^2} c_F \ln N + O(1). \qquad (4.15)$$

Corresponding to (4.13) and (4.14) is the RG equation (choosing $m=\mu$)

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_N\right)\tilde{c}_N = 0. \qquad (4.16)$$

We now show that a specific set of infrared-singular graphs yields the $\ln N$ terms in γ_N. Evidently from (4.11) the large-N momenta receive contributions only from the neighborhood of $\omega = 1$, or, more precisely, $\omega - 1 = O(N^{-1})$. Only one-particle reducible states ought to be important for $\omega \simeq 1$; e.g., $W(\omega, q^2)$ for the bare Born term is proportional to $\delta(1-|\omega|)$. Let us consider the contribution of the one-particle reducible graphs, specifically excluding all propagator corrections and setting the fermion mass to zero:

$$\tilde{T}_{(1)}(q,p) = \frac{1}{(p+q)^2} F^2(p,p+q), \qquad (4.17)$$

where $F(p,p+q)$ is the fermion form factor with one leg off-shell. The infrared-singular part of F is given by the exponential of the one-loop graph; in leading order,

$$F(p,p+q) = \exp\left[\frac{-g^2}{8\pi^2} c_F \omega \ln\left(\frac{-q^2}{\mu^2}\right)\ln\left(\frac{2-\omega}{1-\omega}\right)\right]. \qquad (4.18)$$

For arbitrary ω there is only one $\ln q^2$ in the exponent, but as $\omega\to 1$ the leg $p+q$ goes on-shell and the exponent becomes the Sudakov exponent. Incidentally, (4.18) is found whether μ is treated as the vector meson or as a renormalization point.

Directly from (4.17) and (4.18) we derive the equation

$$\mu\frac{\partial}{\partial\mu}\tilde{T}_{(1)} = \frac{g^2}{2\pi^2} c_F \omega \ln\left(\frac{2-\omega}{1-\omega}\right)\tilde{T}_{(1)}. \qquad (4.19)$$

In the region $\omega \simeq 1$ the same equation holds in leading order for the imaginary part of $\tilde{T}_{(1)}$, denoted $W_{(1)}$, since $\operatorname{Im}\ln(1-\omega)$ is nonleading. Then for the large-N momenta, where $\omega - 1 = O(N^{-1})$, (4.11) shows that

$$\mu\frac{\partial}{\partial\mu} D_N(q^2) = \frac{g^2}{2\pi^2} c_F \ln N\, D_N(q^2), \qquad (4.20)$$

which is the same as (4.15) and (4.16), when $\beta\equiv 0$ and $\tilde{c}_N$ is identified with D_N. *The large-N behavior of the γ_N [to $O(g^2)$ at least] is identified with infrared-singular contributions to the form factor.*

Another way of seeing this connection is to use the Mellin convolution formula to express $W(\omega,q^2)$ in terms of W at some fixed $q_0^2 \gg \mu^2$ (Ref. 33):

$$W(\omega\;\, q^2) = \frac{1}{2\pi i}\int_1^\omega \frac{d\omega'}{\omega'} W\left(\frac{\omega}{\omega'}, q_0^2\right) \times \int dN\, \omega'^{N+1}\left(\frac{q^2}{q_0^2}\right)^{-\gamma_N/2}. \qquad (4.21)$$

Assume that the threshold behavior of W for $q^2 = q_0^2$ is

$$\omega \simeq 1:\;\; W(\omega, q_0^2) \sim (\omega-1)^{P(q_0^2)}, \qquad (4.22)$$

where P is a q-dependent power and factors independent of ω are not written. This behavior is what follows from (4.17) and (4.18). A few simple integrations in (4.21) [using the specific expression (4.15) for γ_N] show that the threshold behavior (4.22) with q_0^2 replaced by q^2 is reproduced if and only if

$$P(q^2) = \tfrac{1}{2}\gamma_N\left(N = \frac{-q^2}{\mu^2}\right) + \text{const}$$
$$= \frac{g^2}{4\pi^2} c_F \ln\left(\frac{-q^2}{\mu^2}\right) + \text{const}\,, \qquad (4.23)$$

precisely as in (4.17) and (4.18). Similar manipulations have been used[35] with an *ad hoc* choice of $P(q_0^2)$, and with asymptotic freedom ($\beta \neq 0$) included, in applications to *hadronic* matrix elements. We specifically exclude such an application, since the Casimir operators $c_F \equiv 0$ for a hadron.

Nonetheless, it is interesting to inquire into the mathematical structure of the RG equation and its solution when nonleading effects are taken into account, such as β and the usual anomalous dimensions Γ_i. Presumably the RG equation simply becomes (aside from gauge terms)

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \sum \Gamma_i(t)\right] S_{\alpha\beta} = 0 \qquad (4.24)$$

for S-matrix elements in the fixed-angle regime. The Γ_i are given by (4.2) (or twice that value, depending on the interpretation of μ) plus nonleading terms, including certain gauge-invariant pieces of the γ_i, and for $|t| \gg \mu^2$ we need only save leading powers in g^2 of these coefficients. The general solution to (4.24) is expressed in terms of the effective coupling constant $\bar{g}(z)$, where $z = \frac{1}{2}\ln(-t/\mu^2)$,

$$S_{\alpha\beta} = R[\bar{g}(z)] \exp\left\{-\int_0^z dz' \sum \Gamma_i[\bar{g}(z'); z - z']\right\} \qquad (4.25)$$

times an arbitrary function of ratios of invariants (e.g., t/s). [ln(4.25), $\bar{g}(z)$ is normalized to g at $z = 0$.] No useful physical information can be gotten from (4.25) until the Γ_i and $\bar{g}(z)$ are known for *all* values of z. However, to indicate the nature of the corrections induced by including $\beta(g)$, consider the asymptotically free case $\beta = -bg^3$, $b > 0$, and the Γ_i as given in (4.2) are assumed to hold for all z. The form factor for large positive z has a term like

$$F_i(t) \sim \exp[-(c_i/8\pi^2 b)\ln t \ln\ln t]\,, \qquad (4.26)$$

where one power of $\ln t$ is replaced by $\ln\ln t$, and the explicit dependence on g^2 is gone. Such terms in the form factor have been deduced earlier[35] by considering the Mellin transform (4.21) with the asymptotically free behavior (4.14) of the $c_N(q^2)$ used, rather than the $\beta \equiv 0$ values (4.13).

It is expected that in the region $t \ll \mu^2$, or $-z \gg 1$, $\bar{g}(z)$ becomes very large. Then, presumably, (4.26) is entirely wrong and the Γ_i cannot be computed from lowest-order processes.

V. PHYSICAL INTERPRETATION

A. Confinement vs dynamical symmetry breakdown

We have described an infrared mechanism by which the presence of non-Abelian massless gauge quanta suppresses the degrees of freedom associated with the group charges to which they couple—an infrared confinement mechanism. In this context, what we call the quark mass and the (vanishing) gluon mass are operationally defined in terms of the singularities in the full propagators of the quark and gluon fields in perturbation theory; in fact, our calculations in the infrared regime are based on assuming a particular pole structure for the bare propagators,[36] and it is an important question whether and under what conditions such assumptions are internally consistent. For example, the alternative possibility exists of a *dynamical symmetry breakdown*[37] (DSB) in which some or all of the gauge mesons become massive; in that case, of course, the associated charges are no longer suppressed.

To the extent that short-distance effects are important an interesting constraint[38] on DSB is this: If the gauge group G (or any of its subgroups) is the group SU(n), and the only fermion representations are $\underline{N}$, $\overline{\underline{N}}$, or $\underline{1}$, then there can be no short-distance DSB of this SU(N). Also, if G (or a subgroup) is purely chiral (all left-handed or all right-handed fermions are group singlets) there is no short-distance DSB of G (or its subgroup).

In a physical model we should clearly require the color group SU(3)$_c$ (a subgroup of G) to remain unbroken, or else the resulting massive vectors will not yield the infrared confinement mechanism we have discussed here. This means, in turn, that only the fermion representations $\underline{3}$, $\overline{\underline{3}}$, and $\underline{1}$ of SU(3)$_c$ may occur (which is, of course, the usual assumption). On the other hand, if G is SU(N), then there must be something more that $\underline{N}$, $\overline{\underline{N}}$, and $\underline{1}$. A mathematical example is Georgi and Glashow's[39] SU(5) which has the fermions in both the 5- and 10-dimensional representations of SU(5) but only in the $\underline{3}$, $\overline{\underline{3}}$, and $\underline{1}$ of SU(3)$_c$.

B. Relation to other confinement ideas

In discussing the infrared regime we introduced a cutoff mass μ and confinement takes place as $\mu \to 0$. It is conceivable that μ^{-1} can be interpreted as the characteristic size of the confinement region in analogy with the existing "bag" models of hadrons.[40] However, we see no relation of these models to our work since, at this stage, no concrete space-time picture of a hadron has emerged from our calculations.

Confinement in strongly coupled gauge theories has been discussed by Wilson[41] who uses a space lattice and techniques from many-body theory. Any relation to our work is clouded by the fact that the non-Abelian nature of the gauge does not seem crucial in Wilson's work.

A soluble field-theoretic model in which confinement takes place is Schwinger's[42] massless QED in two-dimensional space-time. Is there a hint of a two-dimensional structure emerging in our work? Our calculations indicate that the nonforward ($t \neq 0$) quark-quark amplitude vanishes as $\mu \to 0$, but we have had little to say about the near-forward regime ($t=0$) where a singularity like $\delta(t)$, for example, would effectively reduce quark scattering to one space dimension.

In any case, there are one or two other features of two-dimensional gauge theories worth noting. First, in two dimensions it is always possible to eliminate three- and four-vector Yang-Mills couplings by using a gauge such $\eta_\alpha A^\alpha = 0$ for some fixed vector η_α. Note that, as mentioned in Sec. II C such a gauge[43] may be particularly useful for exploring how self-energy insertions to the gluon propagator modify the long-range force between quarks. Second, the properties of exponentiation and factorization (in the fixed-angle regime) have a decidedly eikonal flavor, even though in four dimensions there are complicated cancellations (see Appendix A) before this virtually Abelian structure emerges. Even in an Abelian theory the eikonal structure is an approximation, with one known exception: massless two-dimensional QED, where it is exact. The question thus arises whether confinement in four dimensions occurs by an effective reduction to a two-dimensional theory.[44]

C. Large-momentum-transfer processes

Quite independently of the details of hadron binding and confinement, our results in the fixed-angle regime may have a bearing on large-momentum-transfer hadronic processes. It has been noted that the experimental measurements of exclusive high-energy wide-angle hadronic scattering are compactly summarized by scaling laws[5] of the form

$$\frac{d\sigma}{dt} \sim s^{2-N} f(t/s), \quad s \to \infty\,, \quad s/t \text{ fixed} \tag{5.1}$$

where s and t denote the squared c.m. energy and momentum transfer, respectively and N is the total number of elementary constituents (quarks) of the hadrons according to the quark-model assignments (meson $\sim q\bar{q}$, baryon $\sim qqq$) in the initial and final state. Also, the (spin-averaged) electromagnetic form factor of a hadron behaves like t^{1-n} for large momentum transfer t, where n is the number of hadron constituents. It is interesting to note incidentally, that, even without relating the exponents to the quark content, the power behavior of the exclusive cross section as given in Eq. (5.1) is the same as that of a product of hadron form factors.[13]

The amplitude for the exclusive process is, by definition, the convolution of appropriately irreducible N-quark amplitudes with the hadronic Bethe-Salpeter wave functions. Figure 7 shows a meson-meson amplitude given as the sum of contributions from connected [7(a)], partially connected [7(b), 7(c), 7(d)], and completely disconnected [7(e), 7(f), 7(g)] irreducible quark graphs.

In order to understand the scaling laws in the context of field theory we assume, following Brodsky and Farrar,[5] that the Bethe-Salpeter wave functions vanish at large constituent momenta at least as fast as indicated by the lowest-order interaction kernel. Then an analysis in perturbation theory analogous to that of Sec. II indicates that "end-point" asymptotic contributions of individual graphs do contain infrared-type logarithms of the type $g^2\ln^2 t$ at large momentum transfer t, which, however, *cancel* in the sum of all graphs—essentially because hadrons are color-singlets. An important case is illustrated by the class of Fig. 7(a): with the hadron constituents near the mass shell, we have here what we called a singlet-cluster quark amplitude in the fixed-angle limit.

The infrared-type logarithms do not cancel in "pinch" contributions, which arise in nonplanar graphs representing multiple independent scatterings of quarks from different hadrons. For meson-meson scattering they are depicted in Fig. 7(d). As pointed out by Landshoff[45] this class of

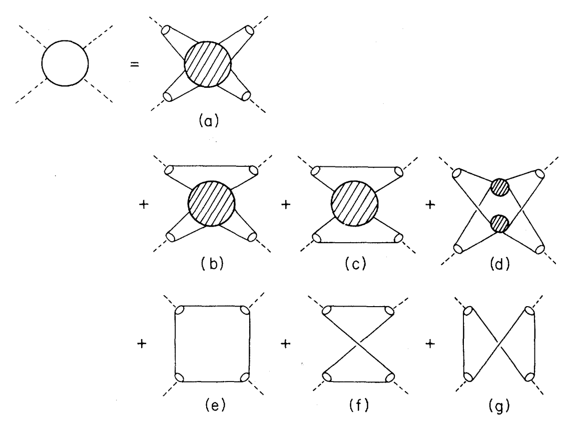

FIG. 7. The meson-meson scattering amplitude expressed in terms of irreducible quark Green's functions as classified according to their connectivity.

graphs gives a pinch contribution to the hadronic amplitude which behaves like $s^{-3/2}$ times the product of the two quark-quark scattering amplitudes (shown as blobs in the figure) evaluated near the mass shell. Since the behavior necessary for the hadronic scaling law is s^{-2}, one concludes that large-angle high-energy scattering of near-mass-shell quarks is somehow damped in energy.[46] This is readily explained in a gauge theory[47]: The "infrared" powers of $g^2\ln^2 t$ accumulate into exponential damping factors for each quark-quark blob separately. We emplasize that the topology of 7(d) is essential for the pinch: the addition, for example, of a virtual gluon exchanged between two quarks belonging to different mesons and colliding at different blobs would eliminate the $s^{-3/2}$ pinch contribution of individual graphs of that type—the resulting graphs would be classified under 7(a). A similar analysis is easily carried out for meson-baryon and baryon-baryon processes.[48]

D. Hadron binding and spectroscopy

The salient features of quark binding inside a hadron may eventually require a new description which our approach, inspired by perturbation theory, cannot express properly at this stage. Nevertheless, it seems worthwhile to push our methods to their limits by looking, for example, at Bethe-Salpeter amplitudes.

The Bethe-Salpeter quark-antiquark wave function of a meson state $|b\rangle$ with a mass $M_b > 2M$ is infrared singular as the momenta of the quark and antiquark approach their mass-shell values. In fact, the leading-logarithm calculation is essentially the same as that of the group-singlet quark form factor [see Eq. (2.20)]. This singularity is reflected in the behavior of the wave function at large spacelike separation x_μ between the constituents: It falls off *faster* than $1/(x^2)^{1/2}$, even though $M_b > 2M$, evidently another manifestation of infrared confinement. Actually, the pure fermion-antifermion amplitude $\langle | T(\psi(x)\bar\psi(0)) | b\rangle$ (for an energy-momentum eigenstate $|b\rangle$ of mass $M_b > 2M$) falls off faster than $1/(x^2)^{1/2}$ even in QED: however, in QED the appropriate gauge-invariant admixture of multiphoton amplitudes restores the $1/(x^2)^{1/2}$ behavior at large spacelike x_μ:

$$\left\langle \left| T\left(\psi(x)\bar\psi(0) \exp\left[ie \int_0^x dz^\mu A_\mu(z) \right]\right) \right| \mathrm{b} \right\rangle \simeq O\left(\frac{1}{(x^2)^{1/2}} \right).$$

This does *not* happen in the non-Abelian theory. We shall present the details of this analysis in a separate publication. It is interesting to note, however, that the infrared behavior is spin independent as well as, neglecting quark-mass differences, unitary spin independent. Thus if an "infrared barrier" is mainly responsible for quark binding, it could account for the SU(6)-like structure of the low-lying hadronic states.

Infrared effects may be important also in connection with Zweig's rule.[49] It has been suggested that this empirical rule has a dynamical origin: Large momenta must flow through gluon lines in graphs for decays violating the rule and one argues that in an asymptotically free theory the effective quark-gluon coupling constant is small at large momenta[50] (e.g., >1 GeV). This asymptotic-freedom-at-short-distance argument may be supplemented by an infrared-suppression-at-long-distance mechanism; Indeed, our work indicates that *connected* irreducible near-mass-shell quark amplitudes are infrared suppressed.

We hope that these and other important aspects of hadron physics will emerge from a further study of non-Abelian gauge dynamics.

APPENDIX A: THE SOFT-MESON FORMULA

The formula states: To $O(\mu^{-1})$, the amplitude for emission of a soft gauge vector (with momentum k of order μ) from any on-shell process is given by

$$\sum_i g \frac{p_\alpha^i}{p^i\cdot k} T^c_{\lambda_i\lambda_i'} \langle \lambda_1 \cdots \lambda_N | \lambda_{N+1} \cdots \lambda_M \rangle' \eta_i\,, \qquad \text{(A1)}$$

where $\langle\{\lambda\}|\{\lambda\}\rangle$ is the on-shell T matrix, evaluated at $k=0$ (notation is explained in Sec. II B). We emphasize the difference between (A1) and the usual Low theorem,[24] which is an expansion in powers of k at *fixed* μ. Thus the function $(p\cdot k)^{-1}\ln(1+p\cdot k\mu^{-2})$ is $O(k^0)$ for small k at fixed μ, but $O(\mu^{-1}\ln\mu)$ when $k \simeq \mu$ and $p \gg k$. Such logarithmic terms appear in each graph (containing loops) which contributes to the on-shell T matrix, but they cancel in the sum of graphs. Thus the result reduces to the conventional Low theorem.

The proof can be carried through to all orders in the Abelian case, using the eikonal approximation, and we sketch here a proof of a related result to fourth order for the non-Abelian case. Figures 8(a)–8(c) show the ways of adding one soft vector to the one-loop graph in QED (the same graphs with the extra photon on the p' line are to be added). Figure 8(c) is nonleading[51] and will be dropped. For 8(a) and 8(b) we use the *eikonal approximation*: All photon momenta (k,q) are neglected in the numerator, and a propagator denominator such as $(p+k+q)^2 - M^2$ is replaced by $2p\cdot(k+q)$, thus dropping quadratic terms in the photon momenta. This approximation leaves the leading infrared singularities unchanged (Appendix B).

Let $\Gamma(p_1,p_2)$ as defined in Appendix B be the one-

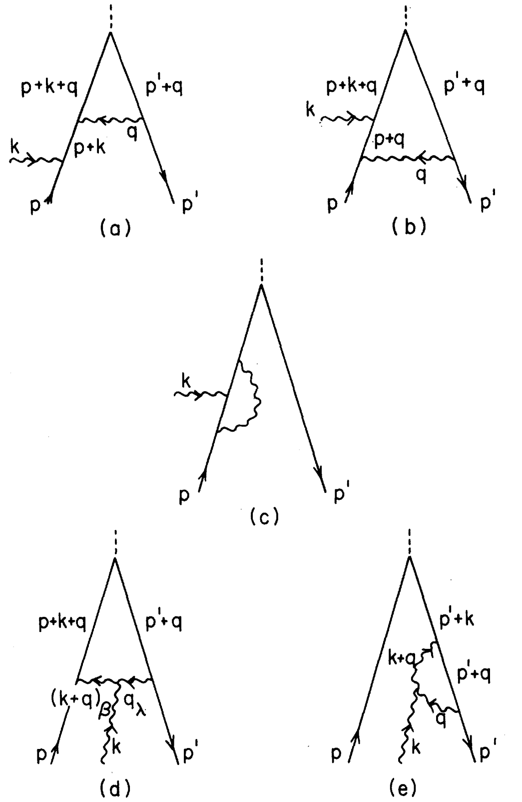

FIG. 8. Emission of one soft gauge meson from the second-order fermion form-factor graphs. The addition of mirror graphs is understood [except for 7(d)].

loop graph for two fermions coupled to a (scalar) current of momentum $p_1 - p_2$. Then Fig. 8(a) involves an *off-shell* vertex $\Gamma(p+k, p')$. But this off-shell behavior is exactly cancelled by Fig. 8(b), as we see by adding before integrating:

$$\frac{1}{p\cdot(k+q)}\left(\frac{1}{p\cdot k}+\frac{1}{p\cdot q}\right)=\frac{1}{p\cdot k}\,\frac{1}{p\cdot q}. \tag{A2}$$

This is the simplest version of the well-known eikonal identity[25] for summing over permutations of eikonalized graphs. It follows from (A2) that the sum of Figs. 8(a) and 8(b), plus their mirror graphs, is

$$F_\alpha^{(a)+(b)}=g\left(\frac{p_\alpha}{p\cdot k}-\frac{p'_\alpha}{p'\cdot k}\right)\Gamma(p,p'), \tag{A3}$$

yielding (A1). A slight familiarity with eikonal techniques will make it obvious that this proof can be extended to all orders and to all processes in QED; we do not dwell on details.

The non-Abelian case has complications, which can be resolved by an extension of the eikonal techniques. What follows is a proof that (A1) is correct at least as a leading-logarithm approximation for the set of graphs in Fig. 8, assuming that $t=(p-p')^2 \gg M^2$ and taking the scalar current with momentum $p-p'+k$ to be a group singlet. Of course, 8(e) is nonleading for large $t=(p-p')^2$ [the other graphs, except 8(c), go like $\ln^2 t$], but we save it in order to cancel a piece of 8(d), *and* because when the vector k is absorbed on the q line, yielding Fig. 1(d), the resulting graph is leading [in contrast to the analogous operation on Fig. 8(c), which yields a nonleading graph].

First note that (A3) no longer holds for Figs. 8(a) and 8(b). In fact, one finds, adding mirror graphs as before,

$$\begin{aligned}F_{\alpha c}^{(a)+(b)}&=t^c c_F\left(\frac{p_\alpha}{p\cdot k}-\frac{p'_\alpha}{p'\cdot k}\right)\Gamma(p,p')\\&\quad+\tfrac{1}{2}t^c c_A\left\{\frac{p_\alpha}{p\cdot k}[\Gamma(p+k,p')-\Gamma(p,p')]\right.\\&\qquad\left.-\frac{p'_\alpha}{p'\cdot k}[\Gamma(p,p'-k)-\Gamma(p,p')]\right\},\end{aligned} \tag{A4}$$

where c is the group index of the vector k_α, c_F the fermion Casimir operator $(t^c)^2$, and c_A the vector Casimir operator. That Fig. 8(b) can be expressed in terms of Γ follows from the eikonal identity (really an eikonalized version of an elementary Ward identity)

$$\frac{1}{p\cdot(k+q)}\,\frac{1}{p\cdot q}=\frac{1}{p\cdot k}\left[\frac{1}{p\cdot q}-\frac{1}{p\cdot(k+q)}\right]. \tag{A5}$$

The one-loop graph is now $c_F\Gamma(p,p')$ instead of just $\Gamma(p,p')$, and so if (A1) is to hold, the c_A term in (A3) must be cancelled out by Figs. 8(d) and 8(e), to which we now turn.

Figure 8(d) has the value

$$F_{\alpha c}^{(d)}=+\frac{ig^2 c_A t^c}{2(2\pi)^4}\times\int dq\,\frac{p_\beta p'_\lambda V_{\alpha\beta\lambda}}{[(k+q)^2-\mu^2](q^2-\mu^2)p'\cdot q\,p\cdot(k+q)}, \tag{A6}$$

where

$$V_{\alpha\beta\lambda}=\overline{V}_{\alpha\beta\lambda}-q_\lambda g_{\alpha\beta}-(k+q)_\beta g_{\alpha\lambda}, \tag{A7}$$

$$\overline{V}_{\alpha\beta\lambda}=(2q+k)_\alpha g_{\beta\lambda}+2k_\beta g_{\alpha\lambda}-2k_\lambda g_{\alpha\beta}. \tag{A8}$$

The decomposition (A7) and (A8) of the Yang-Mills three-vector vertex reveals two terms [the last two in (A7)] which act as pure divergences and which generate the ghostlike lines called Λ lines by 't Hooft.[10] The term $\overline{V}_{\alpha\beta\lambda}$ obeys the naive Ward identity

$$k^\alpha\overline{V}_{\alpha\beta\lambda}=[(k+q)^2-q^2]g_{\beta\lambda} \tag{A9}$$

of the type: divergence of vertex = difference of inverse propagators. Figure 8(e) has an expression similar to (A6) which we do not record, but only note that $V_{\alpha\beta\lambda}$ is multiplied by $p'_\beta p'_\lambda$. Since $p'_\beta p'_\lambda \bar{V}^{\alpha\beta\lambda}$ is nonleading, only the divergence terms in $V^{\alpha\beta\lambda}$ contribute. It is readily verified that the $(k+q)_\beta$ terms in 8(a) and in 8(b) cancel each other while the q_λ terms are nonleading. This cancellation is exactly of the type which yields the Ward identities of Yang-Mills theories.[3]

Now Fig. 8(e) has been completely disposed of and in Fig. 8(d) we need only use the vertex $\bar{V}_{\alpha\beta\lambda}$, which obeys the naive Ward identity. Furthermore, k may be neglected compared to q everywhere in the numerator, since after integration q is replaced by a linear combination of p, p' whose components are much greater than $k \sim \mu$.

The q integration can be done conventionally, but a slightly different approach is instructive. In the propagator denominator $(k+q)^2 - \mu^2$ we may ignore k^2 (and, for that matter, μ^2) compared to $q\cdot k$ for essentially the same reasons as given in the paragraph above. Then applying the eikonal identity

$$\frac{1}{(q^2+2q\cdot k-\mu^2)}\frac{1}{(q^2-\mu^2)} = \frac{1}{2q\cdot k}\left(\frac{1}{q^2-\mu^2}-\frac{1}{q^2+2q\cdot k-\mu^2}\right), \tag{A10}$$

(A6) becomes, with $k^2=0$,

$$F^{(\mathrm{d})}_{\alpha c} = -\frac{ig^2 c_A t^c}{2(2\pi)^4}\int \frac{dq\, p'\cdot p q_\alpha}{p'\cdot q p\cdot(k+q) q\cdot k} \times\left[\frac{1}{q^2-\mu^2}-\frac{1}{(q+k)^2-\mu^2}\right]. \tag{A11}$$

Observe that (A11) obeys the Ward identity which follows from (A9). There is a simple rule for integrating such a form with $q_\alpha (q\cdot k)^{-1}$, when $k^2=0$ (Ref. 43):

$$\int dq \frac{q_\alpha}{q\cdot k} F[(q-Q)^2] = \frac{Q_\alpha}{Q\cdot k}\int dq\, F[(q-Q)^2]. \tag{A12}$$

Then (A11) is

$$F^{(\mathrm{d})}_{\alpha c} = -\tfrac{1}{2} c_A t^c \left(-\frac{g^2 t}{8\pi^2}\right)\int \left(\frac{\beta_1 p_\alpha + \beta_2 p'_\alpha}{\beta_1 k\cdot p + \beta_2 k\cdot p'}\right)\left(\frac{1}{\beta_1\beta_2 t + 2\beta_1 p\cdot k - \mu^2} - \frac{1}{\beta_1\beta_2 t - 2\beta_2 p'\cdot k - \mu^2}\right) d\beta_1 d\beta_2 , \tag{A13}$$

where we have set the Feynman parameter associated with the vector-meson lines equal to one, following Appendix B. To repeat: The result (A13) can be found by conventional integration over q, followed by performing one Feynman-parameter integration explicitly. The short-cuts (A11) and (A12) reveal the nature of the eikonal approximation for Yang-Mills vertices.

To do the integrations over β_1, β_2, observe that β_1 and β_2 must both be small, and that either $\beta_1/\beta_2 \to 0$, or $\beta_2/\beta_1 \to 0$; there is no contribution when β_1/β_2 is finite (see Appendix B). Scaling β_1, β_2 as in Appendix B, we find for (A13)

$$F^{(\mathrm{d})}_{\alpha c} = -\tfrac{1}{2} c_A t^c \left(-\frac{g^2 t}{8\pi^2}\right)\int_0^1 \lambda d\lambda \left\{\frac{p'_\alpha}{p'\cdot k}\int_{x\sim 0} dx\left(\frac{1}{\lambda^2 x t - \mu^2} - \frac{1}{\lambda^2 x t - 2\lambda p'\cdot k}\right) + \frac{p_\alpha}{p\cdot k}\int_{x\sim 0} dx\left[\frac{1}{\lambda^2(1-x)t + 2\lambda p\cdot k} - \frac{1}{\lambda^2(1-x)t - \mu^2}\right]\right\} \tag{A14}$$

$$= -\tfrac{1}{2} c_A t^c \left\{-\frac{p'_\alpha}{p'\cdot k}[\Gamma(p, p'-k) - \Gamma(p,p')] + \frac{p_\alpha}{p\cdot k}[\Gamma(p+k, p') - \Gamma(p,p')]\right\}. \tag{A15}$$

The unwanted term in (A3) is gone, leaving only the result claimed in (A1).

APPENDIX B: THE ONE-LOOP GRAPH

We give some simple results for the one-loop graph for fermions of momenta $p+k, p'-k$ coupled to a scalar form factor. Assume that p and p' are on shell and ignore the fermion mass M everywhere; this does not affect the leading singularity in the asymptotic regime $t=(p-p')^2 \gg M^2$. As usual, the loop momentum q and k are neglected in the numerator. The one-loop graph is defined as

$$\Gamma(p+k, p'-k) = -\frac{ig^2}{(2\pi)^4}\int \frac{dq(4p'\cdot p)}{(q^2-\mu^2)(p+k-q)^2(p'-k-q)^2} \tag{B1}$$

$$= -\frac{g^2 t}{8\pi^2}\int \frac{d\beta_1 d\beta_2 d\beta_3 \delta(1-\beta_1-\beta_2-\beta_3)}{\beta_1\beta_2 t + 2\beta_1\beta_3 p\cdot k - 2\beta_2\beta_3 p'\cdot k - \mu^2}. \tag{B2}$$

In (B2), we have neglected k^2 compared to $p\cdot k$ or $p'\cdot k$. The leading singularities come when β_1 and β_2 are small, so we may set $\beta_3 = 1$ and ignore the δ function. The variables β_1, β_2 are scaled as $\beta_1 = \lambda x$, $\beta_2 = \lambda(1-x)$, $\lambda > 0$, $0 \leqslant x \leqslant 1$. The following results are obtained, in leading order:

$$\Gamma(p+k, p'+k) = -\frac{g^2 t}{8\pi^2}\int_0 \lambda d\lambda \left[\int_{x\sim 0} dx \frac{1}{\lambda^2 xt - 2\lambda p'\cdot k} + \int_{x\sim 1} dx \frac{1}{\lambda^2(1-x)t + 2\lambda p\cdot k}\right], \tag{B3}$$

$$\Gamma(p+k, p') = -\frac{g^2 t}{8\pi^2}\int_0 \lambda d\lambda \left[\int_{x\sim 0} dx \frac{1}{\lambda^2 xt - \mu^2} + \int_{x\sim 1} dx \frac{1}{\lambda^2(1-x)t + 2\lambda p\cdot k}\right], \tag{B4}$$

$$\Gamma(p, p'-k) = -\frac{g^2 t}{8\pi^2}\int_0 \lambda d\lambda \left[\int_{x\sim 0} dx \frac{1}{\lambda^2 xt - 2\lambda p'\cdot k} + \int_{x\sim 1} dx \frac{1}{\lambda^2(1-x)t - \mu^2}\right], \tag{B5}$$

$$\Gamma(p, p') = -\frac{g^2 t}{8\pi^2}\int_0 \lambda d\lambda \left[\int_{x\sim 0} dx \frac{1}{\lambda^2 xt - \mu^2} + \int_{x\sim 1} dx \frac{1}{\lambda^2(1-x)t - \mu^2}\right]. \tag{B6}$$

The integrals are elementary; our point in displaying them in this form is to demonstrate that $\Gamma(p+k,p') - \Gamma(p,p')$ has contributions only from $x \sim 1$, while $\Gamma(p,p'-k) - \Gamma(p,p')$ only has contributions from $x \sim 0$; these facts were used in Appendix A. Also noteworthy is the equation

$$\Gamma(p+k, p'-k) + \Gamma(p,p') - \Gamma(p+k,p') - \Gamma(p,p'-k) = 0\,. \tag{B7}$$

This equation leads to an alternative derivation of (A3). Instead of using (A2) to combine graphs 7(a) and 7(b), these graphs are separately evaluated using (A4). Their sum then yields (A3) when (B7) is used.

In the text we make frequent use of the approximation of ignoring q^2 in fermion denominators like $(p-q)^2 - M^2 = q^2 - 2p\cdot q$. The one-loop integral then becomes superficially divergent, as we see from

$$J(p,p') = -\frac{ig^2}{(2\pi)^4}\int dq \frac{p'\cdot p}{(q^2-\mu^2)p\cdot q p'\cdot q}\,. \tag{B8}$$

This means that the value of J is ambiguous, and other authors have used a different value for J than we have.[12] We argue that $J(p,p')$ and $\Gamma(p,p')$ have the same leading behavior, as we see by regulating (B8) by replacing $2p\cdot q \to \alpha q^2 + 2p\cdot q$, $2p'\cdot q \to \beta q^2 + 2p'\cdot q$, arbitrary α, β. A simple scaling, $p \to \alpha p$, $p' \to \beta p'$, shows that $J_{\text{reg}}(p,p') = \Gamma(p,p')$ for all finite α, β, independent of α and β. Then we define $J(p,p')$ for $\alpha = \beta = 0$ as $\Gamma(p,p')$.

APPENDIX C: THE EXTERNAL-LINE RULE

The following is a heuristic discussion of the external-line rule which we claim (Sec. II C) is appropriate for the infrared limit of (nonforward) amplitudes whose mass-shell quanta are all massive fermions. The rule is as follows:

To obtain the entire set of leading $(L+1)$-loop skeleton graphs, connect by a virtual gluon line a pair of external quark lines of each L-loop skeleton graph in all possible ways. The leading nonskeleton graphs are obtained by vertex and propagator insertions to leading skeleton graphs.

Our argument is not rigorous since we make certain plausible assumptions as we go along. We invite the reader to supply the missing steps.

We begin by noting that the cutoff method is immaterial for leading logarithms: μ can be inserted as mass in the gluon propagator or the external fermion momenta can be taken of the form $p_i + y_i$, where $p_i^2 = M^2$ and y_i is a small departure from mass shell, i.e., $y_i \propto \mu$.

As $\mu \to 0$ a general Feynman integral will behave as

$$(\ln\mu)^\beta c(p_1, p_2, \ldots)\,.$$

Since we are here only interested in the exponent β and not in the (momentum-dependent) coefficient $c(p_1, \ldots)$, we conveniently continue analytically the quark momenta $p_1, p_2, \ldots$ to a Euclidean region while staying on-shell. This can be done by rotating the integration paths of the three momentum components (rather than the energy component as is customary) to the imaginary axis:

$$k^2 = k_0^{\,2} - k_1^{\,2} - k_2^{\,2} - k_3^{\,3} \to k_0^{\,2} + k_1^{\,2} + k_2^{\,2} + k_3^{\,3}\,.$$

Thus, for instance, the quark form factor $\Gamma(p,p')$ is continued to values of the momentum transfer $t = (p-p')^2$ in the unphysical interval $(0, 4M^2)$. This procedure would not work for massless external particles; the available Euclidean region would be "of measure zero" (nor can it be used in the fixed-angle regime).

We shall consider only graphs without fermion loops. Our experience from low-order calculations (up to two loops) indicates that they are nonleading; in fact, it may be that, just as in QED, they are nonleading by a whole power of μ for reasons of gauge invariance.

The leading logarithmic contributions for a given Feynman graph G come from (one or several distinct) regions of integration where a set of L (L = number of independent loops of the graph) independent virtual gluon (or ghost) momenta $k_1, k_2, \ldots, k_L$ approach zero. For nonforward amplitudes this implies that there exist certain other internal lines carrying momenta of the form $\Delta + k$, where Δ is a nonvanishing combination of external momenta and k is some linear combination of $k_1, k_2, \ldots, k_L$. In extracting the leading contributions of the specified subregion of integration we may set $\Delta + k \simeq \Delta$. In so doing these momentum-transfer-carrying lines are removed from under the integral sign. Graphically speaking, these lines are short-circuited to a point and the graph is replaced by one in which all fermion lines meet at this point in a starlike fashion (see Fig. 9). Each fermion ray consists of fermion propagators carrying momentum $p_i + k$, where p_i is the appropriate mass-shell fermion momentum (or near-mass-shell momentum if the off-shell infrared cutoff method is employed) and k is a linear combination of the small momenta $k_1, k_2, \ldots, k_L$. We shall make the following eikonal approximation in fermion propagators:

$$\frac{i(\not p + \not k) + M}{(p+k)^2 + M^2} \to \frac{i\not p + M}{2p \cdot k}.$$

This makes the amplitude a homogeneous function of the quark momenta (and quark masses) of degree zero.

Observe that since the leading infrared logarithms come from all k_i being small, the form and value of the ultraviolet cutoff is immaterial. We choose to simply restrict the integration by

$$k_i^2 < \Lambda^2, \quad i = 1, 2, \ldots, L.$$

We now have an integral of the form

$$T(p_i; \mu) \propto \int_{k_i^2 < \Lambda^2} d^4k_1 \cdots d^4k_L I(p_i; \mu; k_1, \ldots, k_L)$$

$$= \int_{k_i^2 < \Lambda^2/\mu^2} d^4k_1 \cdots d^4k_L I(p_i; 1; k_1, \ldots, k_L).$$

This last equality follows from the fact that I is homogeneous of degree $-4L$ in $k_1, k_2, \ldots, k_L, \mu$. In this last form we see that the infrared behavior is directly related to the ultraviolet limit ($\Lambda \to \infty$) of the integral. The superficial degree of divergence is zero so that for $L = 1$ the integral behaves like $\ln(\Lambda^2/\mu^2)$ and for each additional loop it may

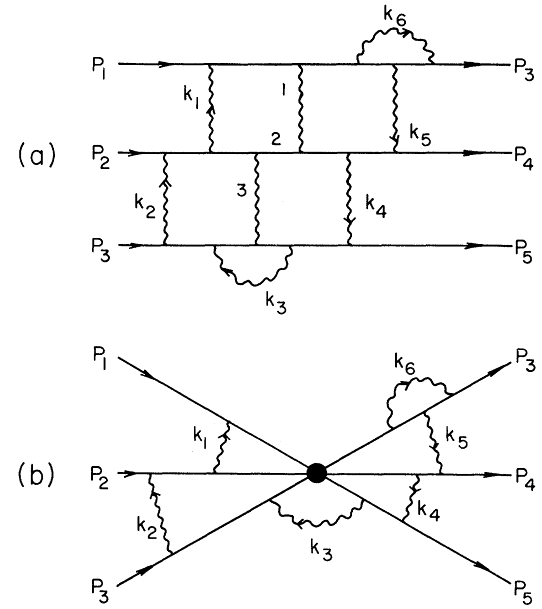

FIG. 9. (a) An example of a fermion graph for the discussion of Appendix C. When the momenta $k_1, \ldots, k_6$ approach zero the lines by (1), (2), and (3) carry nonzero momentum transfers. (b) The same graph with lines (1), (2), and (3) short-circuited.

pick up another power of $\ln(\Lambda^2/\mu^2)$ so that it behaves at most like $[\ln(\Lambda^2/\mu^2)]^L$.

We now focus on that part of the domain of integration where a certain gluon (or ghost) line carries the smallest momentum of all the others. Let k_1 be that momentum; if we choose it as one of the integration momenta we may write the corresponding contribution as

$$T^{(1)} \simeq \int_{k_1^2 < \Lambda^2} \frac{d^4k_1}{k_1^2 - \mu^2} T_1(p_i, \Lambda, k_1), \tag{C1}$$

$$T_1(p_i, \Lambda, k_1) \simeq \int_{\Lambda^2 > k_i^2} d^4k_2 \cdots d^4k_L I_1(p_i; 0; k_1, \ldots, k_L), \tag{C2}$$

where I_1 is associated with the graph G_1 obtained by opening the k_1 line. In (C2) μ was set equal to zero since k_1 provides the necessary infrared cutoff. (C2) can also be written as

$$T_1 \simeq \frac{1}{k_1^2} \int_{\Lambda^2/k_1^2 > k_i^2} d^4k_2 \cdots d^4k_L \times I_1\left(p_i : 0; \frac{k_1}{(k_1^2)^{1/2}}, k_2, \ldots, k_L\right). \tag{C3}$$

If G_1 is a one-particle-irreducible graph [see Fig. 10(a)] the above integral is superifcially convergent (degree of divergence -2). It behaves at

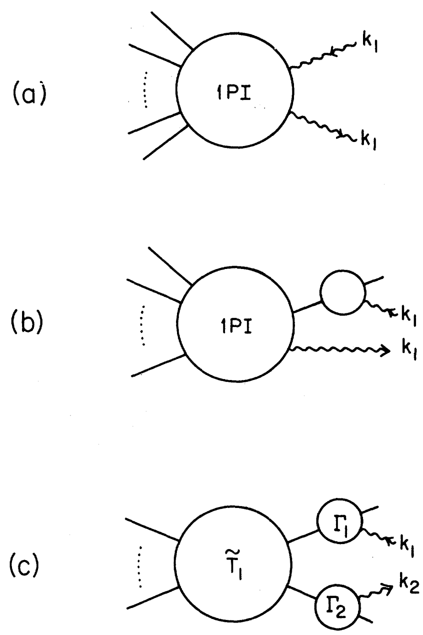

FIG. 10. Upon opening a gluon (or ghost) line carrying momentum k, one obtains (a) a one-particle-irreducible graph, (b) a graph with a single fermion pole, or (c) a graph with two fermion poles.

most like $[\ln(\Lambda^2/k_1^2)]^{L-2}$ so that $T^{(1)}$ behaves at most like

$$\int \frac{d^4k_1}{k_1^2-\mu^2}\frac{1}{k_1^2}\left(\ln\frac{\Lambda^2}{k_1^2}\right)^{L-2} \simeq \left(\ln\frac{\Lambda^2}{\mu^2}\right)^{L-1},$$

i.e., if G_1 is one-particle irreducible the contribution of D_1 is nonleading. The same is true if G_1 has only one fermion pole [see Fig. 10(b)]. Instead of (C3) we have

$$T_1(k_1) \simeq \frac{1}{p\cdot k_1 - i\epsilon}\frac{1}{(k_1^2)^{1/2}}\int d^4k_2\cdots d^4k_L \times I_1\left(p_i;0;\frac{k_1}{(k_1^2)^{1/2}},k_2,\ldots,k_L\right),$$

where again the integral is superficially convergent (degree of divergence -1) and so it behaves at most like $[\ln(\Lambda^2/k_1^2)]^{L-2}$ and correspondingly $T^{(1)}\sim[\ln(\Lambda^2/\mu^2)]^{L-1}$.

If G_1 has *two* fermion poles [see Fig. 10(c)], then

$$T^{(1)} \simeq \int \frac{d^4k_1}{k_1^2-\mu^2}\frac{\Gamma_1(p_1,k_1)}{p_1\cdot k_1 - i\epsilon}\frac{\Gamma_2(p_2,k_1)}{p_2\cdot k_1 - i\epsilon}\tilde{T}_1(p_i,k_1),$$

where Γ_1, Γ_2 are proper or improper vertex graphs and $\tilde{T}_1$ is associated with $\tilde{G}_1$, a graph for the same process as G but with fewer loops. Each of these "component" graphs of G_1 has degree of divergence zero so that together they can provide as many factors of $\ln(\Lambda/k_1)$ as their total number of loops, namely $L-1$. The corresponding contribution to $T^{(1)}$ would then be $[\ln(\Lambda^2/\mu^2)]^L$, i.e., leading. The argument may now be completed by induction. To make it rigorous one should justify (i) the neglect of fermion loops, and (ii) the assertion that as the ultraviolet cutoff Λ approaches ∞, a Feynman integral with L loops behaves at most like $(\ln\Lambda)^L$ or $(\ln\Lambda)^{L-1}$ if its superficial degree of divergence is zero or less than zero, respectively.

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