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Non-Abelian gauge field theories on a lattice and spontaneous breaking of chiral symmetry*

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The perturbation theory for a color-quark gauge theory on a spatial lattice is developed in time-dependent form. Graphical rules are suggested. These methods are then applied to study the nature of the γ_5 symmetry exhibited when the bare-quark-mass term is zero. Investigation of the quark proper self-energy leads to the conclusion that the γ_5 symmetry is realized in the Nambu-Goldstone mode. A Bethe-Salpeter equation for the proper Γ^5 vertex is solved in the ladder approximation, and shown to exhibit zero-mass poles.

I. INTRODUCTION

One aesthetically pleasing and conceptually economical model for hadrons is the color-quark model, in which the quarks couple to vector bosons in a gauge-invariant manner.[1] This model is known to exhibit short-distance behavior consistent with Bjorken scaling, as observed in deep-inelastic scattering experiments.[2] In addition, it is conjectured that the infrared structure of the theory may be such as to permanently "trap" the quarks and gauge bosons, so that only color-singlet bound states of the fundamental fields are observable in isolation.[3]

Unfortunately, in accordance with naive expectations, this trapping is supposed to occur because the effective couplings in the theory become large at large distances. Consequently, a perturbation expansion in $\alpha_s \equiv g^2/4\pi$ is not a useful guide to the infrared behavior of the Green's functions of the theory, and the conjectured trapping properties of the theory cannot be demonstrated by using standard perturbative techniques.

Recently, Wilson has developed new methods[4,5] which may be used to study the infrared properties of gauge field theories. These methods arose from consideration of general renormalization-group concepts. Often, not all the details of the microscopic interaction are relevant for some particular long-range behavior. It may then be reasonable to account for these details as "phenomenological" (though in principle determinable) terms in an effective Hamiltonian, or to neglect them altogether.

The trick is to guess the form of the new effective Hamiltonian without actually going through the process of eliminating the microscopic details, i.e., large-momentum field degrees of freedom, from the original "true" Hamiltonian. One must guess because the task of actually performing the elimination for a gauge-field theory seems rather hopeless at the present time.

Wilson has gone about guessing at the form of the "effective" color-quark theory in a very imaginative manner. One feature of his theory is that the Lagrangian is defined on a space-time lattice, rather than in a cutoff momentum space. This feature allows the most crucial guiding ingredient in the guess, gauge invariance, to be realized in a new and highly useful form. For instance, the "interaction" becomes a combination of kinetic energy and coupling pieces, and the perturbation expansion is defined in powers of $(\alpha_s{}^{-1})$ and of inverse bare quark mass m_0, rather than in powers of α_s.[6]

The details of this construction, as well as other considerations which may guide one to candidates for effective Lagrangians $\mathcal{L}^{(\mathrm{eff})}$ will be presented in the next section.

We will then concentrate our attention on a path-integral formulation of these ideas, using a spatial lattice, while maintaining time as a continuous variable. This variant of the lattice approach to gauge theories is essentially equivalent to the Hamiltonian formalism introduced by Kogut and Susskind (KS).[7] Our starting point will be to postulate a lattice version of the Faddeev-Popov generating functional which reproduces the continuum theory when the lattice spacing a goes to zero.

For convenience, we shall work in a vector gauge $A_0^\alpha(\vec{x}, t) = 0$, in which the nuisance ghost terms are absent. Invariance under time-independent gauge transformations remains,

$$\begin{aligned} &A_j^\alpha(\vec{x}, t) \to A_j^\alpha + \frac{1}{g}\,\partial_j\epsilon^\alpha(\vec{x}) + f^{\alpha\beta\gamma}\epsilon^\beta(\vec{x})A_j^\gamma\,, \\ &\psi \to \left(1 - i\epsilon^\alpha(\vec{x})\frac{\Lambda^\alpha}{2}\right)\psi, \quad \partial_0\epsilon^\alpha(\vec{x}) = 0 \end{aligned} \tag{1.1}$$

and this is implemented on the spatial lattice by Wilson's method.

Although this scheme suffers from the lack of four-dimensional symmetry, it is advantageous in some other respects. In principle only one parameter α_s^{-1} is needed to carry out the perturbation expansion. A perturbation in m_0^{-1}, with m_0 large, which is crucial in Wilson's case, is not required here. Furthermore, the Feynman-Dyson-Wick diagrammatic analysis carries over to a large extent, although there are important differences that we will discuss. In addition, the planar diagrams that emerge bear resemblances to *both* the diagrams of the continuum theory, and to Harari-Rosner-type duality diagrams.

We will not explore all these possibilities in this paper. In Sec. III, only some relations that are relevant for our specific aims will be developed. Given the new formalism of gauge theories on a lattice, one may try to find a way of justifying the ansatz, or of improving upon it, etc.; or, settling upon a given possibility, one may explore its consequences to see if it has anything to do with physics. Here we shall take the latter option. Specifically, we shall investigate the problem of the realization of chiral symmetry in lattice perturbation theory.

Fifteen years ago Nambu and Jona-Lasinio invented a theory of pions as bound states of fermions,[8] whose emission was associated with the conservation of chirality. At present these conjectures enjoy impressive experimental support. Unfortunately, a realistic quark model incorporating these ideas is still lacking, due to the extreme complexity of the problem. The main difficulty stems from the fact that the Nambu-Jona- Lasinio pseudoscalar bosons are massless bound states of fermion fields with zero bare mass. Obviously these objects are hardly tractable in standard renormalizable relativistic quantum field theories.

In this complex situation, the standard strategy is to pursue a modest goal, restricting oneself to the question of compatibility of the conditions for spontaneous γ_5 symmetry violation with the Schwinger-Dyson equations of the quantum field theory. Different approximation schemes have been employed, along with renormalization-group methods, in which at some stage it is necessary to assume smallness of the coupling constant. Investigations along this line have been pursued in Abelian gauge-field theories by Pagels,[9] and in non-Abelian gauge-field theories by Lane.[10] [Recent interest in this reasoning is also due to the possibility of generation of vector particle masses in spontaneously broken gauge-field theories (Ref. 11), but this is a different problem from the one considered in this paper.]

In this paper (Sec. IV) we adopt a similar strategy, in the context of non-Abelian gauge-field theories on a spatial lattice, in which quark confinement is built in. We choose an effective Lagrangian $\mathcal{L}^{(\text{eff})}$, which is manifestly γ_5 invariant due to the absence of the quark mass term, and arrive at a rather remarkable conclusion: In the lowest nontrivial order in α_s^{-1}, the Nambu-Goldstone poles are already revealed.

More specifically, the mass term in the quark self-energy Σ is generated perturbatively, i.e., $\{\Sigma, \gamma_5\}_+ \neq 0$. This ensures the existence of a pole in the quark–antiquark–axial-charge proper vertex owing to the Ward-Takahashi identities. An interesting feature of the lattice theory is that this is *forced* by the requirement that the physical states must be invariant under the residual gauge transformations (1.1). Furthermore, the Bethe-Salpeter equation for the residues of the poles is solved in the ladder approximation. In our case, the approximation of the Bethe-Salpeter kernel by the lowest-order term seems to be legitimate because we are in the *strong*-coupling regime, $\alpha_s \gg 1$.

Of course, in spite of these notable successes, our approach leaves some important problems unsolved. First, there is a problem present in the very formulation of the γ_5-invariant theory on a lattice. This problem is associated with the presence of an energy degeneracy between particles with momentum k and $k \pm \pi/a$ in the fermion sector. It will be specified in more detail in the appropriate place. We shall also discuss ways of living with the problem, although we do not really solve it.

Another important unanswered question is why $\mathcal{L}^{(\text{eff})}$ should have no mass term, even if the true continuum $\mathcal{L}$ had none. At the classical level, if $m\bar{\psi}\psi$ is in $\mathcal{L}^{(\text{eff})}$ it will persist in the continuum limit, as can be seen by dimensional arguments. However, this argument does not rule out such a term in the quantum theory: m_0 in $\mathcal{L}^{(\text{eff})}$ could be entirely quantum mechanical in origin. We cannot exclude this possibility, and believe this important question should be pursued as part of the overall program of investigation using renormalization-group techniques. But our aim here is to explore what happens if the mass term turns out to be absent from $\mathcal{L}^{(\text{eff})}$.

Finally, in Sec. V we will summarize our conclusions, and discuss a few of the remaining problems in greater detail.

II. GAUGE-FIELD THEORY ON A SPATIAL LATTICE

This section contains several parts. In the first part the color-quark gauge field theory in the vector gauge $A_0^\alpha(\vec{x}, t) = 0$ will be described in the continuum limit. This will fix our notations, and make the paper reasonably self-contained. In the remaining parts, the theory will be constructed on a spatial lattice, and a Hamiltonian formalism will be introduced. Generalizations of the Kogut-Susskind formulation will be emphasized, and some departures from their theory are considered in detail.

A. The continuum theory

The non-Abelian gauge theory is based on the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu,\alpha}F^{\mu\nu,\alpha} + i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} + ig\frac{\Lambda^{\alpha}A_{\mu}^{\alpha}}{2}\right)\psi - m_0\bar{\psi}\psi \,, \tag{2.1}$$

where the gauge-covariant tensor

$$F_{\mu\nu}^{\alpha} = \partial_{\mu}A_{\nu}^{\alpha} - \partial_{\nu}A_{\mu}^{\alpha} - gf^{\alpha\beta\gamma}A_{\mu}^{\beta}A_{\nu}^{\gamma} \,. \tag{2.2}$$

The Greek indices α,β,γ are group indices, which take on values $1,\ldots,8$ when the non-Abelian gauge group is SU(3), which we consider exclusively in this paper, although many developments are independent of this choice. The eight 3×3 matrices $\frac{1}{2}\Lambda^{\alpha}$ are the generators of the SU(3) algebra in the fundamental representation. The fields $\psi,\bar{\psi}$ contain "quark" ("antiquark") components which transform according to the $\underline{3}$ ($\underline{3}$) representation. The fields $\psi,\bar{\psi}$ also contain "ordinary" SU(3) indices, which are contracted with the unit matrix in Eq. (2.1). Later in our work, we shall use corresponding 3×3 matrices $\frac{1}{2}\lambda_i$, $i=1,\ldots,8$. [The ordinary SU(3) group will be called the unitary-spin group.] In Eq. (2.2), $f^{\alpha\beta\gamma}$ are the real, antisymmetric structure constants for SU(3).[12]

The canonical quantization of the theory must contend with the problem that only two spatial components of $A_{\mu}^{\alpha}(x)$ are dynamically independent degrees of freedom, because the gauge field is massless. One way to proceed is to select the vector gauge $A_0^{\alpha}(x)=0$, which turns out to be very convenient for further developments. (Notice that we do *not* set $\vec{\nabla}\cdot\vec{A}^{\alpha}=0$.) It can be shown that, in the end, this choice of gauge may be incorporated successfully into the quantization by setting $A_0^{\alpha}=0$ in $\mathcal{L}$. Introducing

$$E_j^{\alpha}(\vec{x},t) = \partial_0 A_j^{\alpha}(\vec{x},t) \,, \tag{2.3}$$

one postulates at equal times

$$[A_i^{\alpha}(\vec{x},t), E_j^{\beta}(\vec{y},t)] = i\delta_{ij}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \,, \tag{2.4a}$$

$$\{\psi_A(\vec{x},t), \psi^{+B}(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y})\delta_A^B \,, \tag{2.4b}$$

where in (2.4b) the Kronecker "δ" on the right-hand side is in all indices A carried by ψ, i.e., the Dirac, unitary, and color indices. The Hamiltonian is then

$$H = \int d^3x : [\tfrac{1}{2}E_j^{\alpha}(\vec{x},t)E_j^{\alpha}(\vec{x},t) + \tfrac{1}{4}F_{jk}^{\alpha}F_{jk}^{\alpha} + i\bar{\psi}(\vec{\gamma}\cdot\vec{\nabla} + ig\vec{\gamma}\cdot\vec{A}^{\alpha}\tfrac{1}{2}\Lambda^{\alpha})\psi + m_0\bar{\psi}\psi] : \,. \tag{2.5}$$

Imposing the vector gauge condition eliminates the timelike mode of excitation, but in Eq. (2.4a) all three spatial degrees of freedom were treated as independent operators. This is legitimate, provided the states of the system are subjected to conditions of constraint. One way of finding the correct equations of constraint is to notice that at the classical level, the Maxwell equations

$$\partial^{\nu}F_{\nu\mu}^{\alpha} = g(j_{\mu}^{\alpha} - f^{\alpha\beta\gamma}F_{\nu\mu}^{\beta}A^{\nu,\gamma}) \,, \tag{2.6}$$

where

$$j_{\mu}^{\alpha} = \bar{\psi}\gamma_{\mu}\tfrac{1}{2}\Lambda^{\alpha}\psi \,, \tag{2.7}$$

include one purely spatial equation in the vector gauge,

$$\operatorname{div}\vec{E}^{\alpha} - g(j_0^{\alpha} - f^{\alpha\beta\gamma}\vec{E}^{\beta}\cdot\vec{A}^{\gamma}) = 0 \,. \tag{2.8}$$

Following Dirac, in passing to a Hamiltonian quantum dynamics these equations should be applied as constraints on states.

The same result is obtained starting from the observation that the vector gauge condition does not specify the gauge completely. The Lagrangian $L=\int d^3x\mathcal{L}$ is still invariant under the restricted local gauge transformations, Eq. (1.1), which are generated by the local operators $Q^{\alpha}(\vec{x})$ in Eq. (2.8). Thus Eq. (2.8) can be also considered as the requirement that the physical states must remain invariant under these residual gauge transformations. This form of the argument will be presented in greater detail when the analog of Eq. (2.8) is derived for the lattice theory.

For completeness, we note that the "gauge conditions" in the quantum theory are consistent with one another. Further, they are consistent with the equations of motion by virtue of current conservation,

$$\partial^{\mu}(j_{\mu}^{\alpha} - f^{\alpha\beta\gamma}F_{\nu\mu}^{\beta}A^{\nu,\gamma}) = 0. \tag{2.9}$$

The canonical gauge-field algebra Eq. (2.4a) may be realized in a space of functionals $\psi(A(x))$, in which the canonical momentum $E_j^{\alpha}(x)$ acts as $-i\delta/\delta A_j^{\alpha}(x)$. In such a realization Eq. (2.8) becomes

$$Q^{\alpha}(\vec{x}) = -i\partial_j\frac{\delta}{\delta A_j^{\alpha}(\vec{x})} + igf^{\alpha\beta\gamma}A_j^{\beta}(\vec{x})\frac{\delta}{\delta A_j^{\gamma}(\vec{x})} - gj_0^{\alpha}(\vec{x}). \tag{2.10}$$

There is one important point to be noted at this stage which is very relevant to our later developments. Since in Eqs. (2.8) and (2.10), $Q^{\alpha}(x)$ is a sum of gluon parts and quark parts, it is possible to try eigenfunctions $\Phi_{\text{phys}} = \Phi(A)\Phi(\psi,\psi^{\dagger})$,

$$\left[-i\partial_j \frac{\delta}{\delta A_j^\alpha} + igf^{\alpha\beta\gamma}A_j^\beta \frac{\delta}{\delta A_j^\gamma} - \rho^\alpha(\vec{x})\right]\Phi(A) = 0, \tag{2.11a}$$

$$[\rho^\alpha(\vec{x}) - gj_0^\alpha(\vec{x})]\Phi(\psi, \psi^\dagger) = 0. \tag{2.11b}$$

If the commutator $[j_0^\alpha(\vec{x}, t), j_k^\alpha(\vec{y}, t)]$ contains a c-number Schwinger term, Eq. (2.11b) does not allow the "separation function" $\rho^\alpha(x)$ to vanish. In that case, the gauge conditions for gluons and quarks cannot be satisfied independently. However, on the lattice it will prove possible to build the Hilbert space off a base state containing no gluons. The lattice analog of Eq. (2.14) can be satisfied with the gauge-field part and the fermion part *separately* annihilating the base state. Consequently, one anticipates this base state may behave pathologically as the lattice spacing is taken to zero. Short-distance anomalous behavior must appear unless the quark's color current algebra contains only operator Schwinger terms.

Finally, we note that the canonical developments above can be carried through with the Faddeev-Popov (FP) ansatz for defining the Feynman path integral.[13] The result is that the action to be used is $S = \int d^4x\mathcal{L}$, where $\mathcal{L}$ is that of Eq. (2.1) with A_0 set equal to zero. There are no FP ghosts in gauges $n_\mu A^\mu = 0$. The path-integral formulation is the most useful for lattice theories from the conceptual as well as from the practical point of view.

B. The lattice theory Lagrangian

The formulation of the gauge field (G) theory on a lattice will be performed following Wilson's original construction.[4] The formalism to be developed subsequently is based on the lattice version of the generating functional W. To do this we introduce anticommuting fermionic sources $\eta(\vec{n})$ at each lattice site $\vec{n} = (n_1, n_2, n_3)$ and gauge-field sources J_L on links $L \equiv (\vec{n}, \hat{l})$, which connect sites $\vec{n}$ and $\vec{n} + \hat{l}$ separated by a unit vector $\hat{l}$. Then W is given by the expression

$$W\{\eta, \tau\} = \int\int D\psi D\overline{\psi} D\mu(G) \exp\left[i\int_{-\infty}^{\infty} d\tau\left(\mathcal{L}_\psi + \mathcal{L}_G + \sum_{\vec{n}} [\overline{\psi}(\vec{n})\eta(\vec{n}) + \overline{\eta}(\vec{n})\psi(\vec{n})] + \sum_L \mathrm{Tr}[J_L^\dagger U(G_L) + U^\dagger(G_L)J_L]\right)\right], \tag{2.12}$$

with

$$\mathcal{L}_\psi = \sum_{\vec{n}}\left[\overline{\psi}(\vec{n}) i\partial_0 \psi(\vec{n}) - \left(\frac{i}{2g^2}\right)\sum_{\hat{l}} \overline{\psi}(\vec{n}) \not{l}\, U(G_{\vec{n},\hat{l}})\psi(\vec{n}+\hat{l}) - m_0\overline{\psi}(\vec{n})\psi(\vec{n})\right], \tag{2.13a}$$

$$\mathcal{L}_G = \sum_L \tfrac{1}{2}\mathrm{Tr}\{\dot{U}(G_L)\dot{U}^\dagger(G_L)] + \frac{1}{g^4}\sum_B \mathrm{Tr}[U(G_{\vec{n},\hat{l}})U(G_{\vec{n}+\hat{l},\hat{j}})U^\dagger(G_{\vec{n}+\hat{j},\hat{l}})U^\dagger(G_{\vec{n},\hat{j}})]. \tag{2.13b}$$

Here $\not{\partial}_0 \equiv \gamma_0\partial_\tau$, $\not{l} = \vec{\gamma}\cdot\hat{l}$, and the time dependences of ψ, G, $\eta(\vec{n}), J_L$ are consistently suppressed. The dimensionless quantities τ, $\vec{n}$, and $\psi(\vec{n})$ represent scaled time $t = a/g^2\tau$, site coordinates $\vec{x} = \vec{n}a$, and fermion fields $\psi(\vec{x}) = a^{-3/2}\psi(\vec{n})$, where a is the lattice spacing. The "bare" mass m_0 is in units of $1/ag^2$. In what follows, the above notations will be explained further, and contact with the continuum theory (2.1) will be established:

(1) The unitary matrices $U(G)$ represent elements of the SU(3) gauge group in the quark representation. The continuum limit is reached by writing

$$U(G_{\vec{n}+\hat{l},\hat{j}}) = \exp[i(\tfrac{1}{2}\Lambda^\alpha)G^\alpha_{\vec{n}+\hat{l},\hat{j}}], \tag{2.14}$$

$$G^\alpha_{\vec{n}+\hat{j},-\hat{j}} = -G^\alpha_{\vec{n},\hat{j}}, \tag{2.15}$$

$$G^\alpha_{\vec{n}+\hat{l},\hat{j}} \equiv gaA_j^\alpha(\vec{n}+\hat{l}), \tag{2.16}$$

$$A_j^\alpha(\vec{n}+\hat{l}) \underset{\substack{a\to 0\\ \vec{x}=\vec{n}a}}{\sim} A_j^\alpha(\vec{x}) + a\partial_l A_j^\alpha(\vec{x}). \tag{2.17}$$

Equation (2.15) expresses the charge-conjugation properties of the lattice gauge fields. Observe that different sets of parameters $\{G_L^\alpha\}$ are attached to distinct links $L = \{\vec{n}, \hat{l}\}$. As $a \to 0$, $A_j^\alpha(\vec{x})$ is interpreted as the vector potential. The time derivative $dU(\tau)/d\tau \equiv \dot{U}_\tau$ is defined by

$$\dot{U}(G_L(\tau)) = \lim_{\delta\to 0}\frac{1}{\delta}[U(G_L(\tau+\delta)) - U(G_L(\tau))]. \tag{2.18}$$

The coefficients in $L = L_\psi + L_G$, Eq. (2.13), are chosen so that as $a \to 0$, L goes into the continuum Lagrangian (2.1) in the vector gauge. L_ψ is obtained by replacing the spatial derivative of ψ by its finite-difference expression,

$$\partial_l\psi(x) \underset{\vec{x}=\vec{n}a}{\longrightarrow} \frac{1}{a}[\psi(\vec{n}+\hat{l}) - \psi(\vec{n})].$$

This expression is made gauge invariant by inserting an exponentiated A_j^α according to Eq. (2.14).

This symbol $\sum_{\hat{l}}$ in Eq. (2.16a) includes summation over all six directions emanating from each site $\vec{n}$. Further, the summations $\sum_L$ and $\sum_B$ in Eq. (2.13b) run over all distinct links and boxes, respectively. An example of the latter is given in Fig. 1, where the wavy lines represent U's connecting four neighboring sites in the clockwise direction.

Apparently, the Lagrangian (2.13) is invariant under transformations $V(g_{\vec{n}})$ which are generated at each site $\vec{n}$ independently:

$$\psi(\vec{n}) \to V^\dagger(g_{\vec{n}})\psi(\vec{n}), \quad \overline{\psi}(\vec{n}) \to \overline{\psi}(\vec{n})V(g_{\vec{n}}), \qquad (2.18')$$

$$U(G_{\vec{n},\hat{l}}) \to V^\dagger(g_{\vec{n}})U(G_{\vec{n},\hat{l}})V(g_{\vec{n}+\hat{l}}) = U(g_{\vec{n}}^{-1}G_{\vec{n},\hat{l}}g_{\vec{n}+\hat{l}}). \qquad (2.18'')$$

Here $g_{\vec{n}}$ is an arbitrary element of the local gauge group $SU(3)_{\vec{n}}$ defined at the site $\vec{n}$. This is a lattice version of the time-independent local gauge transformations Eq. (1.1). We shall dwell upon this invariance in greater detail in Sec. II C.

(2) The integration measure is of central importance to defining the Feynman path integral (2.12). First, we recall that in the functional approach, fermion fields $\psi_A(\vec{n})$, $\overline{\psi}^A(\vec{n})$ are anticommutating objects (elements of a Grassmann algebra) for which integration is defined by the condition

$$\int [\psi_A(\vec{n})]^K d\psi_A(\vec{n}) = \int [\overline{\psi}^A(\vec{n})]^K d\overline{\psi}^A(\vec{n}) = \delta_{K1} \qquad (2.19)$$

for each component A at every site $\vec{n}$.[14]

Note that the fermion part of the measure, which should be read in detail as

$$D\psi D\overline{\psi} \equiv \sum_{\vec{n},e} d\psi_A(\vec{n},\tau_e)d\overline{\psi}^A(\vec{n},\tau_e), \qquad (2.20)$$

is invariant under the local gauge transformations (2.18a). Even the $D\psi$ piece enjoys this property due to the special character of the group ($\det V = 1$) under consideration. Thus the invariance of the theory with respect to the set of transformations (2.18a), (2.18b) will be ensured if the gluon part of the measure

$$D\mu(G) \equiv \sum_{L,e} d\mu(G(L,\tau_e)) \qquad (2.21)$$

(L labels link) shares this property. Since $g_{\vec{n}}$ in (2.18″) are elements of a compact semisimple Lie group, the choice is essentially unique and is given by the Haar measure

$$d\mu(G) = N_G \sum_{\alpha=1}^{8} dG_\alpha [\det M(G)]^{-1}, \qquad (2.22)$$
$$M^{\alpha\beta}(G) = -2i\,\mathrm{Tr}\left[\Lambda^\alpha U^{-1}(G)\frac{\partial U(G)}{\partial G_\beta}\right].$$

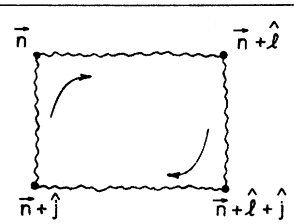

FIG. 1. Gluon self-interaction is represented by a box ($\hat{l} \neq \pm\hat{j}$).

The elements G of the group are assumed to be parametrized according to (2.14), and the completely arbitrary normalization constant N_G will be fixed later. The property alluded to after Eq. (2.21) is the invariance of the expression (2.21) under the left (L) and right (R) shifts separately:

$$d\mu(gG) = d\mu(Gg) = d\mu(G). \qquad (2.23)$$

C. The lattice-theory Hamiltonian

The Hamiltonian formulation of the theory described by the Lagrangian (2.13) turns out to be very helpful in subsequent discussions. First, the Hamiltonian H will be constructed by means of Feynman's path-integral approach. The derivation sketched below yields $H = H_\psi + H_G$,

$$H_\psi = \frac{i}{2g^2}\sum_{\hat{l},\vec{n}} \overline{\psi}(\vec{n})\hat{l}U(G_{\vec{n},\hat{l}})\psi(\vec{n}+\hat{l}) + m_0\overline{\psi}\psi, \qquad (2.24a)$$

$$H_G = \sum_{\hat{l}} [Q_R^\alpha(\vec{n},\hat{l})Q_R^\alpha(\vec{n},\hat{l}) + \tfrac{50}{3}] - \frac{1}{g^4}\sum_B \mathrm{Tr}[U(G_{\vec{n},\hat{l}})U(G_{\vec{n}+\hat{l},\hat{j}})U^\dagger(G_{\vec{n}+\hat{j},\hat{l}})U^\dagger(G_{\vec{n},\hat{j}})], \qquad (2.24b)$$

where the differential operators Q_R^α (Q_L^α) are generators of the group transformations under right (left) multiplication

$$Q_R^\alpha(\vec{n},\hat{l}) = \left(\frac{\Lambda^\alpha}{2}\right)^a_{\ b} U^b_c(\vec{n},\hat{l})\frac{\delta}{\delta U^a_c(\vec{n},\hat{l})} = \mathrm{Tr}\left[\frac{\Lambda^\alpha}{2}U(\vec{n},\hat{l})\frac{\delta}{\delta U(\vec{n},\hat{l})}\right], \qquad (2.25)$$
$$Q_L^\alpha(\vec{n},\hat{l}) = U^a_b(\vec{n},\hat{l})\left(\frac{\Lambda^\alpha}{2}\right)^b_{\ c}\frac{\delta}{\delta U^a_c(\vec{n},\hat{l})} = \mathrm{Tr}\left[U(\vec{n},\hat{l})\frac{\Lambda^\alpha}{2}\frac{\delta}{\delta U(\vec{n},\hat{l})}\right].$$

Here the matrix indices of U are indicated as tensor indices $U_{ab} \equiv U^a_b$. The first operator piece in Eq. (2.24b) represents the sum of SU(3) Laplacians for each link. Recently, Canning and Förster have independently advanced a similar generalization of the KS Hamiltonian.[15] Their method is in the same spirit as ours, but we differ in details.

(1) Schrödinger's equation. Define a state vector $\Phi\{\overline{\psi}^{(u)}(\vec{n}), \psi^{(d)}(\vec{n}), U\}$ as a functional of up $\overline{\psi}^{(u)}(\vec{n})$ and down $\psi^{(d)}(\vec{n})$ components of Dirac fields describing quark and antiquark degrees of freedom, respectively. In Fock space these correspond to the specific realization of the equal-time commutation relation (2.4b) on the lattice

$$\{\hat{\psi}_A(\vec{n},\tau), \hat{\psi}^{\dagger B}(\vec{m},\tau)\} = \delta_A^B\delta_{\vec{n},\vec{m}} \qquad (2.26)$$

where the components

$$\hat{\psi}^{\dagger(u)}(\vec{n},\tau)=\hat{\psi}^{\dagger}(\vec{n},\tau)\tfrac{1}{2}(1+\gamma_0),$$
$$\hat{\psi}^{(d)}(\vec{n},\tau)=\tfrac{1}{2}(1-\gamma_0)\hat{\psi}(\vec{n},\tau) \qquad (2.27)$$

are interpreted as quark and antiquark creation operators. This choice turns out to be only reasonable one in the context of the perturbation theory (see Sec. IV).

Let $A(\hat{\psi}^*, \hat{\psi})$ be a normal-ordered combination of creation $\hat{\psi}^*$ annihilation $\hat{\psi}$ operators defined in the Fock space. Then in the functional approach its action on a state vector $\Phi_1(\psi^*)$ is determined by[14]

$$\Phi_2(\psi_2^*)=\int\int A(\psi_2^*,\psi_1)e^{(\psi_2^*-\psi_1^*)\psi_1}\Phi_1(\psi_1^*)D\psi_1^*D\psi_1. \qquad (2.28)$$

Hence the dynamical principle describing the time development of a system containing fermions along with bosons can be formulated as follows:

$$\Phi(\psi_2^{*(u)},\psi_2^{(d)},U_2)=\int\int \exp[(\psi_2^{*(u)}-\psi_1^{*(u)})\psi_1^{(u)}+(\psi_2^{(d)}-\psi_1^{(d)})\psi_1^{*(d)}-iH_\psi(\psi_2^{*(u)},\psi_2^{(d)},\psi_1^{(u)},\psi_1^{*(d)})\epsilon+i\mathcal{L}_G(\tau_1)\epsilon]$$
$$\times D\psi_1 D\bar{\psi}_1 D\mu(G_1)\Phi(\psi_1^{*(u)},\psi_1^{(d)},U_1), \qquad (2.29)$$

where subscripts 1 and 2 refer to τ_1 and τ_2, respectively, and it is assumed that $\epsilon=(\tau_2-\tau_1)$ is infinitesimal. Now, it is easy to see that this principle with H_ψ defined by Eq. (2.24a) reproduces the generating functional postulated above Eq. (2.12). In passing we note that Eq. (2.29) corresponds to Feynman's continual integral in a phase-space representation in which gluon momenta are integrated out.

Next we turn to the derivation of the gluon part of the Hamiltonian Eq. (2.24b) from the path integral (2.29). It is enough to solve the corresponding problem for a simple system with one link described by the Lagrangian $\mathcal{L}_G^0(\tau)=\frac{1}{2}\mathrm{Tr}(\dot{U}_\tau\dot{U}_\tau^\dagger)$, where U_τ is a shorthand notation for $U(g(\tau))$. Its state vector $\Phi(U_\tau,\tau)$ is defined in a group parameter space, and satisfies the following dynamical equation:

$$\Phi(U_{\tau+\epsilon},\tau+\epsilon)=\int d\mu(g(\tau))\exp\left[i\int_\tau^{\tau+\epsilon}\mathcal{L}_G^0(\tau)\right]\Phi(U_\tau,\tau). \qquad (2.30)$$

This leads to an equation for $\partial\Phi/\partial\tau$ of the Schrödinger form. The linear operator acting on Φ in that equation will be identified as the Hamiltonian. For simple systems the corresponding steps yield the canonical Hamiltonian. In curved spaces noncanonical terms are obtained; the constants in Eq. (2.24b) are such terms because the group space has constant nonzero curvature.

There are four steps in the derivation. First, examine the action, using Eq. (2.18):

$$S\equiv\int_\tau^{\tau+\epsilon}d\tau'\mathcal{L}_G^0(\tau')\cong\frac{1}{2\epsilon}\mathrm{Tr}[2-U(h)-U^\dagger(h)]; \qquad (2.31a)$$

$$U(h)\equiv U^\dagger(g(\tau+\epsilon))U(g(\tau)), \qquad (2.31b)$$

or simply, $h=g^{-1}(\tau+\epsilon)g(\tau)$. With ϵ infinitesimal, $U(h)$ is close to the identity, and may be parametrized by

$$U(h)\equiv 1+i\frac{\eta^\alpha\Lambda^\alpha}{2}-\frac{\eta^\alpha\eta^\beta}{8}\Lambda^\alpha\Lambda^\beta+\cdots. \qquad (2.32)$$

Then

$$S(\tau+\epsilon,\tau)\equiv\eta^2/4\epsilon, \qquad (2.33)$$

where

$$\eta^2=\sum_{\alpha=1}^{8}\eta_\alpha\eta_\alpha.$$

The second step is to account for the time dependence of the coordinates in $\Phi(U_\tau;\tau)$. Inverting Eq. (2.31b), we have

$$\begin{aligned}\Phi(U_\tau;\tau)&=\Phi(U(g(\tau+\epsilon))U(h);\tau)\\&\approx\left[1+U_b^a\left(i\hat{\eta}-\frac{\hat{\eta}^2}{2}\right)_c^b\frac{\delta}{\delta U_c^a}\right.\\&\quad\left.-\frac{1}{2}(U\hat{\eta})_b^a(U\hat{\eta})_d^c\frac{\delta}{\delta U_b^a}\frac{\delta}{\delta U_d^c}\right]\\&\quad\times\Phi(U(g(\tau+\epsilon));\tau)\end{aligned} \qquad (2.34)$$

in which

$$\hat{\eta}=\eta^\alpha\frac{\Lambda^\alpha}{2},\qquad U=U(g(\tau+\epsilon)).$$

Recalling Eq. (2.25), this can be rewritten as

$$\Phi(U_\tau;\tau)\equiv\left(1+i\eta^\alpha Q_R^\alpha-\frac{\eta^\alpha\eta^\beta}{2}Q_R^\alpha Q_R^\beta\right)\Phi(U_{\tau+\epsilon};\tau) \qquad (2.35)$$

since Q_R^α generates the right shifts ($\hat{\eta}$) of a gluon field with parameters $\{g^\alpha\}$.

The third thing we must do is evaluate the measure. As in Eqs. (2.33) and (2.35), only terms to $O(\eta^2)$ need be retained because the phase e^{iS} oscillates very rapidly as $\epsilon\to 0$. Furthermore, it is appropriate to use parameters η in the measure because by (2.31b) we have precisely made a group transformation taking $U(g(\tau))$ into a group element with parameters η. Then, from Eq. (2.22),

$$M^{\alpha\beta}(\eta) \approx \delta^{\alpha\beta} + \mathrm{Tr}\{-2i\hat{\eta}[\Lambda^\alpha, \Lambda^\beta]_- - \tfrac{4}{3}\hat{\eta}\hat{\eta}[\Lambda^\alpha, \Lambda^\beta]_+ + \tfrac{4}{3}\hat{\eta}\Lambda^\beta\hat{\eta}\Lambda^\alpha\}, \tag{2.36}$$

$$(\det M)^{-1} = \exp(-\mathrm{Tr}\ln M) \cong 1 + \tfrac{25}{4}\eta^2 . \tag{2.37}$$

We are now ready to put the pieces together and obtain the Schrödinger equation:

$$\Phi(U_{\tau+\epsilon}; \tau+\epsilon) \cong \Phi(U_{\tau+\epsilon}; \tau) + \epsilon\frac{\partial\Phi(U_{\tau+\epsilon}; \tau)}{\partial\tau} = N_G \int\int \left(\prod_{i=1}^{8} d\eta_i\right)(\det M)^{-1} e^{i\eta^2/4\epsilon} \times \left(1 + i\eta_\alpha Q_R^\alpha - \frac{\eta_\alpha\eta_\beta}{2} Q_R^\alpha Q_R^\beta\right) \times \Phi(U_{\tau+\epsilon}; \tau), \tag{2.38}$$

with $(\det M)^{-1}$ being given by (2.37). The stationary-phase estimate to the integrals is justified for $\epsilon \to 0$. It gives

$$N_G = \left(\frac{i}{64\epsilon}\right)^4, \tag{2.39}$$

$$i\frac{\partial\Phi}{\partial\tau} = H_G^0 \Phi, \tag{2.40a}$$

$$H_G^0 = Q_R^\alpha Q_R^\alpha - \tfrac{50}{3}. \tag{2.40b}$$

Notice that the appearance of Q_R instead of Q_L is due to the way $U(h)$ was introduced in Eq. (2.31b). It is easy to see that $U(h)$ with U_τ and $U_{\tau+\epsilon}$ interchanged yields Q_L^2 in Eq. (2.40b). The equality of the left and right Laplace operators $Q_{R,L}^2$ can be simply checked from Eq. (2.25).

(2) Generators of the local gauge transformations. It has already been indicated [see Eq. (2.18″)] that the gauge field may undergo left (L) and right (R) shifts independently, e.g.,

$$\begin{aligned} U(\vec{n}, \hat{l}) &\to U(\vec{n}, \hat{l}) - ig(\vec{n})U(\vec{n}, \hat{l}), \\ U(\vec{n} - \hat{l}, \hat{l}) &\to U(\vec{n} - \hat{l}, \hat{l}) + iU(\vec{n} - \hat{l}, \hat{l})\, g(\vec{n}), \end{aligned} \tag{2.41}$$

where $g(\vec{n}) = g^\alpha(\vec{n})\frac{1}{2}\Lambda^\alpha$ and $g^\alpha(\vec{n})$ are infinitesimal parameters of the local gauge transformation $V(g_{\vec{n}})$. It is easy to verify that the transformations (2.41) can be realized by differential operators Q_L^α and Q_R^α defined by Eq. (2.25). These act on upper and lower indices, respectively. Since six links emanate from each site, three in positive ($\hat{l}_+$) and three in negative ($\hat{l}_-$) directions, the generators of the local gauge transformation of Eq. (2.18″) are

$$Q_G^\alpha(\vec{n}) = \sum_{\hat{l}_+} Q_L^\alpha(\vec{n} + \hat{l}_+) + \sum_{\hat{l}_-} Q_R^\alpha(\vec{n} + \hat{l}_-). \tag{2.42}$$

Turning to the gauge transformations of the quark fields, we write (2.18′) in infinitesimal form

$$\begin{aligned} \psi_a(\vec{n}) &\to \psi_a(\vec{n}) - ig_a^b(\vec{n})\psi_b(\vec{n}), \\ \bar{\psi}^a(\vec{n}) &\to \psi^a(\vec{n}) + i\bar{\psi}^b(\vec{n})\, g_b^a(\vec{n}). \end{aligned} \tag{2.43}$$

Hence the corresponding generators are

$$Q_\psi^\alpha(\vec{n}) = \psi^\dagger(\vec{n})\frac{\Lambda^\alpha}{2}\psi(\vec{n}) \tag{2.44}$$

This expression is normal-ordered. Its action is determined by Eq. (2.28).

The sum of the generators (2.42) and (2.44) defines the total local color charges

$$Q^\alpha(\vec{n}) = Q_G^\alpha(\vec{n}) + Q_\psi^\alpha(\vec{n}). \tag{2.45}$$

These are lattice counterparts of the continuum charges (2.8), as is straightforwardly checked using Eqs. (2.17) and (2.25) in the limit $a \to 0$.

The invariance of the theory under the time-independent local gauge transformations can now be stated as

$$[H_\psi + H_G, Q^\alpha(\vec{n})] = 0. \tag{2.46}$$

It is interesting to note that the kinetic piece $Q^\alpha Q^\alpha$ of the Hamiltonian H has a symmetry higher than the *combined* left and right shifts generated by (2.45). It is invariant with respect to $Q_L^\alpha(\vec{n}, \hat{l})$ and $Q_R^\alpha(\vec{n}, \hat{l})$ separately. This property is shared by the invariant measure, and is an important ingredient of the formalism below.

(3) *Space of states.* We assume, following KS, that the physical subspace is spanned by eigenstates $\Phi\{\psi, U\}$ of the Hamiltonian (2.24) which are invariant under local gauge transformations (2.45):

$$Q^\alpha(\vec{n})\Phi_{\text{phys}}\{\psi, U\} = 0. \tag{2.47}$$

Obviously in these states all color indices are contracted as, e.g.,

$$\Phi_1(\psi) \sim \bar{\psi}^a(\vec{n})\psi_a(\vec{n}),$$

$$\Phi_2(U) \sim U_b^a(\vec{n}, \vec{n}+\hat{l})U_c^b(\vec{n}+\hat{l}, \hat{j})U_d^{\dagger c}(\vec{n}+\hat{j}, \hat{l})U_a^{\dagger d}(\vec{n}, \vec{n}+\hat{j})|_{\hat{j}\neq\pm\hat{l}},$$

$$\Phi_3(\psi, U) \sim \bar{\psi}^a(\vec{n})U_a^b(\vec{n}, \vec{n}+\hat{l})\psi_b(\vec{n}+\hat{l}),$$

where unitary and Dirac indices are suppressed. We remind the reader that the matrix indices of U are indicated as tensor indices $U_{ab} = U_a^b$ to emphasize the fact that under the left [right] shift $U(g) \to U(g_0 g)$ $[U(gg_0^{-1})]$, U transforms according to the fundamental representation $\underline{3}$ $(\bar{\underline{3}})$ in the index a (b).

We proceed to the normalization of the states. First, recall that in the functional approach, the inner product in the Hilbert space can be defined as[14]

$$(\Phi, \Phi') = \int \prod d\mu(g) \times \prod_i d\psi_i^\dagger d\psi_i e^{-\bar{\psi}_i \psi_i} \Phi^*(\psi, U(g)) \Phi'(\psi, U(g)), \tag{2.48}$$

where the group volume for every link is normalized to 1, $\int d\mu(g) = 1$. (This convention is different from the one used previously [see Eq. (2.39)]. This difference is inconsequential.)

We reiterate that there is a complete equivalence between the Hilbert space constructed above and a specific realization of the space of states when the *fermionic* sector is represented by Fock vectors, using fermion annihilation and creation operators. For the sake of consistency we are treating fermion and gauge fields on the same footing, using the functional realization for both sectors.

Obviously the integrals (in the inner product) factorize into fermionic parts which can be evaluated in a standard way and a product of integrals of gluon fields corresponding to each link. In general, one has to deal with the integrals

$${}^n I\{{}^a_b\} = \int d\mu(g) U^{a_1}_{b_1}(g) \cdots U^{a_n}_{b_n}(g) \tag{2.49}$$

defined on the gauge group manifold.

The general properties of these integrals are discussed in Appendix A, since we shall encounter them quite often.

We conclude this subsection by supplying a few simple examples:

$$\int U^b_a(g) U^{\dagger c}_d(g) d\mu(g) = \tfrac{1}{3} \delta^b_d \delta^c_a, \tag{2.50a}$$

$$\int U^b_a(g) U^c_d(g) U^e_f(g) d\mu(g) = \tfrac{1}{6} \epsilon^{bce} \epsilon_{adf}, \tag{2.50b}$$

$$\begin{aligned}\int U^b_a(g) U^c_d(g) U^{\dagger e}_f(g) U^{\dagger l}_m(g) d\mu(g) &= \tfrac{1}{24}(\delta^b_f \delta^c_m + \delta^c_f \delta^b_m)(\delta^e_a \delta^l_d + \delta^l_a \delta^e_d) \\ &\quad + \tfrac{1}{12}(\delta^b_f \delta^c_m - \delta^c_f \delta^b_m)(\delta^e_a \delta^l_d - \delta^l_a \delta^e_d). \end{aligned} \tag{2.50c}$$

These equations apply to a single link. A general state may, however, contain U's at different links. The reductions explained in Appendix A must be carried out link by link, yielding a product of singlet contractions, one for each link.

D. Remarks on the formalism

It is appropriate to conclude this section by surveying briefly what all this formalism is intended to describe. The points of departure from the KS theory are emphasized in the ensuing discussion.

(1) At this stage, the reader uncomfortable with the path-integral derivation of the Hamiltonian may simply take the operator part of Eqs. (2.24) as the starting point of the theory. The definitions of $Q_{R,L}$ and the prescription for taking $a \to 0$ may be used to verify that this is indeed the continuum "electric" part of the Hamiltonian. In addition, the commutators of $Q^\alpha_{R,L}(\vec{n})$ with U pass to the canonical commutation relations Eq. (2.4a) in the continuum limit.

(2) As it stands, the lattice Lagrangian Eq. (2.13) contains three free parameters, g^2, a, and m_0. The latter two have dimensions, since by scaling the fields ψ, $\psi^\dagger$, the quark-mass parameter retains its natural dimension.

In principle the lattice Lagrangian could contain any or all conceivable interaction terms that respect the internal and gauge symmetries of the continuum theory, and which vanish as $a \to 0$. Each such term allows a variety of new constants to be introduced, some dimensionless and others with dimensions. In practice these are arbitrary constants because their dependence on the genuine parameters in $\mathcal{L}$(continuum) cannot be determined. Thus, each new term in $\mathcal{L}$(lattice) weakens the predictive power of the theory.

As was indicated by KS, and will be discussed at length in Sec. III of this paper, the Hamiltonian [Eq. (2.24)] along with the quark-mass term

$$H^0 = H^0_G + H^0_\psi, \quad H^0_G = Q_{R(L)}{}^2, \quad H^0_\psi = m_0 \bar{\psi}\psi \tag{2.51}$$

may be considered as a zeroth-order approximation for hadron spectroscopy if the color group is SU(3). A gluon on a link carries an energy ϵ, which is proportional to the eigenvalue of the SU(3) Casimir in the $\underline{3}$ (or $\underline{\bar{3}}$) representation:

$$H^0_G U = \epsilon U, \quad \epsilon = \tfrac{4}{3}. \tag{2.52}$$

The quarks carry an energy m_0. Then a hadron's rest energy is, in this approximation,

$$E^{(0)}(\text{hadron}) = I\epsilon + Jm_0 \tag{2.53}$$

if it consists of I gluons and J (quarks + antiquarks).

Thus the energy of the meson in Fig. 2(a) is $2m_0 + \epsilon$. The energy of the baryon in Fig. 2(b) is $3m_0 + 3\epsilon$. Since the energy ϵ and mass m_0 in Eq. (2.53) are in units of g^2/a and $1/ag^2$, respectively, the ratio $\epsilon^{(0)}(\text{meson})/\epsilon^{(0)}(\text{baryon})$ is independent of the lattice constant a. *It is these kinds of dimensionless ratios that one hopes the theory will describe adequately.*

An important point is that, quite independently of whether m_0 and g^2 can be adjusted to give reasonable ratios of masses, one must first justify *additive* energy formulas as in Eq. (2.53). There are, e.g., mesons with masses $2m_0 + n\epsilon$, where $n = 1, 2, 3, \ldots$. Thus ϵ is a level-spacing parameter, but directly in the masses. For contrast,

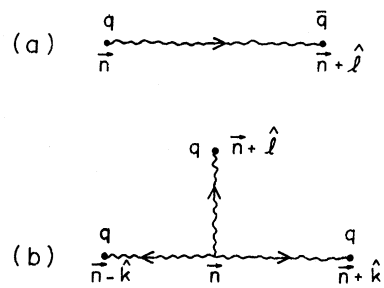

FIG. 2. (a) A meson $\bar{\psi}(\vec{n})\frac{1}{2}\lambda^i \Gamma U(\vec{n},\hat{l})\psi(\vec{n}+\hat{l})$, where λ^i refers to unitary spin, and Γ is a Dirac matrix. (b) A baryon $\bar{\psi}^{ie}_{\alpha}(\vec{n}+\hat{l})\bar{\psi}^{jf}_{\beta}(\vec{n}+\hat{k})\bar{\psi}^{kg}_{\gamma}(\vec{n}-\hat{k}) \times U^a_e(\vec{n},\hat{l})U^b_f(\vec{n},\hat{k})U^c_g(\vec{n},-\hat{k})f_{ijk}g^{\alpha\beta\gamma}\epsilon_{abc}$.

the dual-model spacings refer to (mass)^2. The phenomenological utility of additive quark-mass formulas for low-lying mesons and baryons has been described in Ref. 16.

(3) Our model contains gluon-free particles. Since quarks and antiquarks are defined on all sites, a gauge-invariant gluon-free meson state is of the form

$$M^{i\{\alpha\}}_{\{n\}} = \bar{\psi}(\vec{n})\frac{\lambda^i}{2}\Gamma^{\alpha}\psi(\vec{n}),$$

where i and $\{\alpha\}$ are respectively unitary spin and Dirac indices. The color indices are summed over so the state is locally a color singlet.

Similarly, a gluon-free baryon state is

$$B(\vec{n}) = \bar{\psi}^{ia}_{\beta}(\vec{n})\bar{\psi}^{jb}_{\gamma}(\vec{n})\bar{\psi}^{kc}_{\delta}(\vec{n})\epsilon_{abc}f_{ijk}g^{\beta\gamma\delta},$$

where ϵ_{abc} on the color indices makes the state a color singlet. As in Fig. 2, f_{ijk} and $g^{\beta\gamma\delta}$ schematically represent the unitary spin and Dirac structure of the particular baryon. In this case, $[\bar{\psi}\bar{\psi}\bar{\psi}]$ is a schematic notation for the more careful constructions needed for the description of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ baryons.

Since the crudest estimates of the rest energies of gluon-free meson and baryon states are $2m_0$ and $3m_0$, respectively, one is in an embarrassing situation if $m_0=0$. Now, one expects the interaction to modify these estimates in a number of ways. One set of effects should shift the effective quark mass away from m_0. In addition, since the theory involves vector bosons, one anticipates that the interaction will produce spin splittings, so that the baryon spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decouplet will not remain degenerate. Also, following Ref. 16, it is expected that unitary-spin singlets and octets will be split.

In this paper only the mass-shift effects will be investigated. The whole "quark mass" will be due to self-interactions, and it is not implausible for these effects to be sizable. Thus there are no problems of principle owing to having gluon-free mesons and baryons.

(4) This model also allows for what would normally be termed "exotic" states in unitary symmetry. All gauge-invariant quarkless excitations (GIQE's) are exotic in that sense.

A simple example of a unitary-spin exotic state contains two quarks and two antiquarks:

$$|qq\bar{q}\bar{q}\rangle = \bar{\psi}^{a_1 i_1}(\vec{n}+\hat{j})U^{b_1}_{a_1}(\vec{n}+\hat{j},\vec{n})U^{b_2}_{a_2}(\vec{n},\vec{n}-\hat{j})\bar{\psi}^{a_2 i_2}(\vec{n}-\hat{j})\epsilon^{b_1 b_2 e}U^f_e(\vec{n},\vec{n}+\hat{k})\epsilon_{d_1 d_2 f}\psi_{c_1 l_1}(\vec{n}+\hat{k}+\hat{j})U^{c_1}_{d_1}(\vec{n}+\hat{k}+\hat{j},\vec{n}+\hat{k})$$
$$\times U^{c_2}_{d_2}(\vec{n}+\hat{k},\vec{n}+\hat{k}-\hat{j})\psi_{c_2 l_2}(\vec{n}+\hat{k}-\hat{j}),$$

where unitary indices $i_{1,2}, l_{1,2}$ can be contracted in an arbitrary way. Here Dirac indices have been suppressed. This agrees with expectations that such states are not necessarily totally absent, but are very massive. The GIQE has energy 4ϵ, and the exotic meson above has energy $4m_0+5\epsilon$. Thus if $m_0 \approx \frac{1}{3}\epsilon$ so the lightest baryon mass equals the level spacing, one has $M(\text{GIQE}) \approx 4M(\text{baryon})$, $M(qq\bar{q}\bar{q}) \approx 6.3M(\text{baryon})$. Note that the type of the gauge group [SU(3) in this example] imposes restrictions on the spectrum of exotic states.

III. PERTURBATION THEORY

In this section a systematic perturbation theory is developed, splitting the original Lagrangian Eq. (2.16) into "free" ($\mathcal{L}_0 = \mathcal{L}^0_{\psi} + \mathcal{L}^0_G$) and "interaction" ($\mathcal{L}_I$) parts, as follows:

$$\mathcal{L}^0_{\psi} = \sum_{\vec{n}} [\bar{\psi}(\vec{n})i\not{\partial}_0\psi(\vec{n}) - m_0\bar{\psi}(\vec{n})\psi(\vec{n})], \tag{3.1}$$

$$\mathcal{L}^0_G = \sum_L \{\tfrac{1}{2}\operatorname{Tr}[\dot{U}(G_{\vec{n},\hat{i}})\dot{U}^{\dagger}(G_{\vec{n},\hat{i}})] - \tfrac{50}{3}\}, \tag{3.2}$$

$$\mathcal{L}_I = -\frac{i}{2g^2}\sum_{\vec{n},\hat{l}} \bar{\psi}(\vec{n})\hat{l}U(G_{\vec{n},\hat{l}})\psi(\vec{n}+\hat{l}) + \frac{1}{g^4}\sum_B \operatorname{Tr}[U(G_{\vec{n},\hat{i}})U(G_{\vec{n}+\hat{i},\hat{j}}) \times U^{-1}(G_{\vec{n}+\hat{j},\hat{i}})U^{-1}(G_{\vec{n},\hat{j}})]. \tag{3.3}$$

$\mathcal{L}_I$ will be treated as a perturbation in the strong-coupling regime $g^2 \gg 1$. In $\mathcal{L}^0_G$ a trivial constant term for each lattice link is introduced to cancel the zero-point lattice energy of the Hamiltonian Eq. (2.24b).

Our approach has many features in common with the perturbation theory of Feynman, Dyson, and Wick. However, a difference arises because of the presence of gauge fields $U(g)$ which are group elements rather than algebraic objects. Nevertheless, it will be proven that the following important results still hold:

(a) the linked cluster expansion, i.e., a cancellation of vacuum loops in Green's functions;

(b) the Dyson equation for quark propagators;

(c) The Bethe-Salpeter equation in ladder form for four-point quark Green's functions.

The method of semi-invariants is employed to prove these results. In the past this method has been used efficiently in a different context in statistical physics.[17]

A. Green's functions

Green's functions are given by the variational derivatives of the generating functional W constructed from the Feynman kernel Eq. (2.12):

$$W\{\eta,J\}=\int\int D\psi D\bar{\psi}D\mu(g)\exp\left(i\int_{-\infty}^{\infty}d\tau\left\{\mathcal{L}_0+\mathcal{L}_I+\sum_n[\bar{\psi}(\vec{n})\eta(\vec{n})+\bar{\eta}(\vec{n})\psi(\vec{n})]+\sum_L \mathrm{Tr}[J_L^\dagger U(g_L)+U^\dagger(g_L)J_L]\right\}\right). \tag{3.4}$$

Here $\eta(\vec{n},\tau)$, $\bar{\eta}(\vec{n},\tau)$ and $[J_L(\tau)]^a_c$, $[J_L^\dagger(\tau)]^a_c$ are quark and gluon sources at each site $\vec{n}$ and link L, respectively.

It is convenient first to consider the Green's functions of the free-gluon theory, generated by

$$W_G^0\{J\}=\int\int D\mu(g)\exp\left(i\int_{-\infty}^{\infty}d\tau\left\{\mathcal{L}_G^0(\tau)+\sum_L \mathrm{Tr}[J_L^\dagger U(g_L)+U^\dagger(g_L)J_L]\right\}\right). \tag{3.5}$$

The incorporation of quarks will be straightforward. Let us define an $(n+m)$-point free-gluon propagator

$$D_{n+m}(\tau_1\cdots\tau_{n+m})=\frac{1}{W_G^0\{J\}}\,\frac{\delta^{n+m}W_G^0\{J\}}{\delta J_{L_1}(\tau_1)\cdots\delta J_{L_n}(\tau_n)\delta J^\dagger_{L_{n+1}}(\tau_{n+1})\cdots\delta J^\dagger_{L_{n+m}}(\tau_{n+m})}\bigg|_{J=J^\dagger=0}. \tag{3.6}$$

Explicitly, this is

$$D_{n+m}(\tau_1\cdots\tau_{n+m})=\frac{1}{W_G^0\{J=0\}}\int\int\prod_k D\mu(g(\tau_k))U(g_{L_1}(\tau_1))\cdots U(g_{L_{n+m}}(\tau_{n+m}))\exp\left[i\sum_k\int_{\tau_k}^{\tau_{k+1}}d\tau\,\mathcal{L}_G^0(\tau)\right]. \tag{3.6$'$}$$

For a given ordering $\tau_1>\tau_2>\cdots>\tau_{n+m}$, one has

$$\begin{aligned}D_{n+m}(\tau_1\cdots\tau_{n+m})W_G^0\{J=0\}=&\int\int_{g=g(\tau_{i,f})} D\mu(g)K_G(\tau_f,\tau_1)D\mu(g(\tau_1))\\&\times U(g_{L_1}(\tau_1))K_G(\tau_1\tau_2)D\mu(g(\tau_2))\cdots D\mu(g(\tau_{n+m}))U^\dagger(g_{L_{n+m}}(\tau_{n+m}))K_G(\tau_{n+m},\tau_i),\end{aligned} \tag{3.7}$$

in terms of the Feynman kernel

$$K_G(\tau_{k+1},\tau_k)=\int\int D\mu(g(\tau))\exp\left[i\int_{\tau_k}^{\tau_{k+1}}d\tau\,\mathcal{L}_G^0(\tau)\right]. \tag{3.8}$$

Note that usually the end-point coordinates $g(\tau_{i,f})$ in the generating functional are left unintegrated—one considers them to be fixed. In the limit $\tau_{i,f}\to(\mp)\infty$, they become coordinates of the ground state (vacuum) which in our case is assumed to be gauge invariant [see Eq. (2.47)]. Unlike the usual practice, we have performed the integration over $g\equiv g(\tau_{i,f})$. The integration is well defined since the group is compact. The utility of this step will appear subsequently.

To simplify Eq. (3.7) we make use of the Schrödinger equation (2.41c) in integral form,

$$\Phi\{U(g(\tau_f)),\tau_f\}=\int\int K_G(\tau_f,\tau_i)D\mu(g(\tau_i))\,\Phi\{U(g(\tau_i)),\tau_i\}. \tag{3.9}$$

In other words,

$$\Phi\{U(g),\tau_f\}=e^{-i\hat{H}_G^0(\tau_f-\tau_i)}\,\Phi\{U(g),\tau_i\}. \tag{3.9$'$}$$

This allows us to recast Eq. (3.7) into the form

$$D_{n+m}(\tau_1\cdots\tau_{n+m})W_G^0\{J=0\}=\iint_{\tau_{i,f}=\mp\infty} D\mu(g(\tau_f))K_G(\tau_f,0)\times D\mu(g)\tilde{U}(g_{L_1},\tau_1)\cdots\tilde{U}^\dagger(g_{L_{n+m}},\tau_{n+m})K_G(0,\tau_i)D\mu(g(\tau_i))|_{g_{L_k}\equiv g_{L_k}(0)}\,. \quad (3.10)$$

The symbol $\tilde{U}$ in Eq. (3.10) indicates that an interaction representation was introduced,

$$\tilde{U}(g,\tau)=e^{i\hat{H}_G^0\tau}U(g)e^{-i\hat{H}_G^0\tau}\,,$$
$$\hat{H}_G^0=\hat{Q}_R^\alpha(g)\hat{Q}_R^\alpha(g)\,. \quad (3.11)$$

It is important to notice that $\tilde{U}(g)$ is an operator, and that the commutator $[H_G^0, U(g)]$ is q number rather than c number times U as it is in ordinary field theory. This is a special feature of the lattice gauge fields which we have to take into account.

It is easy to see that the expressions

$$\iint D\mu(G_f)K_G(\tau_f,0),\quad \iint K_G(0,\tau_i)D\mu(G_i)$$

are independent of $g_L\equiv g_L(0)$, and so

$$W_G\{\tau=0\}=\iint_{\tau_{i,f}=\mp\infty} D\mu(g(\tau_f))K(\tau_f,\tau_i)D\mu(g(\tau_i))$$
$$=\iint_{\tau_{i,f}=\mp\infty} D\mu(g(\tau_f))K(\tau_f,0)\times D\mu(g)K(0,\tau_i)D\mu(g(\tau_i))\,.$$

Consequently, for Green's functions we have

$$D_{n+m}(\tau_1\cdots\tau_{n+m})=\iint D\bar{\mu}(g)T(\tilde{U}(g_{L_1},\tau_1)\cdots\tilde{U}^\dagger(g_{L_{n+m}},\tau_{n+m}))\,, \quad (3.12)$$

where the measure is now normalized as $\int d\bar{\mu}(g)=1$ [compare this with the previous convention, Eq. (2.39)]. The presence of the T product accounts for the arbitrary ordering of $\tau_1\cdots\tau_{n+m}$.

The simplest gluon Green's function is (for $\tau>0$)

$$D_{L_1L_2}(\tau)\equiv D_{1+1}^{(L_1L_2)}(\tau,0)=\delta_{L_1L_2}\iint D\bar{\mu}(g)e^{i\hat{H}_G^0\tau}U_b^a(g)e^{-i\hat{H}_G^0\tau}U_d^{\dagger c}(g)\,. \quad (3.13)$$

Since only the singlet combination of $U(g)$ and $U^\dagger(g)$ contributes, applying Eq. (2.50a) one gets after time-ordering

$$D_{bd}^{ac}(\tau)=\tfrac{1}{3}\delta_d^a\delta_b^cD(\tau),\quad D(\tau)=e^{-i\epsilon|\tau|}\,, \quad (3.13')$$

where ϵ was introduced in Eq. (2.52). The Fourier transform of $D(\tau)$ is

$$D(\omega)=\int_{-\infty}^{\infty}D(\tau)e^{+i\omega\tau}d\tau=\frac{2\epsilon i}{\omega^2-\epsilon^2+i\delta}\,. \quad (3.14)$$

Thus ϵ plays the role of an effective gluon bare mass.

At this point, notice from Eq. (3.12) that the higher-order free Green's functions cannot be expressed in terms of $D(\tau)$ alone. The Wick theorem is invalid in its usual form. We will discuss this point fully in Sec. III B.

Clearly the free-fermion sector is a normal field theory, and is spared from these complications. The fermion two-particle Green's function is calculated in the ordinary manner:

$$S_{A,\vec{k},\vec{n}}^B(\tau)W_\psi^0=\iint D\bar{\psi}D\psi\,\psi_A(\vec{k},\tau)\bar{\psi}^B(\vec{n},0)\times\exp\left[i\int_{-\infty}^{\infty}d\tau\,\mathcal{L}_\psi^0(\tau)\right]\,, \quad (3.15)$$

where

$$W_\psi^0=\iint D\bar{\psi}D\psi\exp\left[i\int_{-\infty}^{\infty}d\tau\,\mathcal{L}_\psi^0(\tau)\right].$$

Hence we find

$$S_{A,\vec{k},\vec{n}}^B(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{A,\vec{k},\vec{n}}^B(\omega)e^{-i\omega\tau}d\omega\,,$$
$$S_{A,\vec{k},\vec{n}}^B(\omega)=\left(\frac{i}{\not{\omega}-m_0+i\delta}\right)_A^B\delta_{\vec{k},\vec{n}}\,. \quad (3.16)$$

Since $\not{\omega}=\gamma_0\omega$, in a basis with γ_0 diagonal, $S(\omega)$ is diagonal in all indices ($A=B$).

B. Generalized Wick's theorem

The semi-invariants are Green's functions defined by [compare with Eq. (3.6)]

$$D_{n+m}^c\equiv D_{(L_1\cdots L_nL_1'\cdots L_m')}^c(\tau_1\cdots\tau_{n+m})=\frac{\delta^{n+m}\ln W_G^0\{J\}}{\delta J_{L_1}(\tau_1)\cdots\delta J_{L_n}(\tau_n)\delta J_{L_1'}^\dagger(\tau_{n+1})\cdots\delta J_{L_m'}^\dagger(\tau_{n+m})}\,. \quad (3.17)$$

Here c stands for connected; its meaning will be clarified soon. First note the following obvious relations:

$$D_{(L)} = D^c_{(L)} = 0\,, \tag{3.18a}$$

$$D_{(L_1L_2)} = D^c_{(L_1L_2)} + D^c_{(L_1)}D^c_{(L_2)} = D^c_{(L_1L_2)} = 0\,, \tag{3.18b}$$

$$D_{(L_1L_1')} = D^c_{(L_1L_1')} + D^c_{(L_1)}D^c_{(L_1')} = \delta_{L_1L_1'}D^c_{1+1}\,. \tag{3.18c}$$

It is straightforward to prove by induction that

$$D_{(L_1\cdots L_nL_1'\cdots L_m')} = \sum_{(p)}\prod_{\alpha\beta} D^c_{(L_{\alpha_1}\cdots L_{\alpha_i}L'_{\beta_1}\cdots L'_{\beta_j})} \left(\sum i = n,\ \sum j = m\right), \tag{3.19}$$

where the summation extends over all partitions p of $n+m$ elements (U's and $U^\dagger$'s). Since a contribution to D_{n+m} from distinct links is factorizable, i.e., the corresponding gluons are uncorrelated, one infers from Eq. (3.19) that

$$D^c_{(L_1\cdots L_iL_1'\cdots L_j')} = D^c_{i+j}\,\delta_{L_1L_2}\cdots\delta_{L_iL_1'}\cdots\delta_{L'_{j-1}L'_j}\,. \tag{3.20}$$

The relation (3.19), together with (3.20), forms the content of the generalized Wick theorem. Calling a contraction of $i+j$ gluons D^c_{i+j}, the procedure can be stated in the following form: A vacuum expectation of gluons [see Eq. (3.12)] can be expressed in terms of all possible contractions (two, three, four, etc.) of gluons, contractions of gluons attached to different links being zero. It should be emphasized that unlike the ordinary case, the contractions are not restricted to only two fields, as is the case for the quark sector. There is a good reason for this—the free lattice gauge fields are not systems of harmonic oscillators. Put differently, the measure in the functional integral is not Gaussian.

Next we turn to the linked cluster expansion theorem. Consider a vacuum expectation of an arbitrary operator $\hat{O}\{\psi, U\}$,

$$\langle\hat{O}\rangle = \frac{\int\int D\psi D\bar{\psi} D\mu(g)\hat{O}\{\psi, U(g)\}\exp[i\int_{-\infty}^{\infty} d\tau\,\mathcal{L}(\tau)]}{W\{\eta = J = 0\}}\,. \tag{3.21}$$

Examine the perturbation expansion in L_I of the numerator and denominator separately. In every order the quark and gluon functional integrals factorize. Then one can apply the Wick theorem to the quark fields and the generalized Wick theorem to the gluon fields, performing all possible contractions. In the numerator, there appear factors which are *not connected* with $\hat{O}$ through quark or gluon contractions [Eqs. (3.16) and (3.17)]. Using the standard combinatoric arguments (see, e.g., Ref 18, p. 187), these factors can be shown to cancel the denominator W, yielding

$$\langle\hat{O}\rangle_c = \int\int_c D\psi D\bar{\psi} D\mu(g)\,\hat{O}\{\psi, U(g)\} \times\sum_n \frac{i^n}{n!}\int d\tau_1\cdots d\tau_n\,\mathcal{L}_I(\tau_1)\cdots\mathcal{L}_I(\tau_n) \times\exp\left[i\int_{-\infty}^{\infty} d\tau\,\mathcal{L}_0(\tau)\right]. \tag{3.22}$$

Here the subscript c on the functional integration symbol denotes that only *connected* parts are included. The c appearing in Eq. (3.17) can now be viewed as the specific case that $\hat{O}$ is itself a $\tilde{U}$ or $\tilde{U}^\dagger$ in the *free* theory. The present discussion extends that limited meaning so that c means connected with $\hat{O}$, which is any operator in the *interacting* theory.

We can go further and establish Dyson's equation for the quark propagator [compare Eq. (3.15)]:

$$S'^B_{A,kn}(\tau) = \frac{\int\int D\psi D\bar{\psi} D\mu(g)\psi_A(\vec{k},\tau)\bar{\psi}^B(\vec{n},0)\exp[i\int_{-\infty}^{\infty} d\tau\,\mathcal{L}(\tau)]}{W\{\eta = J = 0\}}\,. \tag{3.23}$$

First, notice that due to the local lattice gauge invariance,

$$S'^B_{A,\vec{k},\vec{n}}(\tau) = S'^A_B(\tau)\,\delta_{\vec{k},\vec{n}}\,. \tag{3.24}$$

Next, define the Fourier transform

$$S'(\omega) = \int e^{i\omega t} S'(\tau)\,d\tau\,, \tag{3.25}$$

and repeat the well-known arguments (see, e.g., Ref. 18, p. 285) to derive the equation

$$S'^{-1}(\omega) = S^{-1}(\omega) - \Sigma(\omega)\,, \tag{3.26}$$

where all indices are omitted. The quark proper self-energy $\Sigma(\omega)$ is defined as a graph with external lines removed, which cannot be divided into two disjoint parts by a removal of a single quark line.

In the same way, one establishes the Bethe-Salpeter equation in a restricted form, for quark-quark scattering. It will be discussed and employed in Sec. IV.

C. Perturbative evaluation of $\Sigma(\omega)$

In this subsection the formalism developed so far is utilized for the low-order perturbative evaluation of the quark self-energy $\Sigma(\omega)$. For illustrative purposes simple diagrammatic rules are also suggested. For our immediate goals the general diagrammatic technique, which appears to be rather complicated, will not be needed (see Ref. 17 and references therein for a discussion of related combinatoric problems).

Examine the quark propagator (3.23) in the lowest nontrivial order:

$$S'_{(0)} = S(\tau)\,, \tag{3.27a}$$

$$S'_{(2)}(\tau) = \frac{i^2}{2!}\iint_c D\psi\, D\bar{\psi}\, D\bar{\mu}(g)\, \psi(\vec{n},\tau)\bar{\psi}(\vec{n},0) \times \int_{-\infty}^{\infty} d\tau_1\, d\tau_2\, \tilde{\mathcal{L}}_I(\tau_1)\, \tilde{\mathcal{L}}_I(\tau_2) \times \exp\left[i\int_{-\infty}^{\infty} d\tau\, \mathcal{L}_\psi^0(\tau)\right]. \tag{3.27b}$$

A first-order term $\int d\bar{\mu}(G)\mathcal{L}_I(\tau)$ vanishes because $\int d\bar{\mu}(g)\, \tilde{U}(g) = 0$. Here $\tilde{\mathcal{L}}_I(\tau)$ is given by Eq. (3.3) with $U(G)$ being substituted for by $\tilde{U}_\tau(G)$ according to the result (3.11). In Eq. (3.27b) a contribution from the denominator W cancels out a disconnected piece in the numerator. The latter has the form

$$S(\tau)\iint D\psi\, D\bar{\psi}\, D\bar{\mu}(g) \times \int_{-\infty}^{\infty} d\tau_1\, d\tau_2\, \mathcal{L}_I(\tau_1)\, \mathcal{L}_I(\tau_2) \exp\left[i\int_{-\infty}^{\infty} d\tau\, \mathcal{L}^0(\tau)\right], \tag{3.28}$$

which, after inserting expression (3.3) for $\mathcal{L}_I$, becomes

$$\frac{1}{2!}\iint d\bar{\mu}(G)\int_{-\infty}^{\infty} d\tau_1\, d\tau_2 \Bigg[\left(\frac{1}{2g^2}\right)^2 \sum_{\substack{\vec{n},\hat{l}\\ \vec{m},\hat{j}}} \bar{\psi}^{\cdot}_{\tau_1}(\vec{n})\hat{l}\, U_{\tau_1}(G_{\vec{n},\hat{l}})\, \psi^{\cdot}_{\tau_1}(\vec{n}+\hat{l})\, \bar{\psi}^{\cdot\cdot}_{\tau_2}(\vec{m})\hat{j}\, U_{\tau_2}(G_{\vec{m},\hat{j}})\, \psi^{\cdot\cdot}_{\tau_2}(\vec{n}+\hat{j}) - \left(\frac{i}{g^4}\right)^2 \sum_{B_1} \mathrm{Tr}[\tilde{U}\tilde{U}\tilde{U}^\dagger\tilde{U}^\dagger]_{\tau_1} \sum_{B_2} \mathrm{Tr}[\tilde{U}\tilde{U}\tilde{U}^\dagger\tilde{U}^\dagger]_{\tau_2}\Bigg]. \tag{3.29}$$

The "superscript dot" contraction notation has been used for the quarks,

$$\psi^{\cdot}_{\tau_1}(\vec{n}_1)\,\bar{\psi}^{\cdot}_{\tau_2}(\vec{n}_2) \equiv S(\tau_1-\tau_2)\,\delta_{\vec{n}_1,\vec{n}_2} \tag{3.30}$$

Here, we pause for a moment to introduce some notation to describe Eqs. (3.27b) and (3.29) in terms of graphs. Represent a lattice site and a time collectively, $(\vec{n},\tau)$, by a point. Quarks at different points $(\vec{n}_1,\tau_1)$ and $(\vec{n}_2,\tau_2)$ can be contracted according to Eq. (3.30). This will be represented by a solid directed line connecting these points [see Fig. 3(a)].

Since gluons, such as $\tilde{U}_\tau(G_{\vec{n},\vec{n}+\hat{l}})$ and $\tilde{U}_\tau^\dagger(G_{\vec{n},\vec{n}+\hat{l}})$, are bilocal objects joining two sites (here $\vec{n}$ and $\vec{n}+\hat{l}$) separated by a positive unit vector $\hat{l}$, we will represent them by wavy lines connecting points $(\vec{n},\tau)$ and $(\vec{n}+\hat{l},\tau)$ in the positive and negative directions, respectively, as in Fig. 3(b). Then it is natural to describe the quark-gluon interaction vertex by a gluon line, with entering and departing quark lines attached at the end points $(\vec{n},\tau)$ and $(\vec{n}+\hat{l},\tau)$. This is illustrated in Fig. 3(c), where a dashed line is drawn to denote a contraction of gluons joining the *same* pair of lattice sites $\vec{n}$ and $\vec{n}+\hat{l}$ at *different* times $\tau_1, \tau_2, \ldots$. With each vertex is associated a factor $(1/2g^2)\hat{l}$.

Finally, the purely gluonic term of the interaction Eq. (3.3) can be described by four points $(\vec{n},\tau)$, $(\vec{n}+\hat{l},\tau)$, $(\vec{n}+\hat{l}+\hat{j},\tau)$, and $(\vec{n}+\hat{j},\tau)$ joined with each other to form an oriented box as in Fig. 3(d) (compare Fig. 1). With each box is associated a factor (i/g^4). All contractions of gluons from a given box vanish, since they join *different* pairs of lattice sites.

Now we are in a position to return to the discussion of the quark proper self-energy, and to

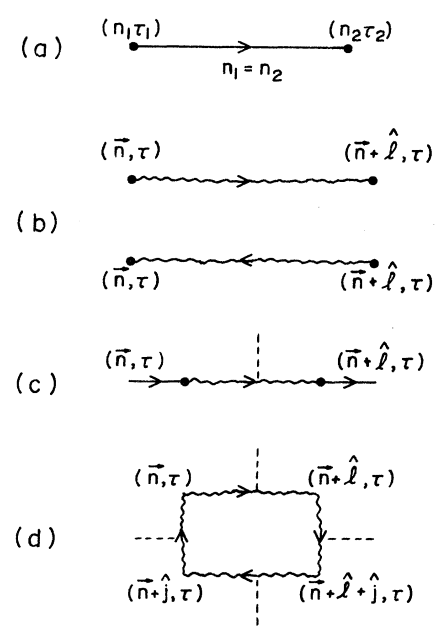

FIG. 3. (a) Quarks at points $(n_1\tau_1)$ and $(n_2\tau_2)$ are contracted, which is possible only if $n_1 = n_2$. (b) Gluons connect two neighboring points $(\vec{n},\tau)$ and $(\vec{n}+\hat{l},\tau)$ in positive and negative directions. (c) Quark-gluon interaction vertex. (d) Gluon self-interaction vertex.

represent Eqs. (3.27b) and (3.29) graphically. They are illustrated in Fig. 4. An extension to higher orders is straightforward. First, note that there is no contribution to $S'(\tau)$ from the third order of L_I. This is because, e.g., from a connected part containing three gluons one cannot construct a closed path in a lattice space. Recall that a quark leaving a given site should return to it at the end of its path. The fourth-order terms $S'_{(4)}$ are illustrated in Fig. 5, which represent reducible [5(a)] and irreducible [5(b)–5(f)] diagrams. The contributions of diagrams 5(b)–5(e) are of order of $(1/g^2)^4$, whereas 5(f) is of order of $(1/g^4)^2(1/g^2)^2$, and so it is suppressed by a factor $(1/g^4)$. By definition, the quark self-energy $\Sigma(\omega)$ in orders $(1/g^2)^2$ and $(1/g^2)^4$ are given respectively by the diagrams in Fig. 5(a) and Figs. 5(b)–5(e) with amputated quark legs. It is not difficult to write down the corresponding analytic expressions. One has

$$\Sigma^{(2)}(\omega) = \frac{1}{2\pi}\left(\frac{1}{2g^2}\right)^2 \sum_{\hat{l}} \int \hat{l} S(\omega')(-\hat{l}) D(\omega-\omega')\, d\omega' , \tag{3.31}$$

where $D(\omega)$ is the simplest semi-invariant, Eqs. (3.13)–(3.14), and arises from the contraction of

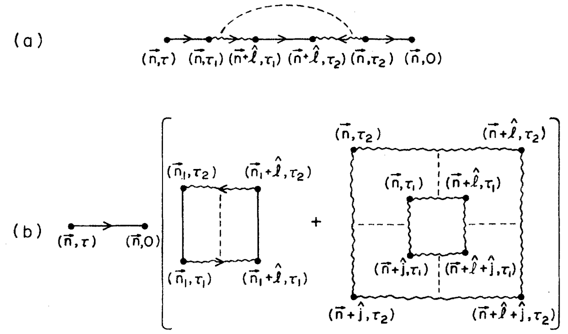

FIG. 4. (a) Quark propagator in the second order. (b) Disconnected part in the numerator of the quark propagator in the second order.

two gluons. The quark Green's function $S(\omega)$ is as given in Eq. (3.16). A straightforward evaluation of the above integral yields

$$\Sigma^{(2)}(\omega) = \left(\frac{1}{2g^2}\right)^2 (3i) \times \left[\frac{1-\gamma_0}{\omega-(\epsilon+m_0-i\delta)} - \frac{1+\gamma_0}{\omega+(\epsilon+m_0-i\delta)}\right]. \tag{3.32}$$

Turning to Figs. 5(b)–5(d), we find

$$\Sigma^{(4)}_{(b)}(\tau_1-\tau_4) = \left(\frac{1}{2g^2}\right)^4 \sum_{\hat{l},\hat{j}} \int \hat{l} S(\tau_1-\tau_2)\hat{j} S(\tau_2-\tau_3)\hat{j} D(\tau_2-\tau_3) S(\tau_3-\tau_4)\hat{l} D(\tau_1-\tau_4)\, d\tau_2\, d\tau_3 , \tag{3.33}$$

$$\Sigma^{(4)}_{(c)}(\tau_1-\tau_4) = -\frac{1}{3}\left(\frac{1}{2g^2}\right)^4 \sum_{\hat{l}} \int \hat{l} S(\tau_1-\tau_4)\hat{l}\, \mathrm{Tr}\{\hat{l} S(\tau_2-\tau_3)\hat{l} S(\tau_3-\tau_2) D(\tau_1-\tau_2) D(\tau_3-\tau_4)\, d\tau_2\, d\tau_3 , \tag{3.34}$$

$$\Sigma^{(4)}_{(d)}(\tau_1-\tau_4) = (1/2g^2)^4 \sum_{\hat{l}} \int \hat{l} S(\tau_1-\tau_2)\hat{l} S(\tau_2-\tau_3)\hat{l} S(\tau_3-\tau_1)\hat{l}\left[\tfrac{1}{3} D^{(c)abde}_{(2+2)eabd}(\tau_1\tau_2\tau_3\tau_4)\right] d\tau_2\, d\tau_3 , \tag{3.35}$$

where $D^{(c)}_{(2+2)}$ is a fourth-order semi-invariant with contracted indices. It can be expressed in terms of integrals over the group manifold using relations (3.19) and (3.12),

$$\begin{aligned} D^{(c)}_{(2+2)} = {} & \int d\bar{\mu}(g)\, T(\tilde{U}^a_e(\tau_1)\tilde{U}^{\dagger b}_a(\tau_2)\tilde{U}^d_b(\tau_3)\tilde{U}^{\dagger e}_d(\tau_4)) \\ & - \int d\bar{\mu}(g)\, T(\tilde{U}^a_e(\tau_1)\tilde{U}^{\dagger b}_a(\tau_2)) \int d\bar{\mu}(g)\, T(\tilde{U}^d_b(\tau_3)\tilde{U}^{\dagger e}_d(\tau_4)) \\ & - \int d\bar{\mu}(g)\, T(\tilde{U}^a_e(\tau_1)\tilde{U}^{\dagger e}_d(\tau_4)) \int d\bar{\mu}(g)\, T(\tilde{U}^{\dagger b}_a(\tau_2)\tilde{U}^d_b(\tau_3)) , \end{aligned}$$

which can be rewritten in a concise form as

$$D^{(c)}_{(2+2)} = \int d\bar{\mu}(g)\, T(\tilde{U}^a_e(\tau_1)\tilde{U}^{\dagger b}_a(\tau_2)\tilde{U}^d_b(\tau_3)\tilde{U}^{\dagger e}_d(\tau_4)) - 3D(\tau_1-\tau_2)D(\tau_3-\tau_4) - 3D(\tau_1-\tau_4)D(\tau_2-\tau_3) . \tag{3.36}$$

Let us examine the integrand appearing in Eq. (3.35) more closely. This analysis will be needed for Sec. IV. Using the definition (3.11), for the particular ordering $\tau_1 > \tau_2 > \tau_3 > \tau_4$ one gets

$$D(\tau_1\tau_2\tau_3\tau_4) \equiv \int d\bar{\mu}(g)\, T(\tilde{U}^a_e(\tau_1)\,\tilde{U}^{\dagger b}_a(\tau_2)\,\tilde{U}^d_a(\tau_3)\,\tilde{U}^{\dagger e}_d(\tau_4))$$

$$= \int d\bar{\mu}(g)\,(e^{+i\hat{H}^0_G\tau_1}\,U^a_e\,e^{-i\hat{H}^0_G(\tau_1-\tau_2)}\,U^{\dagger b}_a\,e^{-i\hat{H}^0_G(\tau_2-\tau_3)}\,U^d_b\,e^{-i\hat{H}^0_G(\tau_3-\tau_4)}\,U^{\dagger e}_d\,e^{-i\hat{H}^0_G})$$

$$= e^{-i\epsilon(\tau_1-\tau_2)}\,e^{-\epsilon(\tau_3-\tau_4)}$$

$$= 3D(\tau_1-\tau_2)\,D(\tau_3-\tau_4)$$

due to the unitarity condition $U^d_b U^{\dagger e}_d = \delta^e_b$ and the relation $H^0_G U^a_b = \epsilon U^a_b$. In a similar way we derive

$$D(\tau_1\tau_2\tau_3\tau_4) = \begin{cases} 3D(\tau_1-\tau_2)\,D(\tau_3-\tau_4) & \text{for}\ \ \underline{\tau}_{12} > \overline{\tau}_{34}\ \ \text{or}\ \ \underline{\tau}_{34} > \overline{\tau}_{12} \\ 3D(\tau_1-\tau_4)\,D(\tau_2-\tau_3) & \text{for}\ \ \underline{\tau}_{14} > \overline{\tau}_{23}\ \ \text{or}\ \ \underline{\tau}_{23} > \overline{\tau}_{14}\,, \end{cases}$$

where shorthand notations $\min(\tau_i\tau_j) = \underline{\tau}_{ij}$ and $\max(\tau_i\tau_j) = \overline{\tau}_{ij}$ are used. The remaining cases $\underline{\tau}_{13} > \overline{\tau}_{24}$ and $\underline{\tau}_{24} > \overline{\tau}^{13}$ are less trivial, e.g., for $\tau_1 > \tau_3 > \tau_2 > \tau_4$,

$$D(\tau_1\tau_2\tau_3\tau_4) = \int d\bar{\mu}(g) e^{i\hat{H}^0_G\tau_1}\,U^a_e\,e^{-i\hat{H}^0_G(\tau_1-\tau_3)}\,U^d_b\,e^{-i\hat{H}^0_G(\tau_3-\tau_2)}\,U^{\dagger b}_a\,e^{-i\hat{H}^0_G(\tau_2-\tau_4)}\,U^{\dagger e}_d\,e^{-i\hat{H}^0_G\tau_4}$$

$$= e^{-i\epsilon(\tau_1-\tau_3)-i\epsilon(\tau_2-\tau_4)} \int d\bar{\mu}(g) U^a_e U^d_b\,e^{-i\hat{H}^0_G(\tau_3-\tau_2)}\,U^{\dagger b}_a U^{\dagger e}_d\,.$$

To realize the exponential operator we perform the decomposition $\underline{3}\otimes\underline{3} = \underline{6}\oplus\underline{\overline{3}}$, i.e.,

$$U^{\dagger b}_a U^{\dagger e}_d = \tfrac{1}{2}(U^{\dagger b}_a U^{\dagger e}_d + U^{\dagger e}_a U^{\dagger b}_d) + \tfrac{1}{2}(U^{\dagger b}_a U^{\dagger e}_d - U^{\dagger e}_a U^{\dagger b}_d).$$

Here the terms on the right-hand side transform according to representations $\underline{6}$ and $\underline{\overline{3}}$, respectively. Substituting this back into the previous expression we derive

$$D(\tau_1\tau_2\tau_3\tau_4) = 3D(\tau_1-\tau_3)D(\tau_2-\tau_4)[\,2D_6(\tau_3-\tau_2) - D(\tau_3-\tau_2)] \quad \text{for}\ \ \tau_1 > \tau_3 > \tau_2 > \tau_4. \tag{3.37}$$

where $D_6(\tau) = e^{-i\epsilon_6|\tau|}$, ϵ_6 being the eigenvalue of H^0_G in representation $\underline{6}$. Expressions for the remaining

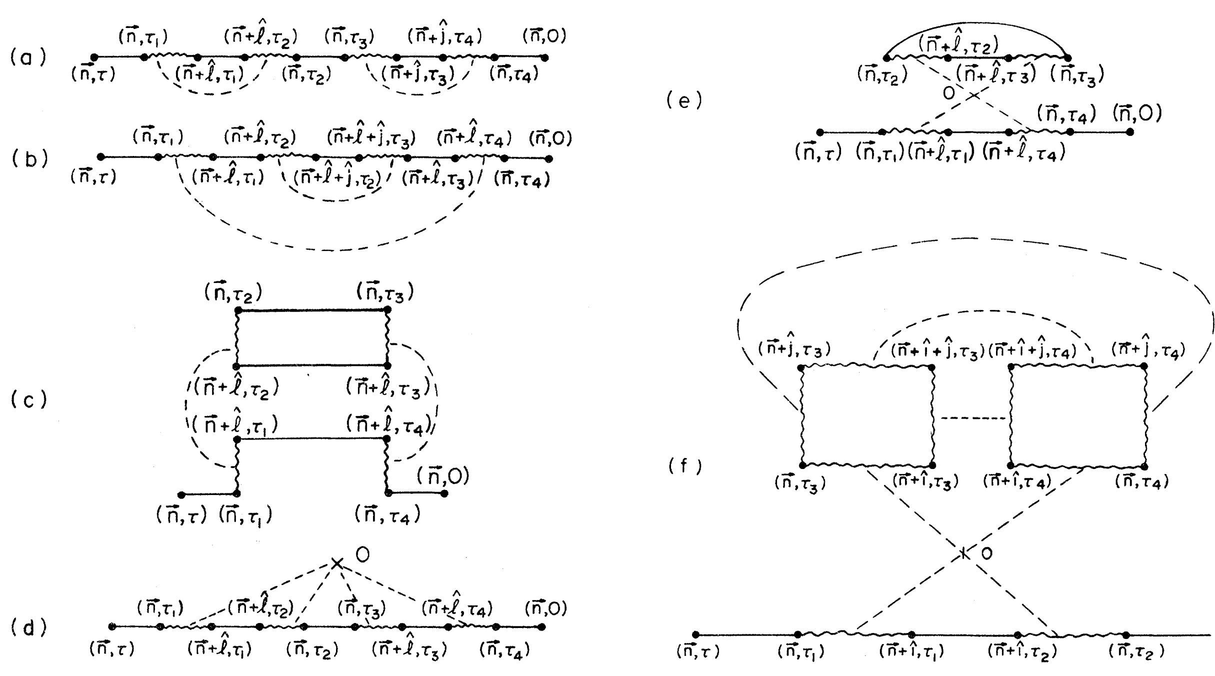

FIG. 5. (a) Reducible diagram; (b)–(f) irreducible diagrams. (b) and (c) Irreducible diagrams contributing to S' in fourth order of $\mathcal{L}_I$. (f) Connected irreducible diagram arising in second order. In (d), (e), and (f) the vertex O describes the four-gluon contraction.

cases $\underline{\tau}_{13}>\overline{\tau}_{24}$ and $\underline{\tau}_{24}>\overline{\tau}_{13}$ can be obtained from Eq. (3.37) by obvious permutations of τ_i's.

Returning to the fourth-order semi-invariant, we have

$$D_{2+2}^{(c)}(\tau_1\tau_2\tau_3\tau_4)=\begin{cases} -3D(\tau_1-\tau_4)D(\tau_2-\tau_3) \text{ for } \underline{\tau}_{12}>\overline{\tau}_{34} \text{ or } \underline{\tau}_{34}>\overline{\tau}_{12} \\ -3D(\tau_1-\tau_2)D(\tau_3-\tau_4) \text{ for } \underline{\tau}_{14}>\overline{\tau}_{23} \text{ or } \underline{\tau}_{23}>\overline{\tau}_{14}. \end{cases} \tag{3.38}$$

Unfortunately for the remaining orderings $\underline{\tau}_{13}>\overline{\tau}_{24}$ and $\underline{\tau}_{24}>\overline{\tau}_{13}$, the semi-invariant given by Eq. (3.36) cannot be cast into a concise form. Observe that in this case $D_6(\tau)$ appears owing to the non-Abelian nature of the gauge group. This can be visualized as a subsequent emission of two gluons forming together representations $\underline{3}$ and $\underline{6}$ in the intermediate state, but the analogy should not be taken too literally. The analysis of $D_{(2+2)}^{(c)}$ will be needed in Sec. IV. A similar analysis can be carried out for $\Sigma^{(4)}$ without any difficulty.

D. Quark confinement

Let us compare our formalism with Wilson's manifestly covariant one. An important difference is reflected in the quark propagator (3.24), which is a Kronecker δ function of the spatial arguments of the fields, but with nontrivial dependence on the time difference. In the Wilson formalism it is a simple Kronecker δ function of the spatial and time arguments of the quark fields. This can be interpreted as the absence of free propagation of quarks, as isolated entities, in space and time. On the spatial lattice, quarks are long-lived.

A crucial feature of both formalisms is that "quarks are infinitely heavy." That is, one can have a state with quark quantum numbers that satisfies the gauge conditions (2.47). This is a quark with an infinitely long gluon tail,

$$\overline{\psi}_{\vec{n}}U(\vec{n},\hat{l})U(\vec{n}+\hat{l},\hat{l})U(\vec{n}+2\hat{l},\hat{l})\cdots. \tag{3.39}$$

Its energy diverges linearly owing to (2.53). The confinement of quarks is intimately related to this fact. Indeed, quarks and antiquarks produced from color-singlet initial states would be observable separately if they traveled macroscopic distances from each other before forming hadrons. However, in this intermediate stage a gluon string of macroscopic length must be stretched between them to ensure the local-gauge-invariance condition (2.47). Hence one anticipates that these configurations of quark and antiquark states will be strongly suppressed.

These arguments can be recast into quantitive form by perturbative evaluation of the contribution to the current correlation function $\langle J(\vec{n},\tau)J(\vec{n},\tau')\rangle$ for the configuration under consideration. The latter can be thought of as quark (q) -antiquark ($\overline{q}$) creation at the point $(\vec{n},\tau)$, with subsequent annihilation at the point $(\vec{n}',\tau')$. Here we follow Wilson's heuristic arguments. For convenience the quark propagators are represented by circles at the points $(\vec{n}_l,\tau_l)$ on the boundary lines in Fig. 6. The boundaries are formed from the gluons in the first term in L_I [Eq. (3.3)]. Each of them should match to at least one GIQE (second term in L_I) to give rise to a nonvanishing result, etc. Recall that the integral over a single gluon $\int d\mu(g)U(g)$ vanishes. It is not difficult to find by means of simple counting that the contribution from the configuration with a perimeter (Pa) and area (Sa^2) is proportional to $(1/g^2)^{P+2S}$. Notice that in the Wilson model the corresponding suppression factor is $(1/ma)^P(1/g^2)^{2S}$, where m stands for a quark mass.

IV. SPONTANEOUS BREAKING OF CHIRAL SYMMETRY

In this section we shall investigate the nature of the chiral symmetry of the Lagrangian Eq. (2.13) in the limit $m_0\to 0$. This symmetry is generated by the global axial charges

$$Q^{5i}=\sum_n Q^{5i}(\vec{n}),\quad Q^{5i}(\vec{n})=\overline{\psi}(\vec{n})\gamma_0\gamma_5\frac{\lambda^i}{2}\psi(\vec{n}). \tag{4.1}$$

We will employ the methods of ordinary field theory. It will be shown that the symmetry is spontaneously broken, and that this is manifested by the existence of a pseudoscalar octet of massless bosons, in accordance with Goldstone's theorem. This is established by means of a Ward-Takahashi identity, using the relation $\{\gamma_5,\Sigma(\omega)\}_+\neq 0$

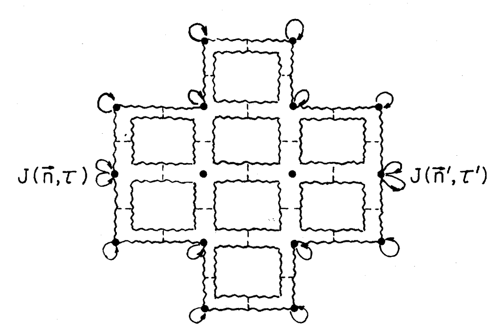

FIG. 6. Diagram contributing to the correlation function of currents. Boundaries are fixed by q and $\overline{q}$ paths.

which holds *perturbatively* in our model. The latter phenomenon will be called a spontaneous generation of the quark mass. Further, for residues of the massless poles, a homogeneous integral equation will be derived, which will turn out to be exactly solvable.

A. Spontaneous generation of the quark mass

Let us analyze the quark self-energy $\Sigma(\omega)$ in the limit $m_0 \to 0$. According to Eq. (3.32), we have

$$\Sigma_\pm(\omega) = \left(\frac{1 \pm \gamma_0}{2}\right) i\Sigma(\omega), \tag{4.2a}$$

$$\Sigma_\pm^{(2)}(\omega) = 6\left(\frac{1}{2g^2}\right)^2 \frac{1}{\pm\,\omega + \epsilon - i\delta}, \tag{4.2b}$$

$$\{\Sigma_\pm^{(2)}(\omega), \gamma_5\}_+ \neq 0. \tag{4.2c}$$

Substituting this approximation to Σ into the Dyson equation (3.26), one finds

$$\begin{aligned} S'^{(2)}(\omega) &= \tfrac{1}{2}(1-\gamma_0)S_-'^{(2)}(\omega) + \tfrac{1}{2}(1+\gamma_0)S_+'^{(2)}(\omega), \\ S_\pm'^{(2)}(\omega) &= i[\pm\omega - \Sigma_\pm^{(2)}(\omega)]^{-1} \end{aligned} \tag{4.3}$$

which has poles at

$$\omega \equiv m_q^\pm = -\tfrac{1}{2}\epsilon \pm [\tfrac{1}{4}\epsilon^2 + 6(1/2g^2)^2]^{1/2}, \quad \epsilon = \tfrac{4}{3}$$

The quantity m_q can be called the quark mass for convenience in analogy with the ordinary definition of particle mass. However, it should be emphasized that the existence of the pole at some fixed value $\omega = m_q$ is not relevant for our later arguments, provided $m_q \neq 0$. Only the relation (4.2c) will be needed.

This result is rather unusual. It follows from the specific prescription of introducing the symmetry-breaking term $m_0 \neq 0$, and taking the limit $m_0 \to 0$. Our approach is perturbative, and it is gratifying that this prescription can be motivated in the context of perturbation theory. The reason for this will become more clear if we note that taking $m_0 = 0$ from the outset, one finds $\{\Sigma(\omega), \gamma_5\}_+ = 0$.

Let us start at the canonical, or zeroth-order, level. It was emphasized in Sec. II that the physical states, including the vacuum state ϕ^0_{phys}, should obey the gauge conditions (2.47). Since H_0, Eqs. (3.1) and (3.2), is locally gauge invariant the bare states also satisfy this condition. In particular, the bare vacuum ϕ^0 is annihilated by the local gauge-group generators,

$$Q_G^\alpha(\vec{n})\Phi^0 = 0,$$

$$Q_\psi^\alpha(\vec{n})\Phi^0 = 0.$$

These equations are trivially satisfied for $\Phi^0 = \text{const}$ with the choice of fermion variables (2.27) which diagonalizes the fermion charges $Q_\psi^\alpha(\vec{n})$.

In addition, an evaluation of $S(\tau)$ by Eq. (3.15), assuming $m_0 = 0$ and still making use of the basis (2.27), gives

$$S_{m_0=0}(\tau) = \tfrac{1}{2}(1+\gamma_0)\theta(\tau) + \tfrac{1}{2}(1-\gamma_0)\theta(-\tau), \tag{4.4}$$

which is the Fourier transform of $S_{m_0\neq 0}(\omega)$ [see Eq. (3.16)] in the limit $m_0 = 0$. This indicates that there is no inconsistency in our prescription of taking $m_0 = 0$ at the end of calculations. However, it does not demonstrate that this prescription is unique. A formal, canonical proof of the unitary equivalence of a general basis for ψ to the basis (2.27) is supplied in Appendix B.

Encouraged by the fact that the lowest nontrivial order in perturbation theory gives results that are in accord with the formal arguments, we proceed to examine the next higher order. The necessary expressions for Σ were given in Sec. III, Eqs. (3.33)–(3.35). We will specifically focus on the infrared ($\omega \to 0$ at $m_0 = 0$) behavior of $\Sigma(\omega)$.

First we shall evaluate $\Sigma_b^{(4)}(\omega)$. This turns out to be more convenient to carry out in the τ representation. Using the expressions (3.13′) and (4.6) for $D(\tau)$ and $S_{m_0=0}(\tau)$, respectively, one finds for $\tau_4 = 0$

$$\Sigma_b^{(4)}(\omega) = 36\left(\frac{1}{2g^2}\right)^4 \int e^{+i\omega\tau_1} d\tau_1 d\tau_2 d\tau_3 e^{-i\epsilon(|\tau_2-\tau_3|+|\tau_1|)} \times (\theta_{21}\theta_{23}\theta_{43}\gamma_+ + \theta_{12}\theta_{32}\theta_{34}\gamma_-),$$

where $\theta_{ij} \equiv \theta(\tau_i - \tau_j)$, $\gamma_\pm = \tfrac{1}{2}(1 \pm \gamma_0)$, and ϵ stands for $(\epsilon - i\delta)$. A subsequent integration in variables τ_1, τ_2, τ_3 gives

$$i\Sigma_b^{(4)}(\omega) = \frac{1+\gamma_0}{2}\Sigma_{b+}^{(4)}(\omega) + \frac{1-\gamma_0}{2}\Sigma_{b-}^{(4)}(\omega),$$

with $\Sigma_{b+}^{(4)}(\omega) = \Sigma_{b-}^{(4)}(-\omega)$ and

$$\Sigma_{b+}^{(4)}(\omega) = -36\left(\frac{1}{2g^2}\right)^4 \left[\frac{1}{(\omega+\epsilon)\epsilon}\left(\frac{1}{\epsilon} + \frac{1}{\omega+\epsilon}\right) + \frac{1}{(\omega-\epsilon)\epsilon}\left(\frac{1}{\epsilon} + \frac{1}{2\epsilon-\omega}\right)\right].$$

We see that $\Sigma_b^{(4)}$ is regular at $\omega = 0$ with a radius of convergence equal to ϵ. Thus it does not suffer from infrared problems.

In the case of $\Sigma_c^{(4)}$ and $\Sigma_d^{(4)}$ the evaluation of the corresponding integrals is straightforward but somewhat tedious, and we merely state the results. Like the previous case, $\Sigma_{d,e}^{(4)}(\omega)$ are analytic function with poles at $\omega = \pm\epsilon$, 2ϵ, etc. They are also regular at $\omega = 0$ essentially because they receive no contribution from the pure three-quark state $(qq\bar{q})$.

There is one other graph in fourth order which exhibits qualitative features of a general nature which are different from those above. Namely, it has a intermediate state with the quantum numbers of a quark, and which contains no gluons. Accordingly, it will be infrared divergent in the limit of massless quarks.

This is easy to verify by an explicit calculation. For a reason which will be clear later, the calculation will be carried out for a general gauge group SU(N).

Let us define the lowest-order polarization operator as

$$\pi^{(2)}(\tau_2-\tau_3)=\left(-\frac{1}{N}\right)\left(\frac{1}{2g^2}\right)^2 \times \mathrm{Tr}[\vec{\gamma}S(\tau_2-\tau_3)(-\vec{\gamma})S(\tau_2-\tau_3)], \tag{4.5}$$

whose Fourier transform is given by

$$\pi^{(2)}(\omega)=\left(-\frac{1}{N}\right)\left(\frac{1}{2g^2}\right)^2\frac{4m_0 i}{\omega^2-(2m_0-i\delta)^2}\,. \tag{4.5'}$$

Returning to Eq. (3.34) one derives

$$\Sigma_{c\pm}^{(4)}(\omega)=\frac{3}{\pi}\left(\frac{1}{2g^2}\right)^2\int d\omega' iS_{\mp}(\omega-\omega')D(\omega')\pi^{(2)}D(\omega'). \tag{4.6}$$

Hence for $m_0\to 0$ one discovers the infrared catastrophe

$$\Sigma_{c\pm}^{(4)}(0)=\left(\frac{1}{N}\right)\left(\frac{1}{2g^2}\right)^4\left(\frac{2}{\epsilon}\right)^2\frac{2}{m_0}\to\infty. \tag{4.6'}$$

In higher orders these divergences will become more and more severe. Consider the diagram in Fig. 7 which is obtained from Fig. 4(a) by inserting n quark loops in the bare gluon propagator. Its contribution is given by

$$\Sigma_{\pm}^{(2n+2)}(\omega)=3\left(\frac{1}{2g^2}\right)^2 \times\int d\omega' iS_{\mp}(\omega-\omega')D(\omega')[D(\omega')\pi^{(2)}(\omega')]^n, \tag{4.7}$$

with $\pi^{(2)}(\omega)$ defined in Eq. (4.5′).

The leading infrared-divergent term of Eq. (4.7) is given by

$$\Sigma_{\pm}^{(2n+2)}(0)=C\left(\frac{1}{2g^2}\right)^{2n+2}\left(\frac{2}{\epsilon N}\right)^{n+1}\left(\frac{1}{m_0}\right)^n N, \tag{4.8}$$

where the constant C is independent of N. The

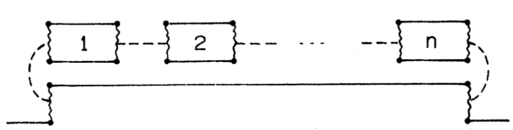

FIG. 7. Contribution from n quark loops to the quark self-energy $\Sigma(\omega)$.

systematic appearance of different factors in the above expression is not accidental. They arise as follows:

(a) The diagram is of order $n+1$ in the expansion parameter $(1/2g^2)^2$.

(b) Each of $n+1$ second-order gluon semi-invariants supplies a factor $2/\epsilon N$ according to Eqs. (3.13′) and (3.14). Notice that the first Casimir eigenvalue in the fundamental representation is $\epsilon\equiv\epsilon(N)=(N^2-1)/2N\sim\frac{1}{2}N$ [cf. Eq. (2.52) for $N=3$].

(c) Every intermediate state with an odd number of fermions and no gluons gives a factor $1/m_0$. The most divergent behavior of a diagram is determined by the maximum number of such states, which is n in this case. Hence the factor $(1/m_0)^n$ results.

(d) A summation over the color indices around any closed index loop gives rise to a factor N. There is only one such loop in Fig. 7.

The problem of infrared divergences must be dealt with to ascertain the validity of Eq. (4.2c) in the context of perturbation theory. Fortunately, the way to proceed is suggested by the pattern of divergences discerned above, and by the fact that, according to Eq. (4.2b),

$$\Sigma_{\pm}^{(2)}(0)\equiv m^{(2)}=\frac{6}{\epsilon}\left(\frac{1}{2g^2}\right)^2\neq 0.$$

We shall rearrange the perturbation series as in ordinary field theory by defining a "renormalized mass" by $m=\Sigma(0)$ in the zero bare-mass limit $m_0=0$. This corresponds to redefining

$$\mathcal{L}_0\to\mathcal{L}_0'=\mathcal{L}_0-m\sum_n\bar\psi(\vec{n})\psi(\vec{n}),$$
$$\mathcal{L}_I\to\mathcal{L}_I'=\mathcal{L}_I+m\sum_n\bar\psi(\vec{n})\psi(\vec{n}) \tag{4.9}$$

in Eqs. (3.1)–(3.3). As usual, this introduces "mass insertion" counterterms for each self-energy subgraph.

Use of this procedure will convert infrared-divergent factors (i.e., ${m_0}^{-1}$) into powers of $(1/2g^2)^{-2}$. Thus a given diagram will be effectively of lower order in $(1/2g^2)^2$ than its superficial degree. However, in Eq. (4.8) there is a second parameter $1/N$ in terms of which an expansion can be developed. The original perturbation series in the single parameter $(1/2g^2)^2$ will be rearranged into a double expansion in $(1/2g^2)^2$ *and* $1/N$. In particular this applies to the quark self-energy $\Sigma(m,\omega)$; the "renormalized mass" m is calculable from the consistency equation

$$m=\Sigma(m,\omega)|_{\omega=0} \tag{4.10}$$

as a double expansion, with $m^{(2)}$ being the leading term. These points will be illustrated by examining a few diagrams.

First let us recalculate Eq. (4.7) according to the above scheme. In $S_\pm(\omega)$ we replace m_0 by m. To leading order of $(1/2g^2)^2$ this modifies Eq. (4.8) to

$$\Sigma_\pm^{(n,l)}(0) = C\left(\frac{1}{2g^2}\right)^{2n+2}\left(\frac{2}{\epsilon N}\right)^{n+1}\left(\frac{1}{m}\right)^n N \approx C\left(\frac{1}{2g^2}\right)^2 \frac{2}{\epsilon}\left(\frac{1}{3N}\right)^n, \qquad (4.11)$$

where the superscript (n,l) on $\Sigma_\pm^{(n,l)}$ designates n quark loops. In passing from the first to second line we approximated m by $m^{(2)}$. We see that in our scheme these would-be infrared-divergent diagrams have all been promoted to the same effective order in $(1/2g^2)^2$. In addition, each quark loop introduces a suppression factor $1/3N$; the origin of the factor $\frac{1}{3}$ will be elaborated below. It should be evident from the consistency equation (4.10) that the one-quark loop diagram results in a correction of order $1/N$ to $m^{(2)}$, i.e.,

$$m^{(2)} \to m^{(2)}\left(1+\frac{2}{3}\frac{1}{3N}\right).$$

Examples of diagrams which are promoted to the order $(1/2g^2)^2$ but which are higher order in $1/N$ are given by Fig. 7 with $n \geq 2$ and Fig. 8(a). The first one was evaluated above. It is easy

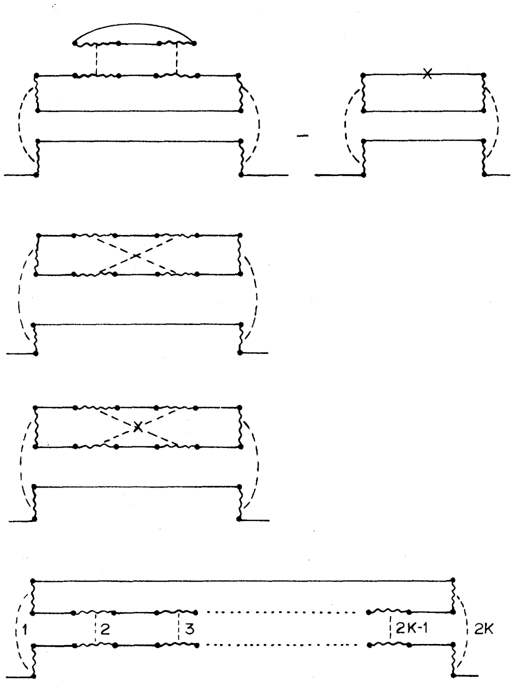

FIG. 8. (a)–(c) Contributions from the fourth-order polarization operator $\pi^{(4)}$ to the quark self-energy $\Sigma(\omega)$. In (a) the self-energy insertion into the quark loop is accompanied by its counterterm. (d) A quark self-energy diagram with $2k$ rungs.

to ascertain that Fig. 8(a) gives a contribution of $(1/2g^2)^2(1/N)^2$. Here the subtraction term is that appropriate for the self-energy insertion displayed in the quark loop, i.e., $\Sigma^{(1l)}(0)$ (just as $m^{(2)}$ subtractions accompany the $\Sigma^{(2)}$ which are not shown).

Notice that the diagrams in Figs. 8(b) and 8(c) are of higher order in $(1/2g^2)^2$ although they have the same superficial degree as the one in Fig. 8(a).

Next we turn to a class of infrared-divergent diagrams, Fig. 8(d), which behave differently from those discussed above. The diagrams in this figure are of order $(1/N)(1/2g^2)^2$ independent of k. Indeed, each rung in the ladder introduces a new loop. The index sum around the loop supplies a factor N which is compensated by $1/N$ arising from the gluon contraction in the new rung. Furthermore, as before, each additional infrared-divergent factor $1/m^{(2)}$ promotes the diagram by $(1/2g^2)^2$. Explicitly, the leading contribution from Fig. 8(d) is

$$\Sigma_\pm^{(2k,r)}(0) \equiv m^{(2)}\left(\frac{1}{N}\right)\left(\frac{1}{i\pi}\right)\left(\frac{1}{3}\right)^{2k} L_k I_k, \qquad (4.12)$$

where

$$L_k = (-1)^k \int_{-\infty}^{\infty} d\eta \frac{1}{\eta - 1 + i}\left(\frac{1}{\eta + 2 - i\delta}\right)^k \times \left(\frac{1}{\eta - 2 + i\delta}\right)^{k-1}$$

and

$$I_k = \sum_{m+n+l=k} \frac{(2k)!}{(m!n!l!)^2}.$$

In Eq. (4.12) the superscript $(2k,r)$ on $\Sigma_+^{(2k,r)}$ stands for $2k$ rungs. The combinatoric factor I_k counts the number of closed paths containing $2k$ links in a three-dimensional lattice. It arises from the summation over the directions of the $2k$ unit vectors introduced by $\mathcal{L}_I$. The factor $(\frac{1}{3})^{2k}$ in Eq. (4.12) results, as in Eq. (4.11), from factors 6 and 2 appearing in $m^{(2)}$ and $D(\omega)$, respectively.

Evidently a consistent way to proceed is to sum the entire class of diagrams in Fig. 8(d). To this end we have evaluated the first four terms $(k=1,\ldots,4)$ numerically. The remaining contribution has been estimated by asymptotic expansions (for large k)

$$L_k = 4i\sqrt{\pi}\frac{1}{2^{2k}\sqrt{k}},$$

$$I_k = \frac{1}{8}\left(\frac{3}{\pi}\right)^{3/2}\frac{6^{2k}}{k^{3/2}},$$

which are derived using the Stirling formula and the method of steepest descent. The result is

$$\Sigma_{\pm}^{(r)}(0) \equiv \sum_{k=1}^{\infty} \Sigma_{\pm}^{(2k,r)}(0) = (0.72)\left(\frac{1}{N}\right) m^{(2)} . \quad (4.12')$$

It is interesting to note that $\Sigma_{\pm}^{(2k,r)}(0)$ decreases with k even for $k \leq 4$ and that the net contribution of the remaining terms is only ~10% of the result (4.12′). Thus, despite the fact that the class of ladder diagrams have a different qualitative behavior, their sum is finite and is of order $1/N$.

In summary, our analysis indicates that the expansion in two parameters $(1/2g^2)^2$ and $1/N$ is a viable computational scheme. In particular it provides a well-defined procedure for a *perturbative* evaluation of the dynamically generated quark mass (4.10).

B. Fermion degeneracy

At this point, it is appropriate to return to the problem of fermion degeneracy first noted in the Introduction. The reader may wish to amuse himself by calculating the energy-momentum dispersion relation for free massive fermions using lattice perturbation theory. [That is, treat $i\partial_0 - \beta m$ as H_0 and the rest of the kinetic energy as a perturbation, summing all orders.] The result,

$$\omega^2(K) = \sum_{i=1}^{3} (\sin K_i a)^2 + m^2 ,$$

may also be obtained directly by solving a difference equation. Unfortunately, there is a degeneracy under $K \to K \pm \pi/a$.

This particular degeneracy is not special to Dirac particles on a lattice, but is due to the first-order character of the difference equation they satisfy. The result is unphysical, and is an artifact of the lattice.

In a spatial lattice realization the degeneracy is due to the invariance of the Lagrangian under $\psi(\vec{n}) \to (-1)^n \gamma_0 \psi(\vec{n})$, implemented by

$$\exp\left(\frac{i\pi}{2} \sum_{\vec{n}} \bar{\psi}(\vec{n})\psi(\vec{n})\right) \exp\left(i\pi \sum_{\vec{n}} n\, \psi^\dagger(\vec{n})\psi^\dagger(\vec{n})\right).$$

Several modifications of the Lagrangian have been proposed to remove this degeneracy.[5,7]

From the γ_5 invariance point of view, however, these modifications are unacceptable. Of course, this does not necessarily mean that formal γ_5 invariance is incompatible with single-valuedness of the energy. One possibility is to construct other candidates for $\mathcal{L}$ that lift the degeneracy, but which respect γ_5 symmetry. An example of such a term is

$$\alpha \int d^3x\, \bar{\psi} i \vec{\partial\!\!\!/} \vec{\partial}^{\,2} \psi + \text{H.c.}$$

$$\to i\alpha \sum_{\vec{n},\hat{k},\hat{l}} \bar{\psi}(\vec{n}) [\psi(\vec{n}+\hat{k}+2\hat{l}) - 2\psi(\vec{n}+\hat{k}+\hat{l}) + \psi(\vec{n}+\hat{k})] .$$

This term gives rise to the dispersion

$$\epsilon^2(k) = m^2 + \sum_{m_+} \sin^2 k_m \left(1 - 8\alpha \sum_{l_+} \cos k_l \sin^2 \frac{k_l}{2}\right)^2 .$$

This term is not degenerate under $k \to k \pm \pi$ except at $k=0$. The residual degeneracy at $k=0$ has to be considered spurious since it occurs at discrete points in momentum space.

The above discussion refers to the free field theory on the lattice. The hope may arise, therefore, that the fermion degeneracy problem will disappear in the exact interacting theory. However, the invariance of the free theory persists unless terms such as the one above, rendered gauge invariant by insertion of U's, are introduced. It is straightforward to give a formal proof in the present framework that the boson spectrum is not affected by this invariance and only the baryon sectors of the theory would be doubled were there no symmetry-breaking term.

We conclude, therefore, that the degeneracy problem is legitimate, but not fatal for chiral symmetry. Wilson's alternative, that of introducing new terms in $\mathcal{L}$ when absolutely necessary, allows for the freedom of inventing appropriate chirally symmetric terms. Indeed, if the lattice can be viewed as having any kind of exact correspondence with a renormalizable field theory, it should sensibly respect, or at least break in a controlled fashion, those invariances of the initial theory which are most physically relevant for the problem under investigation. We shall proceed with the point of view that the degeneracy has been lifted in principle by a term such as the one displayed above, with a very small coefficient. In fact, it turns out that the degeneracy never hampers our calculations, nor our conclusions.

C. Ward-Takahashi identities

We now proceed to the derivation of the lattice version of the Ward-Takahashi identities satisfied by the axial charges (4.1). Consider a vacuum expectation value of three quark fields in the presence of fermionic sources [compare Eq. (2.12)]:

$$\langle \psi_\gamma(\vec{m},\tau_1) M_{\beta,\vec{n}}^{\alpha,i}(\tau,0)\rangle_\eta \equiv W^{-1}\{0\} \int\int d\psi\, d\bar{\psi} D\mu(G) \psi_\gamma(\vec{m},\tau_1) M_{\vec{n},\beta}^{\alpha}(\tau,0) \exp\left\{ i \int_{-\infty}^{\infty} d\tau \left[\mathcal{L}_\psi + \mathcal{L}_G + \sum_{\vec{n}} [\bar{\psi}(\vec{n})\eta(\vec{n}) + \bar{\eta}(\vec{n})\psi(\vec{n})] \right] \right\},$$

(4.13)

with

$$M^{\alpha;i}_{\beta,n}(\tau,0)=\bar{\psi}^{\alpha}(\vec{n},\tau)\lambda^i\psi_\beta(\vec{n},0). \tag{4.14}$$

Only Dirac indices α,β,γ are displayed in the above equations. By definition (2.12), the integral over the anticommuting objects ψ is invariant with respect to translations of integration variables $\psi,\bar{\psi}$. Consequently replacing $\psi(\vec{m},\tau)$ by $\psi(\vec{m},\tau)+\lambda^j\gamma_5\alpha(\vec{m},\tau)$ in Eq. (4.13) and performing a subsequent expansion with respect to $\alpha(\vec{m},\tau_2)$, we find in the first order of $\alpha(\vec{m},\tau_2)$

$$\Big\langle\int_{-\infty}^{\infty}d\tau_2\Big\{i\Big[i\bar{\psi}(\vec{m},\tau_2)\partial\!\!\!/_{\tau_2}\lambda^j\gamma_5\alpha(\vec{m},\tau_2)-\frac{i}{2g^2}\sum_{\hat{l}}\bar{\psi}(\vec{m}-\hat{l},\tau_2)\hat{l}\!\!\!/\,U_{\tau_2}(\vec{m}-\hat{l},\vec{m})\lambda^j\gamma_5\alpha(\vec{m},\tau_2)$$
$$-m_0\bar{\psi}(\vec{m},\tau_2)\lambda^j\gamma_5\alpha(\vec{m},\tau_2)\Big]\psi_\gamma(\vec{m},\tau_1)M^{\alpha;i}_{\beta,n}(\tau,0)$$
$$+[\lambda^j\gamma_5\alpha(\vec{m},\tau_2)]_\gamma\delta(\tau_2-\tau_1)M^{\alpha;i}_{\beta,n}(\tau,0)+\psi_\gamma(\vec{m},\tau_1)\bar{\psi}^{\alpha}(\vec{n},\tau)\lambda^i\lambda^j[\gamma_5\alpha(\vec{m},\tau_2)]_\beta\delta(\tau_1)\Big\}\Big\rangle_{\eta=0}=0.$$

Since $\alpha(\vec{m},\tau_2)$ is an arbitrary anticommuting object, one infers

$$\Big\langle\Big\{\Big[-\partial\!\!\!/_{\tau_2}\bar{\psi}(\vec{m},\tau_2)\gamma_0\gamma_5\lambda^j\psi(\vec{m},\tau_1)-\frac{1}{2g^2}\sum_{\hat{l}}\bar{\psi}(\vec{m}-\hat{l},\tau_2)\hat{l}\!\!\!/\lambda^j\gamma_5U\tau_2(\vec{m}-\hat{l},\vec{m})\psi(\vec{m},\tau)+im_0\bar{\psi}(\vec{m},\tau_2)\lambda^j\gamma_5\psi(\vec{m},\tau_1)$$
$$+\delta(\tau_2-\tau_1)\operatorname{Tr}(\lambda^j\gamma_5)\Big]M^{\alpha;i}_{\beta,n}(\tau,0)-\delta(\tau_2)\bar{\psi}^{\alpha}(\vec{n},\tau)\lambda^i\lambda^j[\gamma_5\psi(\vec{m},\tau_1)]_\beta\Big\}\Big\rangle_{\eta=0}=0.$$

In a similar way the invariance of $\langle\bar{\psi}(\vec{m},\tau_1)M^{\alpha;i}_{\beta,n}(\tau,0)\rangle$ with respect to translation $\bar{\psi}(\vec{m},\tau)\to\bar{\psi}(\vec{m},\tau)+\bar{\alpha}(\vec{m},\tau)\lambda^j\gamma_5$ gives

$$\Big\langle\Big\{\Big[-\bar{\psi}(\vec{m},\tau_1)\gamma_0\gamma_5\lambda^j\partial_{\tau_2}\psi(\vec{m},\tau_2)+\Big(\frac{1}{2g^2}\Big)\sum_{\hat{l}}\bar{\psi}(\vec{m},\tau_1)\hat{l}\!\!\!/\lambda^j\gamma_5U_{\tau_2}(\vec{m},\vec{m}+\hat{l})\psi(\vec{m}+\hat{l},\tau_2)$$
$$+im_0\bar{\psi}(\vec{m},\tau_1)\lambda^j\gamma_5\psi(\vec{m},\tau_2)+\delta(\tau_2-\tau_1)\operatorname{Tr}(\lambda^j\gamma_5)\Big]M^{\alpha;i}_{\beta,n}(\tau,0)-\delta(\tau_2-\tau_1)[\bar{\psi}(\vec{m},\tau_1)\gamma_5]^{\alpha}\lambda^j\lambda^i\psi_\beta(\vec{n},0)\Big\}\Big\rangle_{\eta=0}=0.$$

Combining the last two equations at $\tau_1=\tau_2$, we arrive at the desired identity ($\operatorname{Tr}\lambda^j\gamma_5=0$)

$$\Big\langle\Big\{\partial_{\tau_1}Q^{5j}_m(\tau_1)+\frac{1}{2g^2}\sum_{\hat{l}}[Q^{5j}_{\vec{m}-\hat{l},\hat{l}}(\tau_1)-Q^{5j}_{\vec{m},\hat{l}}(\tau_1)-m_0i\bar{\psi}(\vec{m},\tau_1)\lambda^j\gamma_5]\psi(\vec{m},\tau_1)\Big\}M^{\alpha;i}_{\beta,n}(\tau,0)\Big\rangle_{\eta=0}$$
$$=-\delta^{ij}\delta_{mn}[\gamma_5S'(-\tau)\delta(\tau_1)+S'(-\tau)\gamma_5\delta(\tau-\tau_1)]^{\alpha}_{\beta}$$

where the bilocal object

$$Q^{5j}_{m,\hat{l}}(\tau)=\psi(\vec{m},\tau)\hat{l}\!\!\!/\frac{\lambda^j}{2}\gamma_5U_\tau(\vec{m},m+\hat{l})\psi(\vec{m},\hat{l},\tau)$$

represents a lattice version of the spatial component of the axial-vector current. Its Fourier transform is

$$\int e^{-i\eta\tau+i(\omega+\eta)\tau_1}d\tau_1\,d\tau\sum_{\vec{m}}e^{i\vec{k}\cdot\vec{m}}\Big\langle M^{\alpha;i}_{\beta,n}(\tau,\tau_1)\Big[i\omega Q^{5j}_m(0)+\sum_{\hat{l}}\Big(\frac{e^{i\vec{k}\cdot\hat{l}}-1}{2g^2}\Big)Q^{5j}_{\vec{m},\hat{l}}(0)-m_0i\bar{\psi}(\vec{m},0)\lambda^j\gamma_5\psi(\vec{m},0)\Big]\Big\rangle_{\eta=0}$$
$$=-\delta^{ij}e^{i\vec{k}\cdot\vec{n}}[\gamma_5S'(\eta)+S'(\omega+\eta)\gamma_5]^{\alpha}_{\beta}.$$

These are the Ward-Takahashi identities for the *improper* vertex function, which is diagrammatically represented in Fig. 9. For later purposes it is convenient to reexpress these in terms of *proper* vertex functions Γ by removing the quark legs (henceforth the limit $m_0=0$ is assumed):

$$i\omega\Gamma^{5ij}(\omega+\eta,\eta|\vec{k})-\sum_{\hat{l}}(1-e^{i\vec{k}\cdot\hat{l}})\Gamma^{5ij}_{\hat{l}}(\omega+\eta,\omega|\vec{k})=-\delta^{ij}[S'^{-1}(\omega+\eta)\gamma_5+\gamma_5S'^{-1}(\eta)], \tag{4.15}$$

$$[\Gamma^{5ij}(\omega+\eta,\eta|\vec{k})]^{\alpha}_{\beta}=\int e^{-i\eta\tau}e^{i(\omega+\eta)\tau_1}d\tau\,d\tau_1\sum_{m}e^{i\vec{k}\cdot(\vec{m}-\vec{n})}[S'^{-1}(\omega+\eta)]^{\alpha}_{\beta}\langle M^{\alpha;i}_{\delta,m}(\tau,\tau_1)Q^{5j}_m(0)\rangle[S'^{-1}(\eta)]^{\alpha}_{\beta}, \tag{4.16a}$$

$$[\Gamma^{5ij}_{\hat{l}}(\omega+\eta,\eta|\vec{k})]^{\alpha}_{\beta}=\frac{1}{2g^2}\int e^{-i\eta\tau}e^{i(\omega+\eta)\tau_1}d\tau\,d\tau_1\sum_{m}e^{i\vec{k}\cdot(\vec{m}-\vec{n})}[S'^{-1}(\omega+\eta)]^{\alpha}_{\beta}\langle M^{\gamma;i}_{\delta,n}(\tau,\tau_1)Q^{5j}_{m,\hat{l}}(0)\rangle[S'^{-1}(\eta)]^{\delta}_{\beta}; \tag{4.16b}$$

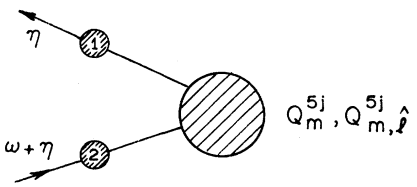

FIG. 9. Reducible quark-quark-axial-vector-current improper vertex function. The blobs 1 and 2 are full quark propagators.

M_n^i is given Eq. (4.14).

It is instructive to verify Eq. (4.15) in lowest-order perturbation theory. The second-order contribution to the vertex function Γ^{5ij} is [see Fig. 10a)]

$$\Gamma^{5ij}_{(2)}(\omega+\eta,\eta\,|\vec{k}) = +\left(\frac{1}{2g^2}\right)^2\frac{3\delta^{ij}}{2\pi} \times \int \sum_{\hat{l}} e^{i\vec{k}\cdot\hat{l}}\hat{l}\,S(\omega+\eta-\omega')\gamma_0\gamma_5 \times S(\eta-\omega')(-\hat{l})D(\omega')d\omega'. \tag{4.17}$$

A comparison with Eq. (3.31), making use of the identity

$$-i\omega S(\omega+\eta)\gamma_0\gamma_5 S(\eta) = S(\omega+\eta)\gamma_5+\gamma_5 S(\eta),$$

yields

$$i\omega\Gamma^{5ij}_{\hat{l}(2)}(\omega+\eta,\eta\,|\,k) = +\left(\frac{1}{2g^2}\right)^2\frac{3\delta^{ij}}{2\pi} \times \int d\omega' D(\omega')[\hat{l}\lambda_5 S(\eta-\omega')(\hat{l})S(\omega+\eta-\omega')\hat{l}\gamma_5].$$

We see that due to Eq. (4.2c), the vertex Γ^{5ij} $(\omega+\eta,\eta)$ acquires a pole at $\omega=0$ already in the lowest nontrivial order. Note also that Eq. (4.15) for $\vec{k}=0$ is identically satisfied.

To consider the case $\vec{k}\neq 0$, we evaluate $\Gamma^{5ij}_{\hat{l}}$ in the lowest order. It is given by a sum of two terms [see Fig. 10b)]

$$\Gamma^{5ij}_{\hat{l}(2)}(\omega+\eta,\eta\,|k) = +\left(\frac{1}{2g^2}\right)\frac{3\delta^{ij}}{2\pi} \times \int d\omega' D(\omega')[\hat{l}\gamma_5 S(\eta-\omega')(\hat{l}) + e^{i\vec{k}\cdot\hat{l}}(-\hat{l})S(\omega+\eta-\omega')\hat{l}\gamma_5]. \tag{4.18}$$

Writing it in the form

$$\Gamma^{5ij}_{\hat{l}(2)}(\omega+\eta,\eta\,|k = -\frac{\delta^{ij}}{2}[\gamma_5\Sigma^{(2)}(\eta)+e^{-i\vec{k}\cdot\hat{l}}\Sigma^{(2)}(\omega+\eta)\gamma_5] \tag{4.19}$$

we immediately infer that Eq. (4.15) for $\vec{k}\neq 0$ is also satisfied. Observe that $\Gamma^{5ij}_{\hat{l}(2)}$ has no pole at $\omega=0$. Evidently such a pole cannot appear in any finite order of the perturbation expansion. Indeed, its presence would not be compatible with Eq. (4.15) for $\vec{k}\neq 0$, $\omega\to 0$ since the right-hand side, $\{\Sigma(\eta),\gamma_5\}_+$, is assumed to be regular for $|\eta|<\epsilon$.

On the other hand, the pole at $\omega=0$, $(\vec{k}=0)$ found in the second order of Γ^{5ij} [Eq. (4.17)] will persist in higher orders, again due to the Ward-Takahashi identity (4.15). It has to be interpreted as a Goldstone particle at zero momentum, $\vec{k}=0$.

Unfortunately, the lack of relativistic covariance does not allow us to verify the dispersion law, $\omega=\omega(k)$, for these particles, although from Eq. (4.15) one expects the result $\omega(k)\sim k$ for $|\vec{k}|$ much less than the cutoff momentum $(1/a)$. This can be ascertained only by summing an infinite class of diagrams, since in any finite order the $\vec{k}$ dependence cannot appear in the denominator. We shall not pursue this problem in its general form. Instead, a supporting argument will be presented by solving a homogeneous integral equation for Γ^{5ij} and $\Gamma^{5ij}_{\hat{l}}$ at ω, $|\vec{k}|\to 0$, resulting from a summation of ladder diagrams.

D. Bethe-Salpeter equation

Let us consider two-particle irreducible diagrams. As usual, these are connected diagrams which cannot be divided into two disjoint parts by removal of two quark lines. We can represent the lattice version of the Bethe-Salpeter equation for proper vertex functions (4.16a), (4.16b) diagrammatically. For Γ^{5ij}, see Fig. 11, where the kernel K is assumed to be a two-particle irreduci-

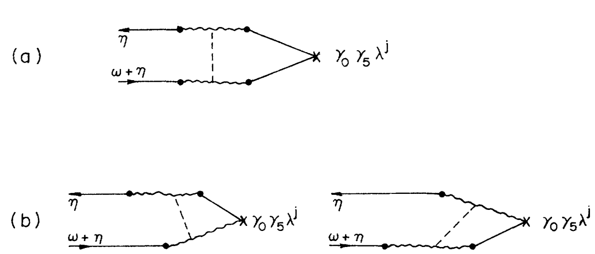

FIG. 10. (a) Quark-antiquark vertex Γ^{5ij} in second order of perturbation theory. (b) The same for the vertex $\Gamma^{5ij}_{\hat{l}}$.

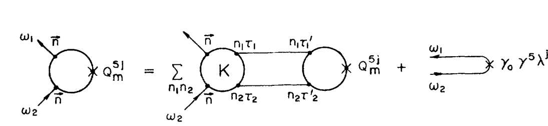

FIG. 11. Bethe-Salpeter equation for the vertex $\Gamma^{5ij}_{(\omega_1,\omega_2)}$.

ble diagram describing quark-quark scattering. The equation for $\Gamma_{\hat{i}}^{5ij}$ differs from the one drawn only by a more complicated inhomogeneous term given analytically by Eq. (4.19). The points $n_i\tau_i$ and $n_i\tau_i'$, with $i=1,2$, are connected by full quark propagators as shown in Fig. 11.

We see that on the right-hand side of the equation in Fig. 11, along with the vertex Γ^{5ij} corresponding to the contribution from $n_1=n_2$, there are other unknown functions due to the terms $n_1\neq n_2$. However, the latter do not appear when the kernel K is restricted to the lowest-order term. Below we restrict ourselves to studying the approximate scheme in which the equation in Fig. 11 for $\Gamma^{5j}=\sum_{i=0}^{8}\Gamma^{5ij}\lambda^i$ becomes

$$\Gamma^{5j}(\omega_1,\omega_2|\vec{K}) = (\text{inhomogeneous term}) + \int\sum_{\hat{i}} K_{\hat{i}}^{(2)}(\omega_{1,2},\omega'_{1,2}|K)\delta(\omega_1-\omega_2-\omega_2')d\omega_1'd\omega_2'\hat{\not{l}}S'(\omega_1')\Gamma^{5j}(\omega_1'\omega_2'\vec{K})S'(\omega_2')\hat{\not{l}}, \tag{4.20}$$

with

$$K_{\hat{i}}^{(2)}(\omega_{1,2},\omega'_{1,2}|K) = \left(\frac{1}{2g^2}\right)^2(e^{-i\hat{\vec{k}}\cdot\hat{i}})\left(-\frac{1}{2\pi}\right)D(\omega_1-\omega_1'). \tag{4.21}$$

A similar equation holds for $\Gamma_{\hat{i}}^{5j}\equiv\sum_i\Gamma_{\hat{i}}^{5ij}\lambda^i$ with a different inhomogeneous term, which is irrelevant to what follows. The homogeneous integral equations corresponding to Γ^{5j} and $\Gamma_{\hat{i}}^{5j}$ coincide and can be written as

$$R(\eta|k,\omega) = \left(-\frac{1}{2\pi}\right)\left(\frac{1}{2g^2}\right)^2\int D(\eta-\eta')d\eta'\sum_{\hat{i}} e^{i\vec{k}\cdot\hat{i}}\hat{\not{l}}S'(\eta'+\omega')R(\eta'|k,\omega)S(\eta'-\omega)\hat{\not{l}}. \tag{4.22}$$

New variables $\omega=\frac{1}{2}(\omega_1-\omega_2)$ and $\eta=\frac{1}{2}(\omega_1+\omega_2)$ have been introduced. In this equation, if the vertices Γ^{5j} and $\Gamma_{\hat{i}}^{5j}$ have a pole, e.g., at $(\omega,k)=(\omega_0,k_0)$, then for these values of (ω,k) a nontrivial solution to Eq. (4.22) must exist. This fixes the η dependence of the corresponding residues.

A simplified version of the above equation at $\omega\sim 0$ will be solved when the quark propagator $S'(\omega)$ is replaced by $S(\omega)$. First rewrite $R(\eta|k,\omega)$ in the form

$$R(\eta|k,\omega) = \tfrac{1}{2}(R_+(\eta|k,\omega)+R_-(\eta|k,\omega)\gamma_5+\tfrac{1}{2}[R_+(\eta|k,\omega)-R_-(\eta|k,\omega)]\gamma_0\gamma_5, \tag{4.23}$$

which reduces Eq. (4.22) to the system of homogeneous integral equations,

$$R_\pm(\eta|k,\omega) = \left(-\frac{1}{2\pi}\right)\chi(x)\int D(\eta-\eta')d\eta'\,\mathrm{Tr}\left[\left(\frac{1\mp\gamma_0}{2}\right)S(\omega+\eta')S(\omega-\eta')\right]R_\mp(\eta'|k,\omega), \tag{4.24}$$

with

$$\chi(k) = \frac{1}{2}\left(\frac{1}{2g^2}\right)^2\sum_{\hat{i}} e^{-i\vec{k}\cdot\hat{i}}. \tag{4.25}$$

It is convenient to perform the Fourier transform to recast Eq. (4.24) into the form

$$\tilde{R}_\pm(\tau|k,\omega) = -\frac{\chi(k)}{2\pi}D(\tau)\int d\tau'\tilde{R}_\mp(\tau'|k,\omega)\int d\eta' e^{i\eta(\tau-\tau')}\,\mathrm{Tr}\left[\left(\frac{1\pm\gamma_0}{2}\right)S(\omega+\eta)S(\omega-\eta)\right], \tag{4.26}$$

with

$$\tilde{R}_\pm(\tau|k,\omega) = \frac{1}{2\pi}\int R_\pm(\eta|k,\omega)e^{-i\eta\tau}d\eta. \tag{4.27}$$

Making use of the formula (4.6) and replacing $D(\tau)$ by $e^{-i\epsilon|\tau|}$ one arrives at

$$\begin{aligned}\tilde{R}_+(\tau|k,\omega) &= -\chi D(\tau)\int d\tau'dt\,e^{i\omega(\tau-\tau'+2t)}\theta(-t)\theta[+(\tau'-\tau-t)]\tilde{R}_-(\tau'|k,\omega),\\ \tilde{R}_-(\tau|k,\omega) &= -\chi D(\tau)\int d\tau'dt\,e^{i\omega(\tau-\tau'+2t)}\theta(+t)\theta[-(\tau'-\tau-t)]\tilde{R}_+(\tau'|k,\omega).\end{aligned} \tag{4.27'}$$

In detail, this is

$$\tilde{R}_+(\tau)e^{i\epsilon|\tau|} = -\chi \int_{-\infty}^{\tau} dt \int_t^{\infty} d\tau' \tilde{R}_-(\tau'), \qquad (4.28a)$$

$$\tilde{R}_-(\tau)e^{i\epsilon|\tau|} = -\chi \int_{\tau}^{\infty} dt \int_{-\infty}^{t} d\tau' \tilde{R}_+(\tau'), \qquad (4.28b)$$

where the limit $\omega \to 0$ is taken and $\tilde{R}_\pm(\tau) \equiv \tilde{R}_\pm(\tau|\omega, k)|_{\omega=0}$. Recall that ϵ is assumed to have an infinitesimal negative imaginary part, i.e., it stands for $\epsilon - i0$. Differentiating twice with respect to τ we derive a system of second-order differential equations

$$\frac{d^2}{d\tau^2}(\tilde{R}_\pm(\tau)e^{i\epsilon|\tau|}) = \chi \tilde{R}_\mp(\tau), \qquad (4.29)$$

with boundary conditions

$$R_\pm(\pm\infty) = R'_\pm(\pm\infty) = 0. \qquad (4.30)$$

The change of variable $z = e^{-i\epsilon|\tau|}$ reduces Eq. (4.29) to the Laplace-type equations

$$\left(z\frac{d^2}{dz^2} + \frac{d}{dz} - \frac{\chi}{\epsilon^2}\hat{\sigma}\right)\psi(z) = 0, \quad |z| \leq 1 \qquad (4.31)$$

where matrix notations have been introduced:

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (4.32)$$

$$\psi(z) = e^{i\epsilon|\tau|}\begin{pmatrix} \tilde{R}_+(\tau) \\ \tilde{R}_-(\tau) \end{pmatrix}. \qquad (4.33)$$

The boundary conditions (4.30) become

$$[z\psi(z)]_{z=0} = [z^2\psi'(z)]_{z=0} = 0. \qquad (4.30')$$

Note that the infinite interval $-\infty < \tau < \infty$ is mapped into the interior of the unit circle $|z| \leq 1$, with the origin, $z = 0$, corresponding to $\tau = \pm\infty$. In deriving Eq. (4.31), we ignored terms arising from a discontinuity at $z = 1$ ($\tau = 0$) since in the region $z > 1$ the function $\psi(z)$ can be chosen arbitrarily. The solution to Eq. (4.31) is given by a well-known ansatz,[19]

$$\psi(z) = \int_l F(w)e^{wz}\, dw, \qquad (4.34)$$

with

$$F(w) = \frac{1}{Q(w)} \exp\left[\int \frac{P(w)}{Q(w)}\, dw\right]\begin{pmatrix} C_+ \\ C_- \end{pmatrix}, \qquad (4.35)$$

$$P(w) = w - \frac{\chi}{\epsilon^2}\hat{\sigma}, \quad Q(w) = w^2$$

where $C_\pm$ are arbitrary constants.

The choice of the integration contour l is constrained only by the requirement that the function $e^{wz}Q(w)F(w)$ attains the same value at the end points of l. Evaluating $F(w)$, one finds

$$\psi(z) = \oint_{|w|=1} e^{wz}\frac{dw}{w}\left[\cosh\left(\frac{\chi}{\epsilon^2}\frac{1}{w}\right) + \hat{\sigma}\sinh\left(\frac{\chi}{\epsilon^2}\frac{1}{w}\right)\right] \times \begin{pmatrix} C_+ \\ C_- \end{pmatrix}; \qquad (4.36)$$

for contour l, the unit circle $|w| = 1$. It is easy to recognize a combination of Bessel functions $J_0(x)$ in Eq. (4.36):

$$\psi(z) = \left[\tfrac{1}{2}(1+\hat{\sigma})J_0\left(2\left(-\frac{\chi}{\epsilon^2}z\right)^{1/2}\right) + \tfrac{1}{2}(1-\hat{\sigma})J_0\left(2\left(\frac{\chi}{\epsilon^2}z\right)^{1/2}\right)\right]\begin{pmatrix} C_+ \\ C_- \end{pmatrix}, \qquad (4.37)$$

which satisfies the required boundary condition (4.30′) at the origin.

The second set of solutions satisfying Eqs. (4.30′) are known to be logarithmically singular at $z = 0$. They will not be written down here.

In summary, we have found nontrivial solutions [see Eqs. (4.27), (4.33), and (4.37)] to the homogeneous Bethe-Salpeter equation (4.22), which are necessary for the poles at $\omega \sim 0$ to be present in the vertex functions $\Gamma^{5j}(\omega+\eta, \eta)$ and $\Gamma^{5j}_{\xi}(\omega+\eta, \eta)$. As was argued above, the latter can only have poles which behave as $\omega = \omega(k)|_{k\to 0} \to 0$. It is interesting to note that in the strong-coupling limit, the residues of the poles given by Eqs. (4.27), (4.33), and (4.37) satisfy the Ward-Takahaski identity (4.15). This simply follows from the relation

$$\{\Sigma^{(2)}(\eta), \gamma_5\}_+ \sim D(\eta). \qquad (4.38)$$

V. SUMMARY AND CONCLUSIONS

This paper represents an attempt to study dynamical violation of γ_5 symmetry in a color-quark gauge theory. It was previously argued that this phenomenon *could* occur, on the basis of examination of the formal properties of the Green's functions of the theory. The contribution of the present work to this area of investigation has been to demonstrate quantitatively that the spontaneous breaking *does* occur within a well-defined scheme of approximation. The most crucial part of the approximation is, of course, the use of the lattice as a guide to the infrared behavior of the full theory.

In the course of pursuing the above investigation, we have developed the Hamiltonian formulation of the lattice theory to some extent. The

Feynman rules and diagrammatic analysis of Sec. III may prove useful for studying other features of the Hamiltonian lattice, although more work needs to be done in the gluon sector.

At this stage in papers on lattice theory, one usually expresses hopes that renormalization-group considerations will soon vindicate the approach. In addition to echoing this sentiment, we have indicated in various places throughout the paper specific questions that our theme of research has brought us to focus upon. We reiterate, however, that an important option for the present time is to press the theory for predictions. Let us list a few possibilities for further work.

The ninth axial-vector current was not discussed in Sec. IV. The reason is that one expects on general grounds that it will behave differently from the unitary symmetry octet of axial-vector currents in the continuum limit. For example, it develops the triangle anomaly whereas the octet currents do not. How this different behavior actually occurs in the $a\to 0$ limit of the quantum theory remains an open and very interesting question.

In addition to pushing forward with the perturbation theory, it would be useful to develop a classification scheme for excited states, analogous to the L-S scheme of the naive quark model. In Fig. 12 states (a) and (b) have the same energy. In case (a) the link energy is the length of the string, which is also the separation between q and $\bar{q}$. But in case (b) the length is not equal to the separation. Is that a violation of the notion that $V(r)\propto r$? An alternative possibility is that (a) and (b) mimic states with different orbital angular momenta. As in the Veneziano model, one may have a number of states of different J at a given (mass)^2. At present, one has no idea of the "trajectory structure" or of the level degeneracy of the lattice theory.

Results from the naive quark model suggest that such a classification scheme will be important for discussing real hadrons. These invariably turn out to be complicated mixtures of the so-called current quarks, which enter into the color-quark gauge-field continuum Hamiltonian. Some semblance of simplicity in the spectroscopy might be restored by using a light-cone dynamical hypothesis, cutting off only the transverse momenta.

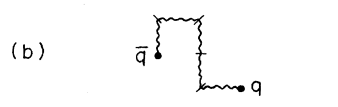

FIG. 12. Equal energy configurations of quark ($\bar{q}$) and antiquark (q) connected by strings of gluons (see text).

In addition, soft-pion emission formulas can usually be derived in a simple manner in a light-cone basis. Unfortunately, our formalism is not adapted to study this interesting question.

Aside from these chiral-symmetry considerations, it would prove of general interest to consider a limit where the number of links between q and $\bar{q}$ goes to infinity, while the lattice spacing shrinks to zero. The Nielsen-Olesen vortex line supports transverse oscillations in the classical limit, the well-known Alfven waves. A very interesting question is whether the non-Abelian "electric" string can also exhibit such excitations. Note in this connection that at the lattice level one may excite a Fig. 12 state either transversely by the action of the purely gluonic part of $\mathcal{L}_I$, or longitudinally by the action of the mixed part of $\mathcal{L}_I$. This is a departure from a pure transverse string behavior which needs to be understood better.

Finally, the lattice theory makes rather definite assertions about the types of quarkless gluon excitations that exist, and how these couple to ordinary hadronic matter. Most theories of composite hadrons admit such exotic states in the spectrum, but only the lattice theory permits a detailed study of their properties from the outset. In the absence of other tractable models, some insight into the subject of GIQE's may be gained from the lattice.

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APPENDIX A

The Haar measure $d\mu(g)$ is left as well as right invariant. Hence the integrals in Eq. (2.49) receives a nonvanishing contribution only from the piece of the integrand invariant with respect to left *and* right shifts. Thus we have to construct singlet combinations of the integrand from the upper (a_i) and lower (b_i) indices separately, remembering that by definition ${}^nI^{\{a\}}_{\{b\}}$ is symmetric under permutation of any pairs (a_i, b_i).

That means that ${}^nI^{\{a\}}_{\{b\}}$ is a singlet of the symmetric group of the elements (a_i, b_i), $i=1,\ldots,n$. The well-known way to construct this is first to symmetrize the upper (a_i) and lower (b_i) indices according to the Young diagram shown in Fig. 13. Then one performs all possible permutations (p) of the elements (a_i, b_i), $i=1,\ldots,n$, in that result.

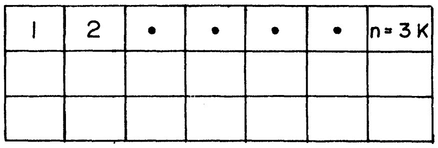

FIG. 13. The Young diagram according to which the indices $\{a_i\}$ and $\{b_i\}$ in Eq. (A2) are symmetrized.

Obviously ${}^{n}I^{\{a\}}_{\{b\}}$ is nonvanishing only for $n=3k$, $k\geq 1$. It is convenient to perform Young's symmetrization by first antisymmetrizing (A) the indices in each column and then symmetrizing the elements in each row. It is remarkable that applying (A) to the integrand of Eq. (2.49), one gets a string of antisymmetric tensors $\epsilon^{a_i a_j a_k}$ $(\epsilon^{123}=1)$. This is due to the special character of the group [SU(3)] under consideration, which enjoys the property

$$A(a_1a_2a_3)U^{a_1}_{b_1}U^{a_2}_{b_2}U^{a_3}_{b_3}=\epsilon^{a_1a_2a_3}\epsilon_{b_1b_2b_3}. \tag{A1}$$

Thus we can summarize our result by the following formula:

$${}^{n}I^{\{a\}}_{\{b\}}=\delta_{n,3k}C(n)p\{a_i,b_i\}S(b)S(a) \times\epsilon^{a_1a_2a_3}\epsilon_{b_1b_2b_3}\cdots\epsilon^{a_{n-2}a_{n-1}a_n}\epsilon_{b_{n-2}b_{n-1}b_n}. \tag{A2}$$

Here the overall coefficient $C(n)$ can be determined by induction, contracting both sides of Eq. (A2) with, e.g., $\epsilon_{a_1a_2a_3}$ and making use of formula (4.1) in the form

$$\epsilon_{a_1a_2a_3}U^{a_1}_{b_1}U^{a_2}_{b_2}U^{a_3}_{b_3}=\epsilon_{b_1b_2b_3}.$$

Owing to the relation

$$\epsilon_{a_1a_2a_3}U^{a_1}_{b_1}U^{a_2}_{b_2}=\epsilon_{b_1b_2b_3}U^{\dagger b_3}_{a_3},$$

the formula (A2) also allows for evaluation of integrals when the integrand contains both U's and $U^\dagger$'s. The examples in Sec. II may be computed straightforwardly using these rules.

APPENDIX B

The notion of massless particles on a lattice is abhorrent at first sight. Notice, however, that in the strong-coupling limit, the typical gauge excitations on the lattice have wavelengths that are considerably shorter than the lattice spacing, i.e., $\epsilon\gg 1/a$. These energies are not gauge-field "rest energies" since there was no gauge-field bare-mass term in the Hamiltonian. The physical excitation spectrum is not identical with the naively conceived continuum gauge-field spectrum, and a similar situation obtains for the quarks.

Formally, however, one does encounter an intermediate problem if $m_0=0$ is specified initially in $\mathcal{L}$. It is that then there is no "free" quark part in $H^{(0)}$, although we do have $(-i\Sigma\psi^\dagger\partial_0\psi)$ in $\mathcal{L}^{(0)}$. Thus

$$[H^{(0)},\psi]=0, \tag{B1}$$

and ψ is any time-independent four-component spinor. We can respect the parity and charge-conjugation properties correctly, and satisfy the canonical anticommutation structure of ψ, by using

$$\psi_\alpha=\frac{1}{\sqrt{2}}(q_\alpha+\bar{q}^\dagger_\alpha), \tag{B2}$$

where q_α and $\bar{q}^\dagger_\alpha$ are each four-component objects. Here α represents Dirac indices. In a γ_5-diagonal basis, e.g., the upper [lower] two components of ψ would correspond to helicity (+) [(−)] quarks in the continuum limit, and similarly for the antiquarks. However, this implies that

$$j^\alpha_0(\vec{n},\tau)|0\rangle=:\psi^\dagger(\vec{n},\tau)\frac{\Lambda^\alpha}{2}\psi(\vec{n},\tau):|0\rangle\neq 0. \tag{B3}$$

(For directness, a Fock-type basis, with $q|0\rangle=\bar{q}|0\rangle=0$, will be employed in this appendix.) On the other hand,

$$Q^i_5|0\rangle=\sum_n:\psi^\dagger(\vec{n},\tau)\gamma_5\lambda^i\psi(\vec{n},\tau):|0\rangle=0. \tag{B4}$$

The vacuum $|0\rangle$ is chirally invariant, but cannot be locally gauge invariant in the fermion and gluon sectors separately.

To maintain the gauge condition in the separable form we have been using, therefore, it is necessary to construct a new state

$$|\Omega\rangle=V|0\rangle, \tag{B5}$$

with $V^\dagger V=1$. The new base state should have the properties that $j^\alpha_0(\vec{n},\tau)|\Omega\rangle=0$, and $Q^\alpha(\vec{n},\tau)|\Omega\rangle=0$, i.e., V can be chosen independent of gauge fields. In addition, V should have the quantum numbers of the vacuum.

A simple candidate for such a V may be constructed out of the q and $\bar{q}$ components introduced in Eq. (B2). It is

$$V(\tau)=\exp\left\{i\theta\sum_{\vec{n}}[q^\dagger(\vec{n},\tau)\gamma^0\bar{q}^\dagger(\vec{n},\tau)-\bar{q}(\vec{n},\tau)\gamma_0 q(\vec{n},\tau)]\right\}. \tag{B6}$$

The charge density $j^\alpha_0(\vec{n},\tau)$ annihilates $|\Omega\rangle$ if $\theta=\pm\frac{1}{4}\pi$.

Now, however, q and $\bar{q}^\dagger$ are no longer lowering and raising operators on the base state $|\Omega\rangle$, $q|\Omega\rangle\neq\bar{q}|\Omega\rangle\neq 0$. One possibility is to rewrite q and $\bar{q}$ so that

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi+\bar{\chi}^\dagger\\ \varphi+\bar{\varphi}^\dagger\end{pmatrix},$$

where χ and ϕ are quark helicity (+) and (−) operators, and $\overline{\chi}^\dagger, \overline{\phi}^\dagger$ are the corresponding antiquark operators. Since V, Eq. (4.6), fills the bare vacuum with pairs, one might expect that some new linear combination of quark and antiquark operators will annihilate $|\Omega\rangle$.

Indeed, with $\theta=\frac{1}{4}\pi$, we find that

$$\begin{aligned}(\overline{\varphi}+\chi^\dagger)|\Omega\rangle&=0,\\(\overline{\chi}+\varphi^\dagger)|\Omega\rangle&=0,\\(\chi-\overline{\varphi}^\dagger)|\Omega\rangle&=0,\\(\varphi-\overline{\chi}^\dagger)|\Omega\rangle&=0.\end{aligned}\tag{B8}$$

The corresponding conjugate operators, e.g., $(\overline{\phi}^\dagger+\chi)$, are raising operators on $|\Omega\rangle$, and

$$\left\{\frac{\overline{\varphi}+\chi^\dagger}{\sqrt{2}},\frac{\overline{\varphi}^\dagger+\chi}{\sqrt{2}}\right\}_{\text{equal time}}=\delta,$$

where δ is in all indices (including spatial position) carried by χ and $\overline{\phi}$.

The crucial observation at this point is that

$$[Q_5,V]\neq 0\tag{B9}$$

where V is given in Eq. (B6), with $\theta=\pm\frac{1}{4}\pi$ and where Q_5 is defined in Eq. (B4). Consequently, $Q_5|\Omega\rangle\neq 0$. But if $m_0=0$, $[Q_5,H^{\text{tot}}]=0$. If we build our space of states off the "vacuum" $|\Omega\rangle$, we seem to be in a position to realize the Goldstone theorem.

It turns out that working directly with $|\Omega\rangle$, and with ψ as in Eq. (B7), is rather cumbersome. In studying observables in perturbation theory, we shall always be interested in quantities of the form

$$O=\langle\Omega|\hat{O}|\Omega\rangle,\tag{B10}$$

where $\hat{O}$ is an operator. In the interaction representation, the ψ and $\psi^\dagger$'s possibly present in $\hat{O}$ will satisfy free equations of motion, and thus be time independent. We may, therefore, equivalently consider

$$O=\langle 0|V^\dagger\hat{O}V|0\rangle.\tag{B11}$$

This means that in calculating O we may use (with $\theta=\frac{1}{4}\pi$)

$$\psi=V^\dagger\left(\frac{q+\overline{q}^\dagger}{\sqrt{2}}\right)V=\frac{1+\gamma_0}{2}q+\frac{1-\gamma_0}{2}\overline{q}^\dagger.\tag{B12}$$

In a β-diagonal basis, as in Eq. (2.27), we have once again

$$\psi=\begin{pmatrix}q\\\overline{q}^\dagger\end{pmatrix},\tag{B13a}$$

with

$$\{q_\alpha,q_\beta^\dagger\}=\left(\frac{1+\gamma_0}{2}\right)_{\alpha\beta},\tag{B13b}$$

etc. Clearly ψ, Eq. (B13), is $\lim_{m_0\to 0}\psi_{(m_0)}$, where $\psi_{(m_0)}$ was defined in Eq. (2.27).

It is straightforward to verify that once again

$$:\psi^\dagger\Lambda^a\psi:|0\rangle=0,\tag{B14a}$$

$$:\psi^\dagger\gamma^5\lambda^i\psi:|0\rangle\neq 0,\tag{B14b}$$

just as if we had started with a massive quark field and then let the bare mass go to zero. It is this property that encourages us to work with a nonzero bare mass, and then take the limit.

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