Particle spectrum in model field theories from semiclassical solutions of the field equations*

Abraham Klein†
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Franz Krejs
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
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Bound-state spectra (of mesons or of solitons-antisolitons) are studied for the sine-Gordon and quartic-coupling nonlinear boson models in one-plus-one dimensions using methods developed previously by the authors for the example of the nonlinear Schrödinger equation. In addition to reproducing the spectra first derived by Dashen, Hasslacher, and Neveu (DHN) up to the order of the first quantum correction, we have also calculated (in the semiclassical approximation) all bound-state form factors as well as the matrix elements of the field between any bound state and any continuum state for the scattering of a meson on a bound state. For the sine-Gordon theory the results have been obtained in two ways: first, by an algorithm, derived in due course, for transcribing an exact classical solution into a quantum operator; second, by a systematic expansion about the weak-coupling limit which requires the techniques used previously for the nonlinear Schrödinger equation. Only this latter technique is available for the quartic model, but its (more complicated) application here leads to an explanation of why in leading order the same form of bound-state spectra are obtained for the two models. Compared to the work of DHN, aside from methodology, the main new results are the matrix elements of the field operators, but we also present a complete quantum interpretation of all their classical calculations as well as an explanation of why our methods are equivalent.

I. INTRODUCTION

This paper continues the study of simple nonlinear field theories based on direct solution for matrix elements of the field equations. In two previous papers[1,2] (referred to as I and II), we have studied the soliton in the quartic interaction model of bosons[1] (building on work of Goldstone and Jackiw[3]) and bound states in the nonlinear Schrödinger equation.[2]

In the present paper we again deal with bound states, this time for the sine-Gordon and the quartic interaction theories in one-plus-one dimensions, precisely the problems dealt with by Dashen, Hasslacher, and Neveu (DHN) in a monumental work.[4,5] It is our purpose (i) to reproduce the bound-state spectra first derived by these authors, (ii) to generate several classes of matrix elements not given in previous work, and (iii) to explain why the two models studied yield the same functional form for the bound-state spectrum.

We start by reviewing the basic equations for the models utilized. The sine-Gordon theory is defined by the Lagrangian density

$$\mathcal{L}(x,t)=\tfrac{1}{2}(\partial_t\phi)^2-\tfrac{1}{2}(\partial_x\phi)^2+\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)-1\right], \tag{1.1}$$

with corresponding Hamiltonian density ($\pi=\partial_t\phi$)

$$\mathcal{H}(x,t)=\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\partial_x\phi)^2+\frac{m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right]. \tag{1.2}$$

Our study is based directly on the solution of the field equations

$$(-\partial_t^{\,2}+\partial_x^{\,2})\phi(x,t)-\frac{m^3}{\sqrt{\lambda}}\sin\left[\frac{\sqrt{\lambda}}{m}\phi(x,t)\right]=0 \tag{1.3}$$

and of the commutation relations

$$[\phi(x,t),\pi(y,t)]=i\delta(x-y). \tag{1.4}$$

On the classical level, the field equation (1.3) possesses space-dependent static solutions

$$\phi_\pm(x)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}[\exp(\pm mx)], \tag{1.5}$$

which, in the quantum theory, are associated with the soliton (+) and antisoliton (−), respectively. Their common mass is, in lowest approximation,

$$M=8m^3/\lambda. \tag{1.6}$$

Our calculation of the first quantum correction[4] to this result is given in Appendix B.

Of the infinitely many time-dependent classical solutions known, only the one which is periodic in time and confined in space (the so-called doublet or breather),

$$\phi_B(x,t)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\left(\frac{m^2-\omega^2}{\omega^2}\right)^{1/2}\times\frac{\cos\omega(t-t_0)}{\cosh(m^2-\omega^2)^{1/2}(x-x_0)}\right], \tag{1.7}$$

will concern us. Here ω is a continuous parameter and t_0, x_0 are arbitrary. Upon semiclassical quantization this gives rise to a spectrum of particle states with masses

$$M_n = 2M \sin \frac{nm}{2M}, \tag{1.8}$$

where M is the soliton mass (1.6) and the number of such states is restricted by the condition

$$n = 1, 2, \ldots, < 8\pi m^2/\lambda. \tag{1.9}$$

In addition to rederiving (1.8) as well as the first quantum correction to it, we have been able to derive matrix elements of the field ϕ connecting the various states $|n\rangle$, the bound states at rest. This has been done by two methods. The first is based on having available the explicit form (1.7). There then exists a simple algorithm for transcribing this solution into a generating operator for all matrix elements of the field among the states $|n\rangle$. The results of this exercise suggest a viable sequence of approximations, of which the first is equivalent to the method applied previously in the solution of the nonlinear Schrödinger equation.[2] By application of this method, it can be seen that (1.8) emerges from the lowest approximation to the semiclassical calculation.

The second example which we shall consider is the quartic-coupling model defined by the Lagrangian density

$$\mathcal{L}(x, t) = \tfrac{1}{2}(\partial_t \phi)^2 - \tfrac{1}{2}(\partial_x \phi)^2 - U(\phi), \tag{1.10}$$

with

$$U(\phi) = \frac{1}{2\lambda}(m^2 - \lambda\phi^2)^2. \tag{1.11}$$

The Hamiltonian density is

$$\mathcal{H}(x) = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_x \phi)^2 + U(\phi), \tag{1.12}$$

and the equation of motion takes the form

$$\partial_t^2 \phi(x, t) - \partial_x^2 \phi(x, t) - 2m^2 \phi(x, t) + 2\lambda \phi^3(x, t) = 0. \tag{1.13}$$

For this equation, the analog of (1.5) and (1.6) has been fully discussed.[6,3] Exact time-dependent solutions are, however, unknown. Instead, to study possible bound-state spectra, we apply the method developed for the sine-Gordon theory. For reasons that are explained, the application requires some care, but the final form of the c-number field equations suggests that the starting Hamiltonian can be replaced by a unitarily equivalent one which has the same weak-coupling form as the sine-Gordon theory. From our previous results this explains as well why the spectrum (1.8), with suitably altered constants, is again found, to the order studied.

Our presentation is arranged as follows: In Sec. II, the semiclassical result is first derived by an irreducibly simple application of Wilson-Sommerfeld quantization to the exact classical solution (1.7), giving rise to the spectrum (1.8). The bulk of this section is then devoted to showing how the same classical solution yields a generating operator for matrix elements of the field between the states in question.

In Sec. III, we develop the method, suggested by the results of the previous section, for studying an arbitrary self-coupled boson model, in semiclassical approximation, using the quartic interaction model and sine-Gordon models as examples, rederiving the exact results for the latter and corresponding results for the former. In Sec. IV we calculate the first quantum corrections for each model by a technique only slightly modified from a similar calculation for the nonlinear Schrödinger equation and thereby reproduce the results of DHN. In Sec. V we calculate once more the first quantum correction to the sine-Gordon model in a quantum version of the corresponding calculation of DHN. We also obtain thereby a generating operator for all transition matrix elements of the field between the bound states and the scattering states of a meson by the bound states.

In Appendix A we confirm the Lorentz invariance of our methods and in Appendix B we collect results on the (static) solitons.

It is appropriate at this point to add a few remarks concerning the connection between our method and that of DHN, which methods appear, at first sight, to be quite distinct, but which culminate in precisely the same final integrals.[7] DHN calculate the trace of the time-development operator e^{-iHT}, where H is the Hamiltonian, over a *subspace* of the eigenstates of H; they convert the trace to a functional integral computed by semiclassical approximations. To understand the connection between the two methods, we need the trivial observation that if H is diagonal, computation of the trace of H suffices to calculate the trace of the time-development operator. In our method we focus on c-number or matrix-element equations following from the field equations. We have shown,[1] however, that these equations are the Euler-Lagrange equations for the trace of the Lagrangian or for the trace of the Hamiltonian, under the constraint that the operator commutation relations be satisfied. Furthermore, the consistent approximate solution of these equations guarantees that H is diagonal to the same approximation.

From the examples given before[1,2] and in this paper it is therefore clear that we are performing exactly the same calculations as DHN, though the technique and mode of reasoning are quite different: We construct semiclassical approximations, or asymptotic solutions, to fully quantum-mechanical equations rather than "quantizing" classical solutions. The only possible advantage for our method, which we might dare to claim so far, is based on the fact that we obviously generate matrix elements as well as energies[8,9] in a natural way. Whether there are other advantages or disadvantages will have to be established by future applications.

II. SEMICLASSICAL QUANTIZATION

A. Wilson-Sommerfeld quantization

Given an *explicit* periodic classical solution, the most direct way to obtain the spectrum is to make use of the correspondence principle, which connects the classical frequency with successive energy differences,

$$\omega(n) \cong E_n - E_{n-1} \cong \frac{dE}{dn}. \tag{2.1}$$

Even though we are dealing with a field theory rather than a particle theory, (2.1) contains the assertion that a simply periodic motion implies the existence of a set of bound states, labeled here by the integer n.

We apply (2.1) in the integrated form[10]

$$n = \int_0^{E_n} \frac{d\epsilon}{\omega(\epsilon)}. \tag{2.2}$$

To apply this condition, we calculate $\omega(\epsilon)$ as follows: We first notice that the field energy calculated from the solution (1.7) is independent of time. (This can be shown from the equation of motion and the vanishing of the spatial derivative at infinity.) We then find

$$\begin{aligned} E_c &= H[\phi_B(x,t)] \\ &= (m^2 - \omega^2)^{1/2}(2M/m). \end{aligned} \tag{2.3}$$

Inverting (2.3) for use in (2.2) yields directly

$$mn = 2M \sin^{-1}(E_n/2M), \tag{2.4}$$

which is (1.8) with the identification $E_n \leftrightarrow M_n$.

B. Quantum significance of the breather solution

Not only does the breather solution lead to the semiclassical approximation, (2.4), to the spectrum, but, as we now proceed to show, it also leads to a similar approximation for a whole array of matrix elements of the field operator. We shall first describe the recipe for obtaining these results and then give the "derivation" of the recipe.

In (1.7), we choose $x_0 = t_0 = 0$ merely as a convenience. From the resulting expression we define an operator $Y(x, \hat{n}, \hat{\theta})$ by first writing

$$\begin{aligned} \phi_B(x,t) &\equiv Y(x, n, \omega t) \\ &\equiv \frac{4m}{\sqrt{\lambda}} \tan^{-1}[y(n,x)\cos\omega t], \end{aligned} \tag{2.5}$$

where

$$y(n,x) = \frac{\tan(nm/2M)}{\cosh\{m[\sin(nm/2M)]x\}}. \tag{2.6}$$

In this first step, we have replaced ω everywhere in ϕ_B except in the combination ωt by $\omega(n)$, where

$$\omega(n) = \frac{dM_n}{dn} = m \cos\frac{nm}{2M}, \tag{2.7}$$

The combination ωt is next replaced by θ. We interpret n and θ as the action-angle variables of an oscillator degree of freedom and carry out a semiclassical quantization by placing carets on these symbols which now operate in the one-dimensional space of states $|n)$ with the rules

$$e^{i\hat{\theta}}|n) = |n-1), \quad \hat{n}|n) = n|n). \tag{2.8}$$

The well-known difficulties[11] in defining a unitary operator by (2.8) need not concern us, since we shall insist on this relation only for $n \gg 1$, where theoretical objections are numerically irrelevant. In the same limit there is also no objection to regarding $\hat{\theta}$ and $\hat{n}$ as canonically conjugate variables. The operator $\hat{Y}$, defined arbitrarily by the ordering exhibited in (2.5), will yield matrix elements whose significance and accuracy will be specified below.

The matrix elements of the operator $\hat{Y}$ between the states (2.8) are obtained straightforwardly by expansion of the inverse tangent

$$\begin{aligned} (n|\hat{Y}|n+2p+1) &= (n+2p+1|\hat{Y}|n) \\ &= (4m/\sqrt{\lambda}) \\ &\quad \times \sum_{l=p}^{p+[n]/2} [\tfrac{1}{2}y(n,x)]^{2l+1} \\ &\quad \times \frac{(-1)^l}{(2l+1)} \binom{2l+1}{l-p}, \end{aligned} \tag{2.9}$$

where $[n] = n$ for n even and $n-1$ for n odd.

But the relationship of these objects to the original field operator has still to be specified: Let $|n(p)\rangle$ be the field-theoretical bound state with center-of-mass momentum p. We shall introduce the notation

$$\begin{aligned} \langle n'|\phi(x)|n\rangle \equiv \lim_{p' \to 0} \int \frac{dp}{2\pi} e^{i(p-p')x} \\ \times \langle n'(p')|\phi(0)|n(p)\rangle \end{aligned} \tag{2.10}$$

to signify an approximation in which the states $|n\rangle$ are assumed so heavy as to be insensitive to recoil in elementary acts of emission and absorption (fixed-source limit). Then we aver that

$$(n|\hat{Y}(x,\hat{n},\hat{\theta})|n') \equiv \langle n|\phi(x)|n'\rangle, \tag{2.11}$$

i.e., the matrix elements of $\hat{Y}$ in the one-particle space equal (in a certain approximation) the matrix elements of the original field operator between the bound states.

The ambiguity of ordering in defining the operator $\hat{Y}$ now means that *each* pair of square brackets in (2.9) is only the leading term of a series with the next term down by at least $(1/n)$. For this reason the matrix elements with $p \sim n$ are uncertain to within their own value, as are the high-order terms of the matrix elements for small p. As we shall see, however, in the regime to which the theory best applies, $\sin(nm/2M) \ll 1$, these terms are themselves small.

We turn to the justification of (2.11). We examine the quantum field equation [cf. (1.3)] in the form, for $t=0$,

$$\partial_x{}^2\hat{\phi}(x) + [[\hat{\phi}(x),H],H] = (m^3/\sqrt{\lambda})\sin[(\sqrt{\lambda}/m)\hat{\phi}(x)]. \tag{2.12}$$

We further simplify matters by restricting ourselves to the no-recoil approximation. (The Lorentz covariance thereby destroyed is regained in Appendix A, where we apply the methods of I to this problem and convince ourselves that recoil is not relevant to the present consideration.) The next step is then to replace the full field equation by its projection on to the states $|n\rangle$. This involves the assumption that when we calculate matrix elements of (2.12) between members of the set $|n\rangle$, matrix elements of products of operators are also evaluated by the sum rule

$$\langle n|\hat{\phi}^2(x)|n'\rangle \cong \sum_{n''} \langle n|\hat{\phi}(x)|n''\rangle\langle n''|\hat{\phi}(x)|n'\rangle. \tag{2.13}$$

Furthermore, we can replace H by its projection h on to this space, where according to (1.8), h can be represented in the form

$$h = h(\hat{n}) = 2M\sin(\hat{n}m/2M). \tag{2.14}$$

In the semiclassical limit we would further replace the double commutation with $h(\hat{n})$ by a double Poisson bracket (PB).

Thus if we are to identify $\hat{Y}(x,\hat{n},\hat{\theta})$ with the projection of $\hat{\phi}(x)$ as required by (2.11), we must show that the classical function $Y(x,n,\theta)$ satisfies

$$\partial_x{}^2 Y + [[Y,h],h]_{\rm PB} = (m^3/\sqrt{\lambda})\sin[(\sqrt{\lambda}/m)Y]. \tag{2.15}$$

Here

$$-i[Y,h]_{\rm PB} = \frac{\partial Y}{\partial\theta}\frac{\partial h}{\partial n} - \frac{\partial Y}{\partial n}\frac{\partial h}{\partial\theta} = \omega\frac{\partial Y}{\partial\theta}, \tag{2.16}$$

and thus

$$[[Y,h],h]_{\rm PB} = -\omega^2\frac{\partial^2 Y}{\partial\theta^2}. \tag{2.17}$$

The replacement $\theta \to \omega t$ now restores us to our starting point (2.5), from which we may advance to (2.15).

This really completes our derivation, which is more aptly termed an *identification* of the quantum significance of the classical breather. In this identification, we have recognized that it is a disguised form of a generating operator for the bound-state form factors (2.10). This identification also permits an alternate derivation of the spectrum (1.8), on a completely quantum basis. The self-consistency of our identification certainly requires that

$$M_n = \langle n|H|n\rangle = (n|H(\hat{Y}(x,\hat{n},\hat{\theta}))|n). \tag{2.18}$$

The energy is thus given by the constant term in the Fourier expansion of $H(Y)$ in powers of $e^{i\theta}$ evaluated for $\hat{n}=n$. This can be calculated equivalently by replacing $\theta \to \omega t$ and averaging the classical expression over a period $T=(2\pi/\omega)$ of the classical motion,

$$\begin{aligned} M_n &= T^{-1}\int_0^T dt\, H(Y(x,n,\omega t)) \\ &= H(Y(x,n,\omega t)) \\ &= (2M/m)[m^2-\omega^2(n)]^{1/2} \\ &= 2M\sin(nm/2M), \end{aligned} \tag{2.19}$$

which follows from the recognition that $H(Y)$ is independent of t, that its value for *arbitrary* ω is given by (2.3), and that in this case $\omega(n)$ is given by (2.7).

To summarize what has been found, we may say that we have identified the quantum field operator in the no-recoil, no-loop approximation by transcription of the breather solution (1.7). In Secs. IV and V, we shall carry through the study of the first quantum corrections to this result. But prior to that we shall deal with a different question. Suppose that we have a general self-coupled boson field in one-plus-one dimensions, where the interaction is given as a power series in $\phi(x)$, and that an exact classical solution analogous to (1.7) is not known. We shall demonstrate in Sec. III that, nevertheless, in a suitable weak-coupling regime, where the interaction is dominated by its lowest-order terms, we can generate a viable sequence of approximations. We shall apply the method[12] to the sine-Gordon theory and show that we generate the leading terms of (2.12) and then show how to apply the method to the quartic interaction model studied in I.

III. SYSTEMATIC WEAK-COUPLING EXPANSION

A. Sine-Gordon theory

We attempt to generate the series (2.9) directly as such. It turns out that our methods yield, fairly naturally, a series in the functions $\sin\alpha/\cosh[m(\sin\alpha)x]$, $\alpha=(nm/2M)$, rather than the functions $y(n,x)$ defined in (1.9) and that the energy is generated as a series in powers of $\sin\alpha$. Only in the sine-Gordon theory do powers beyond the first vanish. The form of (2.9) suggests in any event that the leading approximation should be based on retaining only the amplitude

$$\langle n|\,\hat{\phi}(x)\,|n+1\rangle=(n|\,\hat{Y}\,|n+1)\equiv\psi_n^{(1)}(x)\sim y(n,x) \tag{3.1}$$

and setting all other amplitudes to zero. As indicated, this should yield roughly the first term of the series (2.9). In the second approximation we add the amplitude

$$\langle n|\,\hat{\phi}(x)\,|n+3\rangle=(n|\,\hat{Y}\,|n+3)\equiv\psi_n^{(3)}(x)\sim[y(n,x)]^3\,, \tag{3.2}$$

which should be obtained in leading order at the same time that we obtain the second approximation to (3.1).

To obtain equations for the amplitudes (3.1) and (3.2), we utilize the operator equation (1.2) in expanded form

$$\partial_x{}^2\hat{\phi}(x)+[[\hat{\phi}(x),H],H]=m^2\hat{\phi}-\frac{1}{3!}\lambda\hat{\phi}^3+\frac{1}{5!}\lambda\left(\frac{\lambda}{m^2}\right)\hat{\phi}^5+\cdots\,. \tag{3.3}$$

From the result for $\psi_n^{(1)}$, it will be evident that quintic and higher terms in (3.3) may be ignored during the first go-round. We require the precise definition

$$\omega_n=E_n-E_{n-1}\,, \tag{3.4}$$

as well as the approximate evaluation

$$\begin{aligned}\langle n|\,\hat{\phi}^3|\,n+1\rangle &=\langle n|\,\hat{\phi}\,|n+1\rangle\langle n+1|\,\hat{\phi}\,|\,n\rangle\langle n|\,\hat{\phi}\,|\,n+1\rangle\\ &+\langle n|\,\hat{\phi}\,|\,n+1\rangle\langle n+1|\,\hat{\phi}\,|n+2\rangle\langle n+2|\,\hat{\phi}\,|n+1\rangle\\ &+\langle n|\,\hat{\phi}\,|\,n-1\rangle\langle n-1|\,\hat{\phi}\,|\,n\rangle\langle n|\,\hat{\phi}\,|n+1\rangle\\ &\cong 3[\psi_n^{(1)}(x)]^3\,.\end{aligned} \tag{3.5}$$

[The number-smearing approximation in (3.5) is equivalent to ignoring operator ordering problems encountered in Sec. II. However, here, we are in a position, if we wish, to evaluate correction terms, and the influence of such correction, if any, on the energy will be considered at the appropriate juncture.]

From (3.3) to (3.5) we obtain a familiar equation for $\psi_n^{(1)}$, namely

$$[-\partial_x{}^2+(m^2-\omega_n{}^2)]\psi_n^{(1)}-\tfrac{1}{2}\lambda(\psi_n^{(1)})^3=0\,, \tag{3.6}$$

which has the solution

$$\psi_n^{(1)}(x)=\frac{2}{\sqrt{\lambda}}(m^2-\omega_n{}^2)^{1/2}\frac{1}{\cosh(m^2-\omega_n{}^2)^{1/2}x}\,. \tag{3.7}$$

When we return to the computation of the energy, we shall find the value of ω_n without reference to any previous considerations, but since we know that

$$\psi_n^{(1)}\sim\frac{m}{\sqrt{\lambda}}\sin\alpha\,, \tag{3.8}$$

we may verify that the omitted quintic term in (3.3) is $O(\sin^2\alpha)$ compared to the cubic term. Therefore it will have to be considered only in the next approximation, which we give in bare outline.

We first consider the equation for $\psi_n^{(3)}\sim(\psi_n^{(1)})^3$. With the help of the approximation

$$E_{n+3}-E_n\cong 3\omega_n \tag{3.9}$$

and the approximate evaluations

$$\langle n|\,\hat{\phi}^3|\,n+3\rangle\cong(\psi_n^{(1)})^3+6(\psi_n^{(1)})^2\psi_n^{(3)}\,, \tag{3.10}$$

$$\langle n|\,\hat{\phi}^5|\,n+3\rangle\cong 5(\psi_n^{(1)})^5\,, \tag{3.11}$$

we obtain for $\psi_n^{(3)}$ the equation

$$(-\partial_x{}^2+m^2-9\omega_n{}^2)\psi_n^{(3)}-\frac{\lambda}{6}(\psi_n^{(1)})^3-\lambda(\psi_n^{(1)})^2\psi_n^{(3)}+\frac{\lambda}{24}\left(\frac{\lambda}{m^2}\right)(\psi_n^{(1)})^5=0\,, \tag{3.12}$$

which is solved as

$$\psi_n^{(3)}(x)=-\frac{1}{6\sqrt{\lambda}}\frac{1}{m^2}(m^2-\omega_n{}^2)^{3/2}\times\frac{1}{\cosh^3(m^2-\omega_n{}^2)^{1/2}x}\,. \tag{3.13}$$

Incidentally both $\psi_n^{(1)}$ and $\psi_n^{(3)}$ check with the corresponding leading terms of (2.9) if the known value of ω_n is utilized.

We next turn to the correction to $\psi_n^{(1)}$. With the improved evaluations (which take account of the presence of $\psi_n^{(3)}$ to the required order)

$$\langle n|\,\hat{\phi}^3\,|n+1\rangle \cong 3(\psi_n^{(1)})^3 + 3(\psi_n^{(1)})^2\psi_n^{(3)}\,, \tag{3.14}$$

$$\langle n|\,\hat{\phi}^5\,|n+1\rangle \cong 10(\psi_n^{(1)})^5\,, \tag{3.15}$$

we find

$$(-\partial_x^{\,2} + m^2 - \omega_n^{\,2})\psi_n^{(1)} - \tfrac{1}{2}\lambda(\psi_n^{(1)})^3 - \frac{\lambda}{2}\lambda(\psi_n^{(1)})^2\psi_n^{(3)} + \frac{\lambda}{12}\left(\frac{\lambda}{m^2}\right)(\psi_n^{(1)})^5 = 0\,. \tag{3.16}$$

This equation suffices to determine $\psi_n^{(1)}$ up to second order in the form

$$\psi_n^{(1)}(x) = \frac{a}{\cosh(m^2 - \omega_n^{\,2})^{1/2}x} + \frac{c}{\cosh^3(m^2 - \omega_n^{\,2})^{1/2}x}\,, \tag{3.17}$$

where

$$a = \frac{2}{\sqrt{\lambda}}(m^2 - \omega_n^{\,2})^{1/2}\left[1 + \frac{1}{2}\left(1 - \frac{\omega_n^{\,2}}{m^2}\right)\right] \tag{3.18}$$

and

$$c = -\frac{1}{2\sqrt{\lambda}}\,\frac{(m^2 - \omega_n^{\,2})^{3/2}}{m^2}\,. \tag{3.19}$$

Equations (3.17)–(3.19) again check with (2.9). It should be clear that the evaluation of higher-order terms is only a matter of care and patience.

To complete the present calculation we require a quantization condition. We present one which contains the previous semiclassical limit as a special case: If we evaluate

$$E_n = \langle n|H(\hat{\phi}(x))|n\rangle \tag{3.20}$$

in terms of the solutions found above, we will obtain

$$E_n = E_n(\omega_n) \tag{3.21}$$

as a quantum difference equation for the energy, in consequence of (3.4). We therefore now turn to the calculation of the energy which also will allow us to show that (3.12) and (3.16) can be derived from the trace variational principle studied in I. The energy is evaluated by the technique of intermediate sums familiar by now and turns out to be

$$E_n = \int dx\Big[(\omega_n^{\,2}+m^2)\psi_n^{(1)2} + (9\omega_n^{\,2}+m^2)\psi_n^{(3)2} + (\partial_x\psi_n^{(1)})^2 + (\partial_x\psi_n^{(3)})^2 - \frac{\lambda}{4}\psi_n^{(1)4} - \frac{\lambda}{3}\psi_n^{(1)3}\psi_n^{(3)} - \lambda\psi_n^{(1)2}\psi_n^{(3)2} + \frac{\lambda}{36}\left(\frac{\lambda}{m^2}\right)\psi_n^{(1)6} + \frac{\lambda}{12}\left(\frac{\lambda}{m^2}\right)\psi_n^{(1)5}\psi_n^{(3)}\Big]. \tag{3.22}$$

To make the connection to the trace variational principle studied in I we first note that because we neglect n-fluctuations the trace variational principle becomes a simple variational principle. This holds as long as the trace is taken over a range of states not greater than $\sqrt{n}$ about n, a condition met in the semiclassical limit considered here.

In I we showed that the constrained variational principle is equivalent to an unconstrained variation of the Lagrangian. To obtain the latter variational expression from (3.22) we thus subtract

$$\delta\left(-2\omega_n^{\,2}\int\psi_n^{(1)2} - 18\omega_n^{\,2}\int\psi_n^{(3)2}\right) \tag{3.23}$$

from $\delta E_n = 0$, as this gives just the variation of the expectation value of the Lagrangian $L(\hat{\phi})$. It is then readily verified that variation of the resulting expression with respect to $\psi_n^{(1)}$ and $\psi_n^{(3)}$ yields (3.16) and (3.12), respectively.

As explained above, the solutions (3.17) and (3.13) to these equations can be viewed as a power series in $m^{-1}(m^2 - \omega_n^{\,2})^{1/2} = \sin\alpha$, which is small in the weak-coupling regime. A corresponding power series exists then for the energy of which series we shall consistently only calculate the first two terms. For this it suffices to calculate E_n from the expression

$$E_n \cong \int dx\left[(\omega_n^{\,2}+m^2)\psi_n^{(1)2} + (\partial_x\psi_n^{(1)})^2 - \frac{\lambda}{4}(\psi_n^{(1)})^4\right], \tag{3.24}$$

as all other terms are of higher order in $(m^2 - \omega_n^{\,2})^{1/2}$. To simplify (3.24) we use a virial theorem

$$\int dx\left[(m^2 - \omega_n^{\,2})\psi_n^{(1)2} - (\partial_x\psi_n^{(1)})^2 - \frac{\lambda}{4}(\psi_n^{(1)})^4\right] = 0\,, \tag{3.25}$$

which can be derived from (3.6). Combining (3.24) and (3.25) yields the expression, which is our quantum difference equation,

$$E_n = 2\int dx[m^2(\psi_n^{(1)})^2 - \tfrac{1}{4}\lambda(\psi_n^{(1)})^4] = (2M/m)(m^2 - \omega_n^{\,2})^{1/2}\left\{1 + O\left(\left(1 - \frac{\omega_n^{\,2}}{m^2}\right)^2\right)\right\}. \tag{3.26}$$

Here the term in the curly brackets only indicates the accuracy of the present calculation, since we already know that the exact semiclassical calculation, defined as the one carried out in Sec. II, replaces the term in the curly brackets by unity. It was this calculation which dictated which matrix elements have to be chosen as nonvanishing in the above calculation.

The replacement of ω_n by (dE/dn) in (3.26) yields a differential equation equivalent to the Wilson-Sommerfeld quantization, as one easily verifies. We shall return for a more complete consideration of the structure of the answer after carrying through for the quartic interaction model the calculations analogous to those performed in this subsection for the sine-Gordon model.

B. Quartic interaction model

We refer to the model defined by Eq. (1.10). This model also gives rise to a one-dimensional spectrum of bound states. For this spectrum DHN have given the formula

$$E_n = 2nm\left[1 - \frac{2}{32}\left(\frac{\lambda n}{m^2}\right)^2\right] \cong 2M \sin\frac{nm}{M}, \tag{3.27}$$

where $M = (4m^3/3\lambda)$ is the mass of the soliton for this model, and the factor within the sine reminds us that in this model the elementary particle has mass $2m$. Thus (3.27) is essentially equivalent to the result obtained for the sine-Gordon theory and suggests that we attempt to derive it using the method of this section.

The calculation is, however, slightly less trivial for the present model. Because we are in the vacuum sector, we must deal with the Hamiltonian obtained after carrying out a displacement $\phi \to (m/\sqrt{\lambda} + \phi)$ about the vacuum value of ϕ. This yields

$$H = \int dx[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_x\phi)^2 + 2m^2\phi^2 + 2m^2(\sqrt{\lambda}/m)\phi^3 + 2m^2(\lambda/4m^2)\phi^4], \tag{3.28}$$

and the corresponding equation of motion is

$$-\partial_t^2\phi = -\partial_x^2\phi + 4m^2\phi + 6m^2(\sqrt{\lambda}/m)\phi^2 + 2m^2(\lambda/m^2)\phi^3.$$

Since the ϕ^3 term corresponds to a repulsive interaction, binding, if it occurs, must be due entirely to the ϕ^2 term. The implication is that this term contributes at least at the same order as the ϕ^3 term.

A self-consistent scheme which yields the expected result in the leading approximation requires three nonvanishing amplitudes

$$\psi_n(x) \equiv \langle n-1|\hat{\phi}(x)|n\rangle \sim (m/\sqrt{\lambda})\sin(nm/M), \tag{3.29}$$

$$\sigma_n(x) \equiv \langle n|\hat{\phi}(x)|n\rangle \sim (\sqrt{\lambda}/m)\psi_n^2(x), \tag{3.30}$$

$$\tau_n(x) = \langle n-1|\hat{\phi}(x)|n+1\rangle \sim \sigma_n(x). \tag{3.31}$$

The estimates (3.29)–(3.31) could be made in advance, (3.29) from the work of Sec. III A and (3.30) and (3.31) from the anticipation that ϕ^2 and ϕ^3 would, in leading approximation, contribute on equal footing. We have also verified that additional amplitudes, omitted in the subsequent discussion, such as

$$\mu_n(x) \equiv \langle n-1|\hat{\phi}(x)|n+2\rangle \sim (\lambda/m^2)\psi_n^3(x) \tag{3.32}$$

are too small to be included in the first approximation.

We proceed as follows: If the estimates (3.29)–(3.31) are correct, then the equation for $\psi_n(x)$, derived from (3.29), is

$$0 = (4m^2 - \omega_n^2)\psi_n(x) - \partial_x^2\psi_n(x) + 12m^2(\sqrt{\lambda}/m)[\sigma_n(x) + \tau_n(x)]\psi_n(x) + 6m^2(\lambda/m^2)\psi_n^3(x). \tag{3.33}$$

Similarly, for $\sigma_n(x)$ and $\tau_n(x)$, we find the equations

$$0 = 4m^2\sigma_n(x) - \partial_x^2\sigma_n(x) + 12m^2(\sqrt{\lambda}/m)\psi_n^2(x) + 12m^2(\lambda/m^2)\psi_n^2(x)[\sigma_n(x) + \tau_n(x)], \tag{3.34}$$

$$0 = (4m^2 - 4\omega_n^2)\tau_n(x) - \partial_x^2\tau_n(x) + 6m^2(\sqrt{\lambda}/m)\psi_n^2(x) + 6m^2(\lambda/m^2)\psi_n^2(x)\sigma_n(x) + 12m^2(\lambda/m^2)\psi_n^2(x)\tau_n(x). \tag{3.35}$$

On the basis of the original estimates and the estimate [cf. (3.27)] $\omega_n \sim 2m$, only the first and third terms of (3.34) and (3.35) need be retained for the sufficiently accurate solutions

$$\sigma_n(x) \cong -3(\sqrt{\lambda}/m)\psi_n^2(x), \tag{3.36}$$

$$\tau_n(x) \cong \tfrac{1}{2}(\sqrt{\lambda}/m)\psi_n^2(x). \tag{3.37}$$

With the aid of these last results, we return to (3.33) which becomes

$$0 = (4m^2 - \omega_n^2)\psi_n(x) - \partial_x^2\psi_n(x) - 24\lambda\psi_n^3(x). \tag{3.38}$$

Comparing with (3.6), we see that (3.38) has the solution

$$\psi_n(x) = (12\lambda)^{-1/2}(4m^2 - \omega_n^2)^{1/2} \times \frac{1}{\cosh(4m^2 - \omega_n^2)^{1/2}x}. \tag{3.39}$$

From (3.38) we also have the virial theorem

$$0=(4m^2-\omega_n^2)\int dx\,\psi_n^2(x)-\int dx[\partial_x\psi_n(x)]^2 -12\lambda\int\psi_n^4(x). \qquad (3.40)$$

In (3.39) we have the analog of the quantity $\psi_n^{(1)}(x)$ computed for the sine-Gordon theory. It is also clear that we can generate the corrections of higher order, but that again the procedure would be somewhat more tedious than in the previous example.

We turn to the computation of the energy, where we soon realize that we cannot proceed blithely by insertion of the results (3.36), (3.37), and (3.39) into (3.28). The simplest way to make the necessary point is to remember that the energy is a quantity which is conserved in consequence of the equations of motion. Now since the effective equation of motion (3.38) has the same form as that for the weak-coupling sine-Gordon case, it follows from the effective Hamiltonian

$$H_{\text{eff}}=\int[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\partial_x\phi)^2+(4m^2)\tfrac{1}{2}\phi^2-2\lambda\phi^4], \quad (3.41)$$

where ϕ is now a transformed or effective operator which *possesses the same class of nonvanishing matrix elements as in the sine-Gordon theory*, in particular, no matrix elements of the type (3.36) and (3.37). Another way to express this result is to recognize that H_{eff} is the Hamiltonian that would result *approximately* by carrying out a unitary change of basis designed to eliminate the cubic term from (3.28). Such a transformation cannot be expected in the present theory to lead to a closed result, but would in the end generate an infinite series of interaction terms of higher order. What has been shown is that we can, equally well, generate the results of such a transformation in matrix element form.

The rest of our discussion will be simplified by adherence to the new Hamiltonian (3.41). Thus in Sec. IV we shall base our discussion of one-loop quantum corrections on it. In substantiation of the above remarks, we have, however, verified by a much more elaborate calculation that the same results can be obtained from the original form of the Hamiltonian.

From (3.41) we thus compute

$$\begin{aligned}E_n&=\langle n|H_{\text{eff}}|n\rangle\\&\cong(\omega_n^2+4m^2)\int\psi_n^2(x)+\int[\partial_x\psi_n(x)]^2-12\lambda\int\psi_n^4(x)\\&=8m^2\int\psi_n^2(x)-18\lambda\int\psi_n^4(x),\end{aligned} \quad (3.42)$$

where the second form is implied by (3.40). We must consistently discard the second term in (3.42), moreover, since as our study of the sine-Gordon theory implies, it is of the same order of magnitude as the contributions from the second approximation to $\psi_n(x)$, which we have not computed. We therefore find consistently

$$E_n\cong 8m^2\int\psi_n^2(x)=2M[1-(\omega_n^2/4m^2)]^{1/2}, \qquad (3.43)$$

which yields (3.27).

C. Further consideration of the energy

We conclude this long section with several additional observations concerning the energy quantization condition in the semiclassical approximation. First, our calculations for the two special examples indicates for the self-coupled neutral-boson model in one-plus-one dimensions that the Wilson-Sommerfeld quantization condition will take the form (in dimensionless units)

$$E_n=\left[1-\left(\frac{dE}{dn}\right)^2\right]^{1/2}+a\left[1-\left(\frac{dE}{dn}\right)^2\right]^{3/2}+\cdots. \qquad (3.44)$$

In the present section we have shown that $a=0$ for the sine-Gordon theory, but we have not computed a for the quartic interaction model. It is not difficult to show that the solution to (3.44) is of the form

$$E_n=\sin n+c\sin^5 n+\cdots \qquad (3.45)$$

and that $c\neq 0$ requires $a\neq 0$. Otherwise, the second term in (3.45) is at least the seventh power of the sine. The calculation of c for the general model under discussion appears to be a suitable exercise for an interested reader.

Secondly, it appears appropriate to consider at this point the leading n-fluctuation corrections to (3.44), completely neglected up to this point. These corrections have two sources. One of these arises from a consideration such as

$$\begin{aligned}\langle n|\hat\phi^2(x)|n\rangle&=(\langle n|\hat\phi|n+1\rangle)^2+(\langle n|\hat\phi|n-1\rangle)^2\\&\cong 2(\langle n|\hat\phi|n-1\rangle)^2+\frac{d}{dn}(\langle n|\hat\phi|n-1\rangle)^2.\end{aligned} \qquad (3.46)$$

Applied consistently to (3.41), for instance, we find

$$E_n=E_n^{(0)}+\frac{1}{2}\frac{d}{dn}E_n^{(0)}\equiv E_n^{(0)}+\Delta E_n^{(1)}, \qquad (3.47)$$

where by $E_n^{(0)}$ we mean [in the units of (3.44)]

$$E_n^{(0)} = (1-\omega_n^2)^{1/2}. \tag{3.48}$$

Since, however, $\omega_n = E_n - E_{n-1}$, there turns out to be a compensatory correction[13] obtained by writing

$$\omega_n \cong \frac{dE}{dn} - \frac{1}{2}\frac{d^2E}{dn^2}. \tag{3.49}$$

Inserted into (3.47), this yields a correction term

$$\Delta E_n^{(2)} = \frac{1}{2}\frac{(dE/dn)(d^2E/dn^2)}{[1-(dE/dn)^2]^{1/2}} = -\Delta E_n^{(1)}, \tag{3.50}$$

the complete compensation being a *special* result following from $E_n = \sin n$.

The upshot of this calculation is that quantum corrections arising from n-fluctuation are absent, at least in lowest order. We are finally free to turn to the remaining first quantum corrections, the one-loop effects.

IV. FIRST QUANTUM CORRECTIONS: WEAK-COUPLING LIMIT

A. Associated scattering problem

The considerations here are analogous to those which have been carried out for the nonlinear Schrödinger equation, where the physics has been described completely. To compute one-loop corrections, we need the scattering wave function,

$$\chi_{n,k} \equiv \langle n-1|\hat\phi(x)|n-1,k\rangle, \tag{4.1}$$

involving some suitable (here unspecified) boundary condition on the scattering state and the bound state "correlation amplitude,"

$$\langle n|\hat\phi(x)|n-2,k\rangle = \eta^*_{n,k}, \tag{4.2}$$

which as we learned in II, must be considered on an equal footing. The equations for these quantities will be obtained from H_{eff}, Eq. (3.41), which yields the field equation

$$(\partial_t^2 - \partial_x^2)\hat\phi(x,t) + 4m^2\hat\phi(x,t) - 8\lambda\hat\phi^3(x,t) = 0, \tag{4.3}$$

of which one matrix element, in the form (3.38), has already been computed.

We here apply (4.3) to the computation of the matrix elements (4.1) and (4.2). The special points of the computation are that in the matrix elements of $\hat\phi^3$ we keep only terms linear in the new amplitudes, and in the equation for η^* we encounter the energy difference

$$[-2\omega_n + (k^2+4m^2)^{1/2}]^2 - 4m^2 \cong -k^2 - 8m^2\sin\alpha, \tag{4.4}$$

where $\alpha = (nm/M)$. This evaluation is accurate only for $k^2 \sim m^2\sin^2\alpha \ll m^2$ and therefore must be discussed. Recall that we shall utilize the results of our calculation to compute a correction to the energy, which will involve an integral over k, by no means confined to small k. The integral will, however, turn out to involve only the phase shift and, at least for the sine-Gordon equation, the approximation under study will be verified to yield the exact phase shift.

With this caution, we obtain at this stage the coupled equations

$$k^2\chi_k(x) = -\partial_x^2\chi_k(x) - 48\lambda\psi_n^2(x)\chi_k(x) - 24\lambda\psi_n^2(x)\eta_k^*(x), \tag{4.5}$$

$$(-k^2 - 8m^2\sin^2\alpha)\eta_k^*(x) = -\partial_x^2\eta_k^*(x) - 48\lambda\psi_n^2(x)\eta_k^*(x) - 24\lambda\psi_n^2(x)\chi_k(x), \tag{4.6}$$

or in terms of the dimensionless variables,

$$k = (2m\sin\alpha)\nu, \quad z = (2m\sin\alpha)x, \tag{4.7}$$

$$\nu^2\chi_\nu = -\partial_z^2\chi_\nu - \frac{4}{\cosh^2 z}\chi_\nu - \frac{2}{\cosh^2 z}\eta_\nu^*, \tag{4.8}$$

$$(-2-\nu^2)\eta_\nu^* = -\partial_z^2\eta_\nu^* - \frac{4}{\cosh^2 z}\eta_\nu^* - \frac{2}{\cosh^2 z}\chi_\nu. \tag{4.9}$$

In this last form the equations also apply to the sine-Gordon equation in the weak-coupling limit, except that when we return to dimensional variables, we must replace $2m$ by m and put in the appropriately different value of α.

Equations (4.8) and (4.9) have been encountered in our study of the cubic Schrödinger equation. We may therefore record the solution corresponding to a particle incident from $x = -\infty$:

$$(2E_k L)^{1/2}\chi_\nu = \frac{e^{i\nu z}}{(\nu^2-1)-2i\nu} \times \left(\nu^2 - 1 + 2i\nu\tanh z + \frac{1}{\cosh^2 z}\right), \tag{4.10}$$

$$(2E_k L)^{1/2}\eta_\nu^* = \frac{e^{i\nu z}}{(\nu^2-1)-2i\nu}\,\frac{1}{\cosh^2 z}, \tag{4.11}$$

$$E_k = (k^2+4m^2)^{1/2}. \tag{4.12}$$

In terms of the dimensionless variables, this differs from the nonrelativistic case only in the energy factor $(2E_k)^{1/2}$ required by the Klein-Gordon normalization in a "box,"

$$\lim_{L\to\infty}\int dx[|\chi_\nu(x)|^2 - |\eta_\nu(x)|^2] = \frac{1}{2E_k}. \tag{4.13}$$

From (4.10), we read the phase shift

$$\delta(\nu)=\tan^{-1}\frac{2\nu}{\nu^2-1}. \tag{4.14}$$

B. Calculation of the energy

The one-loop correction, termed $\Delta E^{(3)}$, to the energy, calculated from $H_{\rm eff}$ by the same approximation that led to (4.5) and (4.6) takes the form

$$\Delta E^{(3)}=\sum_\nu\int dx[(k^2+4m^2)\tfrac{1}{2}|\xi_\nu|^2+(4m^2)\tfrac{1}{2}|\xi_\nu|^2+\tfrac{1}{2}|\partial_x\xi_\nu|^2+(-k^2+4m^2-8m^2\sin^2\alpha)\tfrac{1}{2}|\eta_\nu|^2$$
$$+(4m^2)\tfrac{1}{2}|\eta_\nu|^2+\tfrac{1}{2}|\partial_x\eta_\nu|^2-24\lambda(\psi)^2(|\chi_\nu|^2+|\eta_\nu|^2)-12\lambda(\psi)^2(\chi_\nu\eta_\nu+\chi_\nu^*\eta_\nu^*)]. \tag{4.15}$$

This is simplified considerably by combination with a virial theorem derivable from (4.5) and (4.6), namely

$$\sum\int[k^2(|\chi|^2-|\eta|^2)-8m^2\sin^2\alpha|\eta|^2]=\sum\int[(|\partial_x\chi|^2+|\partial_x\eta|^2)-48\lambda\psi^2(|\chi|^2+|\eta|^2)-24\lambda\psi^2(\chi\eta+\chi^*\eta^*)], \tag{4.16}$$

giving

$$\Delta E^{(3)}=\sum\int(k^2+4m^2)(|\chi|^2-|\eta|^2)$$
$$+\sum\int 8m^2\cos^2\alpha|\eta|^2$$
$$=\sum_k\tfrac{1}{2}E_k+\sum\int 8m^2\cos^2\alpha|\eta|^2. \tag{4.17}$$

Here, Eq. (4.13) has been used to simplify the first term.

The final expression for the one-loop quantum correction has three additional contributions: (i) One is a contribution $\Delta E^{(4)}$ which is the change in the energy due to the change in the normalization of the lowest-order function $\psi_n(x)$ induced by the inclusion of one-loop corrections. Neither of two methods described in II for the derivation of this correction is applicable here, and we must have recourse to the variational principle associated with our method. The argument is described in Sec. IV C below. The result is

$$\Delta E^{(4)}=-\sum\int 8m^2\cos^2\alpha|\eta|^2, \tag{4.18}$$

which cancels the second term of (4.17). (ii) We must add a term which guarantees that the vacuum expectation value of $H_{\rm eff}$ vanishes to the first loop approximation. As is well known, this is the expression

$$\Delta E^{(5)}=-\sum_k{}'\tfrac{1}{2}E_k, \tag{4.19}$$

where the distinction between the unprimed (interacting) sum and primed (noninteracting) sum can only be enforced by starting with L finite, as described by DHN. (iii) Finally, we must add a self-energy correction obtained by writing for the lowest-order energy

$$E_n^{(0)}=8\left(\frac{m^2}{\lambda}\right)(m_0{}^2)^{1/2}\sin\alpha$$
$$\cong\frac{8m^3}{\lambda}\sin\alpha-\frac{4m}{3\lambda}\delta m^2\sin\alpha. \tag{4.20}$$

Since

$$\delta m^2=-\frac{3}{2\pi}\lambda\int_{-\Lambda}^{\Lambda}\frac{dk}{(k^2+4m^2)^{1/2}}, \tag{4.21}$$

where Λ is a cutoff, from (4.20) we obtain

$$\Delta E^{(6)}=\frac{2m\sin\alpha}{\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{(k^2+4m^2)^{1/2}}. \tag{4.22}$$

The combination of (4.17), (4.18), and (4.19) first yields[14]

$$\Delta E^{(3)}+\Delta E^{(4)}+\Delta E^{(5)}$$
$$=\sum\tfrac{1}{2}E_k-\sum{}'\tfrac{1}{2}E_k$$
$$=-\int_{-\infty}^{\infty}\frac{dE_k}{2\pi}\delta(\nu)-m$$
$$=-\frac{4m}{\pi}\sin\alpha-\frac{2m}{\pi}\sin\alpha\int_{-\Lambda}^{\Lambda}\frac{dk}{E_k}\,\frac{k^2+4m^2}{k^2+4m^2\sin^2\alpha}, \tag{4.23}$$

which results from an integration by parts.

Finally, combination with (4.22) yields a finite integral

$$\Delta E=-\frac{4m}{\pi}\sin\alpha$$
$$-\frac{2m}{\pi}\sin\alpha\cos^2\alpha\int_{-\infty}^{\infty}\frac{d\mu}{(\mu^2+1)^{1/2}(\mu^2+\sin^2\alpha)}$$
$$=-\frac{4m}{\pi}\sin\alpha-\frac{4m}{\pi}\cos\alpha\left(\frac{\pi}{2}-\alpha\right). \tag{4.24}$$

This is the result for the quartic interaction model. For the sine-Gordon theory, the correct result is obtained by the replacements (quartic → sine-Gordon)

$$m \to \tfrac{1}{2}m, \qquad \frac{nm}{M} \to \frac{nm}{2M}, \tag{4.25}$$

where the same symbol M is used for the soliton mass special to each theory.

To understand the result (4.24), we consider the total energy $E_n^{(0)} + \Delta E_n = E_n$ and demand

$$\frac{dE_n}{dn} = \omega_n = m \cos\alpha$$

$$= 2M \cos\alpha \frac{d\alpha}{dn} + \frac{2m}{\pi}\left(\frac{\pi}{2} - \alpha\right)\sin\alpha \frac{d\alpha}{dn}, \tag{4.26}$$

so that (for the sine-Gordon theory)

$$\frac{d\alpha}{dn} = \frac{m}{2M} + O\left(\left(\frac{m}{M}\right)^3 n\right). \tag{4.27}$$

We therefore take with c a constant of integration

$$\alpha = \frac{m}{2M}(n+c). \tag{4.28}$$

Treating (4.24) as first terms of a Taylor expansion, this allows us to write

$$E_n = 2M' \sin\frac{nm(n-1+c)}{2M'}, \tag{4.29}$$

where M' is the soliton mass including the first quantum correction, namely

$$M' = \frac{8m^3}{\gamma'}, \quad \gamma' = \lambda\left[1 - \left(\frac{\lambda}{8\pi m^2}\right)\right]^{-1}. \tag{4.30}$$

To get a sensible result for $n=1$, i.e., to require that $E_1 \to m$ as $\lambda \to 0$, we choose $c=1$, which is the *same* choice as made by DHN.[15]

For the quartic interaction there is a similar result, only with

$$\gamma' = \lambda\left(1 - \frac{3}{2\pi}\frac{\lambda}{m^2}\right)^{-1}. \tag{4.31}$$

The resulting M' is *not* the mass of the ϕ^4 soliton. Here the effect of higher-order terms needs to be considered.

C. Normalization condition

We have left one loophole in the argument of Sec. IV B, namely, the derivation of condition (4.18). Here the arguments given in II will not work. Fortunately, the variational principle underlying our approach provides the required information. In the semiclassical limit, we have

$$\delta\langle n|L|n\rangle = \delta\langle n|H|n\rangle - \delta\left\langle n\left|\int \pi^2(x)\right|n\right\rangle = 0, \tag{4.32}$$

i.e., as previously remarked, we may drop the trace calculation and be assured that our c-number matrix-element equations follow from (4.32) (as we have indeed shown by repeated example). For a solution, however, the Rayleigh-Ritz principle requires the separate vanishing

$$\delta\left\langle n\left|\int \pi^2(x)\right|n\right\rangle = 0. \tag{4.33}$$

Now a calculation gives

$$\left\langle n\left|\int \pi^2(x)\right|n\right\rangle = \omega_n^2\int \psi_n^2 + \omega_{n+1}^2\int \psi_{n+1}^2 + \sum_k\int E_k^2(|\xi_k|^2 - |\eta_k|^2) + \sum_k\int [E_k^2 + (E_k - 2\omega_n)^2]|\eta_k|^2. \tag{4.34}$$

The condition (4.33) can then be considered for our purposes in two steps. In the no-loop approximation it tells us that

$$\delta\int \psi_n^2 = 0 \quad \text{(no loop)}, \tag{4.35}$$

which is not directly useful because we do not know what the norm is until after we have quantized some other way. However, when we allow one-loop corrections, (4.33) and (4.34) tell us the *change* in the norm due to the one-loop corrections. In fact, remembering (4.13) and the approximation (4.4), we find

$$2\omega^2\delta\int \psi_n^2 = -\sum\int 8m^2\cos^2\alpha\,|\eta_k|^2, \tag{4.36}$$

where the left-hand side is the computed change in the *classical* energy due to one-loop renormalization of the classical solution ψ_n, and thus (4.18) is justified.

V. FIRST QUANTUM CORRECTIONS TO SINE-GORDON THEORY: A COMPLETE CALCULATION

A. The scattering operator

This section can be considered as the logical continuation of Sec. II. First, we construct a semiclassical "scattering operator" $\psi_k^\dagger(x,\hat{n},\hat{\theta})$ defined by the equation [cf. (2.11)]

$$(n|\psi_k^\dagger(x,\hat{n},\hat{\theta})|n') \equiv \langle n|\hat{\phi}(x)|n', k^{(+)}\rangle. \tag{5.1}$$

We then show that the first quantum correction can be calculated from a phase shift associated with this operator.

To find the quantity $\psi_k^\dagger$ we take a matrix of Eq. (2.12) and if in the nonlinear interaction term we retain terms linear in $\psi_k^\dagger$ only and use a semiclassical approximation, assuming that $\psi_k^\dagger$ commutes approximately with Y, we obtain the semiclassical operator equation

$$E_k{}^2\psi_k^\dagger + 2E_k[\psi_k^\dagger, h] + [[\psi_k^\dagger, h], h] + \partial_x{}^2\psi_k^\dagger - m^2\cos[(\sqrt{\lambda}/m)Y(x,\hat{n},\hat{\theta})]\psi_k^\dagger = 0 \quad (5.2)$$

in the space of the model oscillator states (n) [cf. (2.15)]. At this point, it is consistent to view the commutator brackets as Poisson brackets and therefore to view the entire equation as a c-number equation where

$$[\psi_k^\dagger, h] = i\omega\frac{\partial\psi_k^\dagger}{\partial\theta} - i\frac{\partial}{\partial t}\psi_k^\dagger(x,n,\omega t). \quad (5.3)$$

Writing

$$\psi_k^\dagger = X_k e^{ikx}, \quad (5.4)$$

and introducing the explicit form for $Y(x,n,\theta)$, (2.5), we must solve the equation

$$\left(\partial_x{}^2 + 2ik\partial_x - m^2\cos^2\alpha\,\partial_\theta{}^2 + 2imE_k\cos\alpha\,\partial_\theta + \frac{8m^2\cosh^2 z\tan^2\alpha\cos^2\theta}{\cosh^2 z + \tan^2\alpha\cos^2\theta}\right)X_k = 0, \quad (5.5)$$

where

$$z = m\sin\alpha\, x, \quad \alpha = (nm/2M) \quad (5.6)$$

as before.

The exact solution of (5.5) normalized to (see below)

$$\lim_{L\to\infty}\int dx\,|X_k(x,n,\theta)|^2 = (2E_k)^{-1} \quad (5.7)$$

is

$$(2E_kL)^{1/2}X_k(x,n,\theta) = \frac{1}{2}(1+e^{2i\delta}) + \frac{1}{2}\frac{\cosh z\sinh z\,(e^{2i\delta}-1)}{D} - \frac{1}{2}\frac{\tan^2\alpha\cos^2\theta\,(1+e^{2i\delta})}{D} - \frac{1}{4}\frac{\tan^2\alpha(1+e^{-2i\theta})}{D}\left(\rho e^{i\delta} + \frac{1}{\rho e^{i\delta}}\right), \quad (5.8)$$

where

$$D = \cosh^2 z + \tan^2\alpha\cos^2\theta, \quad (5.9)$$

$$\rho = \left[\frac{\sin^2\alpha\, k^2 + (m - E_k\cos\alpha)^2}{\sin^2\alpha\, k^2 + (m + E_k\cos\alpha)^2}\right]^{1/2}, \quad (5.10)$$

and

$$\tan\delta = \frac{2\nu}{\nu^2 - 1}, \quad \nu = \frac{k}{m\sin\alpha} \quad (5.11)$$

is the same phase shift as we found in the approximate theory of Sec. IV.

That Eq. (5.8) satisfies (5.5) we leave as an exercise to the reader. We did not quite discover this solution, but in fact, "translated" it from Appendix C of DHN (up to some factors of 2) using the dictionary described in Sec. II and transforming standing-wave to outgoing-wave solutions. We have verified our normalization by showing that to the required order in k^2 and $m^2\sin^2\alpha$ that

$$(n|\psi_k^\dagger(x,\hat{n},\hat{\theta})|n) \cong \xi_k(x), \quad (5.12)$$

$$(n|\psi_k^\dagger|n-2) \cong \eta_k^*(x), \quad (5.13)$$

and therefore the theory just presented contains the linearized scattering theory of Sec. IV in the limit. [To aid the enterprising reader, we note that the contributions to (5.12), (5.13) come from the first two terms of (5.8) and from the part of the last term proportional to $(\rho e^{i\delta})^{-1}$.]

B. Quantum correction to the energy

We now utilize the scattering operator of Sec. V A to calculate the one-loop contribution to the energy. We start from the expression

$$E_n = \left\langle n\left|\int\left\{\frac{1}{2}[H,\hat{\phi}][\hat{\phi},H] + \frac{1}{2}(\partial_x\hat{\phi})^2 - \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\hat{\phi}\right) - 1\right]\right\}\right|n\right\rangle \quad (5.14)$$

and do the "sum over intermediate states." The purely bound-state contribution, $E_n^{(0)}$, was computed in Sec. II. We wish to add the contribution which is bilinear in ψ_k. By the same approximations used to derive (5.2), we then find the expression

$$\text{“}\Delta E\text{”} = \left(n\left|\int dx\{\tfrac{1}{2}E_k{}^2|\psi_k|^2 + \tfrac{1}{2}[h,\psi_k^\dagger][\psi_k,h] + \tfrac{1}{2}|\partial_x\psi_k|^2 - \tfrac{1}{2}m^2\cos[(\sqrt{\lambda}/m)Y]|\psi_k|^2\}\right|n\right). \quad (5.15)$$

By the same argument as used in Sec. II, this expression can be replaced by a time average over a period. Let angular brackets without further indication refer either to (5.15) or to the associated classical average and we shall also write

$$[\psi_k, h] = i\partial_t \psi_k. \qquad (5.16)$$

We proceed to simplify (5.15). First we combine it with a virial theorem derivable from (5.2), namely

$$0 = \langle\{E_k{}^2|\psi_k|^2 + |\partial_t\psi_k|^2 - |\partial_x\psi_k|^2 - m^2\cos[(\sqrt{\lambda}/m)Y]|\psi_k|^2\}\rangle, \qquad (5.17)$$

thus obtaining

$$\text{“}\Delta E\text{”} = \langle(E_k|\psi_k|^2 + |\partial_t\psi_k|^2)\rangle = \tfrac{1}{2}\sum_k E_k + \langle|\partial_t\psi_k|^2\rangle. \qquad (5.18)$$

This expression is the analog of $\Delta E^{(3)}$, Eq. (4.17). To obtain a final expression which truly represents the first quantum correction, we must take the additional steps which follow (4.17). These steps are the same in detail as in the linearized case except for the argument which cancels the second term of (5.18) (see below). The final result is the same as found previously, since the linearized calculation gave the correct phase shift.

Finally, we give the argument which leads to cancellation of the second term of (5.18). From the classical expression,

$$E_n^{(0)} \equiv E_c = \langle\{\tfrac{1}{2}(\partial_t Y)^2 + \tfrac{1}{2}(\partial_x Y)^2 - (m^4/\lambda)[\cos(\sqrt{\lambda}/m)Y - 1]\}\rangle \qquad (5.19)$$

and

$$\delta E_c = \delta\langle(\partial_t Y)^2\rangle, \qquad (5.20)$$

utilizing the classical field equation for Y. This change due to one-loop effects can be computed from the constancy of

$$\left\langle n\middle|\int \pi^2\middle|n\right\rangle = \langle\{(\partial_t Y)^2 - E_k{}^2|\psi_k|^2 + |\partial_t\psi_k|^2\}\rangle \qquad (5.21)$$

and the separate constancy of the second term on the right-hand side. We thus obtain

$$\delta E_c = \delta\langle(\partial_t Y)^2\rangle = -\langle|\partial_t\psi_k|^2\rangle, \qquad (5.22)$$

as required.

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APPENDIX A: LORENTZ COVARIANCE

In this appendix we shall investigate the Lorentz transformation properties of the states associated with the classical breather solution. These states are interpreted as heavy particles and their energy and momentum should thus transform like the components of a Lorentz vector. In I we extensively considered this question for the soliton of the ϕ^4 theory and established proper Lorentz covariance for that case. As we shall essentially follow the same procedure and use the same methods here, we shall be brief and refer the reader to I for details. As an example we shall treat the sine-Gordon theory in the weak-coupling regime. It is straightforward to extend the results away from this limit and also to the ϕ^4 theory. We shall first derive the equations for ψ_n, χ_{nk}, and η^*_{nk} with the inclusion of recoil and then calculate the energy to demonstrate Lorentz covariance.

In the weak-coupling limit, keeping only the first terms in the expansion of the interaction, the sine-Gordon Hamiltonian becomes

$$H = \int dx\left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\partial_x\hat{\phi})^2 + \frac{m^2}{2}\hat{\phi}^2 - \frac{\lambda}{4!}\hat{\phi}^4\right], \qquad (A1)$$

from which follows the equation of motion

$$\partial_t{}^2\hat{\phi} - \partial_x{}^2\hat{\phi} + m^2\hat{\phi} - \frac{\lambda}{6}\hat{\phi}^3 = 0. \qquad (A2)$$

To find, including recoil effects, the analog to $\psi_n^{(1)}$, Eq. (3.1), we take matrix elements of (A2) between appropriate states of the set $\{|n(p)\rangle\}$. This yields

$$\{[E_{n-1}(p) - E_n(q)]^2 - (q-p)^2 - m^2\}\langle n-1(p)|\hat{\phi}|n(q)\rangle = \frac{\lambda}{6}\langle n-1(p)|\hat{\phi}^3|n(q)\rangle. \qquad (A3)$$

Defining the Fourier transform

$$\psi_{np}(x) = \int\frac{dq}{2\pi}e^{i(q-p)x}\langle n-1(p)|\hat{\phi}|n(q)\rangle, \qquad (A4)$$

(A3) is transformed into

$$-[E_{n-1}(p) - E_n(\hat{p}+p)]^2\psi_{np}(x) - \partial_x{}^2\psi_{np}(x) + m^2\psi_{np}(x) = \frac{\lambda}{6}\int\frac{dq}{2\pi}e^{i(q-p)x}\langle n-1(p)|\hat{\phi}^3|n(q)\rangle, \qquad (A5)$$

where $\hat{p} = -i\partial_x$. For large enough n, $M_n - M_{n-1} = \omega_n \ll M_n$, and since $\langle\hat{p}\rangle \sim m \ll M$, we may expand to find

$$E_{n-1}(p) - E_n(\hat{p}+p) \cong \omega_n(1-v^2)^{1/2} + v\hat{p}, \tag{A6}$$

where

$$v = \frac{p}{E_{n-1}(p)}. \tag{A7}$$

The left hand side of (A5) becomes

$$\left[(1-v^2)\partial_x^{\,2} - 2i\frac{v\omega_n}{(1-v^2)^{1/2}}\partial_x + \omega_n^{\,2} - \frac{m^2}{1-v^2}\right]\psi_{np}. \tag{A8}$$

The right-hand side is treated in the same way as described in detail in I. If, in addition, the phase of ψ_{np} is changed by

$$\psi_{np}(x) = \exp\left[\frac{iv\omega x}{(1-v^2)^{1/2}}\right]X_{np}(x), \tag{A9}$$

so as to eliminate the linear derivative term in (A8), then we obtain as equation for X_{np}

$$[(1-v^2)\partial_x^{\,2} + \omega_n^{\,2} - m^2]X_{np}(x) + \frac{\lambda}{2}|X_{np}(x)|^2 X_{np}(x) = 0. \tag{A10}$$

This equation has the solution

$$X_{np}(x) = X_n^0\left(\frac{x}{(1-v^2)^{1/2}}\right), \tag{A11}$$

where X_n^0 is a solution of (A10) for $v=0$ or, equivalently, is a solution of (3.6).

If the right-hand side of (A3) is evaluated more carefully to the one-loop level, then (A10) is modified to

$$[(1-v^2)\partial_x^{\,2} + \omega_n^{\,2} - m^2]X_{np}(x) + \frac{\lambda}{2}\left\{|X_{np}(x)|^2 + \int \frac{dk}{2\pi}[|\chi_{nk}(x,v,\omega_k)|^2 + |\eta_{nk}(x,v,\omega_k)|^2]\right\}X_{np}(x) = 0, \tag{A12}$$

where the amplitudes χ and η are defined below in (A18) and (A26), respectively.

The equations for χ and η^* are found in an entirely analogous fashion. For χ we consider the matrix element $\langle n-1(p)|\hat{\phi}|n-1(q);k\rangle$, which is an obvious generalization of (4.1). From (A2) follows the equation

$$\{-[E_{n-1}(p) - E_{n-1}(q) - \omega_k]^2 + (p-q-k)^2 + m^2\}\langle n-1(p)|\hat{\phi}|n-1(q);k\rangle = \frac{\lambda}{6}\langle n-1(p)|\hat{\phi}^3|n-1(q);k\rangle, \tag{A13}$$

where

$$\omega_k = (k^2+m^2)^{1/2}. \tag{A14}$$

We define the Fourier transform

$$\psi_n(x,v,\omega_k) = \int \frac{dq}{2\pi}e^{i(q+k-p)x}\langle n-1(p)|\hat{\phi}|n-1(q);k\rangle. \tag{A15}$$

The expansion analogous to (A6) in this case is

$$E_{n-1}(p) - E_{n-1}(\hat{p}+p-k) - \omega_k \cong -\omega_k(1-uv) - v\hat{p}, \tag{A16}$$

where v is given by (A6) and

$$u = \frac{k}{\omega_k}. \tag{A17}$$

The Fourier transform of the left-hand side of (A13) then becomes

$$[(1-v^2)\partial_x^{\,2} - 2iv\omega_k(1-uv)\partial_x + \omega_k^{\,2}(1-uv)^2 - m^2] \times \psi_n(x,v,\omega_k).$$

Treating the right-hand side as in I and introducing

$$\psi_{nk}(x,v,\omega_k) = \exp\left[i\frac{v\omega_k(1-uv)x}{1-v^2}\right]\chi_{nk}(x,v,\omega_k), \tag{A18}$$

we find as the equation for χ_{nk}

$$\left[(1-v^2)\partial_x^{\,2} + \frac{\omega^2(1-uv)^2}{1-v^2} - m^2\right]\chi_{nk}(x,v,\omega_k) = -\lambda X_{np}^{\,2}(x)\chi_{nk}(x,v,\omega_k). \tag{A19}$$

A solution of this equation is

$$\chi_{nk}(x,v,\omega_2) = c\chi_n^0(x',\omega_k'), \tag{A20}$$

where

$$x' = \frac{x}{(1-v^2)^{1/2}}, \tag{A21}$$

$$\omega_k' = \frac{(1-uv)\omega_k}{(1-v^2)^{1/2}}, \tag{A22}$$

and the constant c can be obtained from a consideration of the commutation relation as shown in I. χ_n^0 is a solution of (A19) for $v=0$.

Lastly, to find an equation for η^* we proceed in a similar fashion from the equation

$$\{-[E_n(p) - E_{n-2}(q) - \omega_k]^2 + (p-q-k)^2 + m^2\}\langle n(p)|\hat{\phi}|n-2(q),k\rangle = \frac{\lambda}{6}\langle n(p)|\hat{\phi}^3|n-2(q),k\rangle. \tag{A23}$$

We make use of the Fourier transform

$$Y^*_{nk}(x,v,\omega_k)=\int \frac{dq}{2\pi}e^{i(q+k-p)x}\langle n(p)|\hat\phi|n-2(q),k\rangle \tag{A24}$$

and the expansion

$$E_n(p)-E_{n-2}(\hat p+p-k)\cong -v\hat p+uv\omega_k+2\omega_n(1-v^2)^{1/2}. \tag{A25}$$

Introducing the phase change

$$Y^*_{nk}(x,v,\omega_k)=\exp\left\{[(1-uv)\omega_k-2\omega_n(1-v^2)^{1/2}]\frac{vx}{1-v^2}\right\}\times\eta^*_{nk}(x,v,\omega_k), \tag{A26}$$

we find, without giving details which are entirely analogous to those above,

$$[(1-v^2)\partial_x{}^2+\omega''-m^2]\eta^*_{nk}(x,v,\omega_k)=\lambda X_{np}{}^2(x)\eta^*_{nk}(x,v,\omega_k), \tag{A27}$$

where

$$\omega''=\frac{(1-uv)\omega_k}{(1-v^2)^{1/2}}-2\omega_n. \tag{A28}$$

Again, the function

$$\eta^*_{nk}(x,v,\omega_k)=c'\eta^{0*}_n(x',\omega'_k) \tag{A29}$$

is a solution of (A27). x' and ω'_k are given by (A21) and (A22), respectively, and η^0_n is a solution of (A27) for vanishing velocity. As in I the constant c' can be found by studying the commutation relations.

Leaving aside questions of renormalization we turn to the calculation of the energy to the one-loop level. The methods used, basically evaluation by sum rules, are the same as in I, to which we refer for details. To find the energy $E_n(p)$ we take the expectation value of (A1) in the state $|n(p)\rangle$. In an obvious notation we shall use subscripts x, χ, η to denote from which intermediate states the various contributions to the energy come. For instance,

$$\begin{aligned}\langle n(p)|\hat\pi^2|n(p)\rangle_\chi&=\int\frac{dk}{2\pi}\frac{dq}{2\pi}\langle n(p)|\hat\pi|n(q),k\rangle\times\langle n(q),k|\hat\pi|n(p)\rangle\\&=\int\frac{dk}{2\pi}dx\,\chi^*_{nk}(x,v,\omega_k)\times\left[-v^2\partial_x{}^2+\frac{\omega_k(1-uv)^2}{(1-v^2)^2}\right]\times\chi_{nk}(x,v,\omega_k),\end{aligned} \tag{A30}$$

where use has been made of the inverse of (A15), (A18), and of an expansion analogous to (A16). We only quote the results for the various contributions:

$$\langle n(p)|\partial_x\hat\phi^2|n(p)\rangle_\chi=\int\frac{dk}{2\pi}dx\,\chi^*_{nk}(x,v,\omega_k)\left[-\partial_x{}^2+\frac{v^2\omega_k{}^2(1-uv)^2}{1-v^2}\right]\chi_{nk}(x,n,\omega_k), \tag{A31}$$

$$\langle n(p)|\hat\pi^2|n(p)\rangle_\eta=\int\frac{dk}{d\pi}dx\,\eta_{nk}(x,v,\omega_k)\left\{-v^2\partial_x{}^2+\left[\frac{(1-uv)\omega_k}{(1-v^2)^{1/2}}-2\omega_n\right]^2\frac{1}{1-v^2}\right\}\eta^*_{nk}(x,v,\omega_k), \tag{A32}$$

$$\langle n(p)|(\partial_x\hat\phi)^2|n(p)\rangle_\eta=\int\frac{dk}{2\pi}dx\,\eta_{nk}(x,n,\omega_k)\left\{-\partial_x{}^2+\frac{v^2}{1-v^2}\left[\frac{1-uv}{(1-v^2)^{1/2}}\omega_k-2\omega_n\right]^2\right\}\eta^*_{nk}(x,v,\omega_k), \tag{A33}$$

$$\langle n(p)|\hat\pi^2|n(p)\rangle_x=2\int dx\,X_{np}(x)\left(-v^2\partial_x{}^2+\frac{\omega_n{}^2}{1-v^2}\right)X_{np}(x), \tag{A34}$$

$$\langle n(p)|(\partial_x\hat\phi)^2|n(p)\rangle_x=2\int dx\,X_{np}(x)\left(-\partial_x{}^2+\frac{v^2\omega_n{}^2}{1-v^2}\right)X_{np}(x). \tag{A35}$$

The evaluation of the mass term gives

$$\frac{m^2}{2}\langle n(p)|\hat\phi^2|n(p)\rangle=m^2\int dx\left\{|X_{np}|^2+\frac12\int\frac{dk}{2\pi}[|\chi_{nk}(x,v,\omega_k)|^2+|\eta_{nk}(x,v,\omega_k)|^2]\right\}, \tag{A36}$$

and for the quartic term we obtain

$$U_4\equiv-\frac{\lambda}{4!}\langle n(p)|\hat\phi^4|n(p)\rangle=-\frac{\lambda}{2}\int dx\,X^*_{np}(x)\left\{\frac12|X_{np}(x)|^2+\int\frac{dk}{2\pi}[|\chi_{nk}(x,v,\omega_k)|^2+|\eta_{nk}(x,v,\omega_k)|^2]\right\}X_{np}(x). \tag{A37}$$

Putting everything together, the energy is found to be

$$E_n(p) = \int dx X_{np}^*(x)\left[(1+v^2)\left(-\partial_x^2 + \frac{\omega_n^2}{1-v^2}\right) + m^2\right] X_{np}(x) - \frac{\lambda}{4}\int dx |X_{np}(x)|^4$$
$$+\frac{1}{2}\int \frac{dk}{2\pi} dx\, \chi_{nk}^*(x,v,\omega_k)\left\{(1+v^2)\left[-\partial_x^2 + \frac{(1-uv)^2\omega_k^2}{(1-v^2)^2}\right] + m^2\right\}\chi_{nk}(x,v,\omega_k)$$
$$-\frac{\lambda}{2}\int \frac{dk}{2\pi} dx |X_{np}(x)|^2 |\chi_{nk}(x,v,\omega_k)|^2$$
$$+\frac{1}{2}\int \frac{dk}{2\pi} dx\, \eta_{nk}(x,v,\omega_k)\left((1+v^2)\left\{-\partial_x^2 + \left[\frac{1-uv}{(1-v^2)^{1/2}}\omega_k - 2\omega_n\right]^2 \frac{1}{1-v^2}\right\} + m^2\right)\eta_{nk}^*(x,v,\omega_k)$$
$$-\frac{\lambda}{2}\int \frac{dk}{2\pi} dx |X_{np}(x)|^2 |\eta_{nk}(x,v,\omega_k)|^2. \qquad \text{(A38)}$$

To finally show that the energy (A37) has the correct Lorentz transformation properties we need a virial theorem, which we shall derive from the equations of motion (A12), (A19), and (A27). Multiplying (A12) by $(\partial_x X_{np})$ and using (A37) yields

$$(1-v^2)\partial_x(\partial_x X_{np})^2 + (\omega_n^2 - m^2)\partial_x X_{np}^2 = \frac{\partial U_4}{\partial X_{np}}(\partial_x X_{np}^*). \qquad \text{(A39)}$$

In a similar way, but in addition integrating over momenta, we get from (A19) and (A27), respectively,

$$\frac{1}{2}\int \frac{dk}{2\pi}\left\{(1-v^2)\partial_x|\partial_x\chi_{nk}|^2 + \left[\frac{(1-uv)^2\omega_k^2}{1-v^2} - m^2\right]\partial_x|\chi_{nk}|^2\right\} = \frac{\partial U_4}{\partial \chi_{nk}}(\partial_x \chi_{nk}^*), \qquad \text{(A40)}$$

$$\frac{1}{2}\int \frac{dk}{2\pi}\left((1-v^2)\partial_x|\partial_x\eta_{nk}|^2 + \left\{\left[\frac{1-uv}{(1-v^2)^{1/2}}\omega_k - 2\omega_n\right]^2 - m^2\right\}\partial_x|\eta_{nk}|^2\right) = \frac{\partial U_4}{\partial \eta_{nk}^*}(\partial_x \eta_{nk}). \qquad \text{(A41)}$$

Summing (A39)–(A41) and integrating gives the desired virial theorem. With the help of (A11), (A20), and (A29) all quantities can be expressed in the rest frame and the virial theorem reads

$$\int dx\left(X_n^{0*}(-\partial_x^2 + \omega_n^2)X_n^0 + \frac{1}{2}\int \frac{dk}{d\pi}\{\chi_{nk}^{0*}(-\partial_x^2 + \omega_k^2)\chi_{nk}^0 + \eta_{nk}^0[-\partial_x^2 + (\omega_k - 2\omega_n)^2]\eta_{nk}^{0*}\}\right)$$
$$= m^2\int dx\left[|X_n^0|^2 + \frac{1}{2}\int \frac{dk}{2\pi}(|\chi_{nk}^0|^2 + |\eta_{nk}^0|^2)\right] + \int dx\, U_4(X_n^0, \chi_{nk}^0, \eta_{nk}^0). \qquad \text{(A42)}$$

The mass of the heavy particle associated with the state $|n(p)\rangle$ is found by taking the energy E_n, Eq. (A38), in the rest frame where $v=0$. Making use of the virial theorem (A42), M_n takes the form

$$M_n = \int dx\left(2X_n^{0*}(-\partial_x^2 + \omega_n^2)X_n^0 + \int \frac{dk}{d\pi}\{\chi_{nk}^{0*}(-\partial_x^2 + \omega_k^2)\chi_{nk}^0 + \eta_{nk}^0[-\partial_x^2 + (\omega_k - 2\omega_n)^2]\eta_{nk}^0\}\right). \qquad \text{(A43)}$$

By using (A11), (A20), (A21), and (A29) the energy $E_n(p)$, Eq. (A38), can be written in terms of quantities taken in the rest frame. With the expression (A43) for the mass M_n, we find for $E_n(p)$

$$E_n(p) = (1-v^2)^{1/2}M_n + v^2\frac{M_n}{(1-v^2)^{1/2}}$$
$$= \frac{M_n}{(1-v^2)^{1/2}}, \qquad \text{(A44)}$$

which shows that it indeed transforms as is appropriate for the energy of a particle. In an analogous way the correct transformation properties of the momentum density can be established.

APPENDIX B: QUANTUM CORRECTIONS TO THE SOLITON MASS

For the sake of completeness we shall in this appendix briefly compile the results of calculating the quantum corrections to the soliton mass, cf. Ref. 6. We shall first consider the ϕ^4 theory. To include the self-energy correction we write for the soliton mass

$$M_0 = \frac{4}{3}\frac{m_0^3}{\lambda}$$
$$\cong \frac{4}{3}\frac{m^3}{\lambda} - \frac{2m}{\lambda}\delta m^2, \qquad \text{(B1)}$$

where δm^2 is given by (4.21). We next consider the contribution of the excited states to the soliton mass, called ΔE_3 in Sec. IV. These states satisfy the equation

$$\left[-\frac{d}{dx^2}-2m^2+6\lambda\phi_c^{\,2}(x)\right]\psi_k(x)=\omega_k^{\,2}\psi_k(x), \tag{B2}$$

where the classical solution ϕ_c is given by

$$\phi_c(x)=\frac{m}{\sqrt{\lambda}}\tanh mx. \tag{B3}$$

The spectrum associated with (B2) consists of a discrete state with $\omega_b^{\,2}=3m^2$ and a continuum with $\omega_k^{\,2}=k^2+4m^2$, to be interpreted as states of a soliton and a free meson. From (B2) the following virial theorem can be derived:

$$\sum_k \omega_k^{\,2}\int dx|\psi_k(x)|^2=\sum_k\int dx[|\partial_x\psi_k(x)|^2-2m^2|\psi_k(x)|^2+6\lambda\phi_c^{\,2}(x)|\psi_k|^2]. \tag{B4}$$

The contribution to the soliton mass which results from the inclusion of these states is

$$\begin{aligned}(\Delta M)_1&=\sum_k\tfrac{1}{2}(\omega_k^{\,2}-2m^2)\int dx|\psi_k(x)|^2\\&+\tfrac{1}{2}\sum_k\int dx|\partial_x\psi_k(x)|^2\\&+3\lambda\sum_k\int dx\ \phi_c^{\,2}(x)|\psi_k(x)|^2.\end{aligned} \tag{B5}$$

Using the virial theorem (B4) and the normalization of ψ_k, $(\Delta M)_1$ becomes

$$\begin{aligned}(\Delta M)_1&=\sum_k\omega_k^{\,2}\int dx|\psi_k(x)|^2\\&=\sum_k\tfrac{1}{2}\omega_k.\end{aligned} \tag{B6}$$

In order to insure a vanishing expectation value of the Hamiltonian in the vacuum we have to subtract the corresponding expression for the noninteracting system,

$$(\Delta M)_2=\sum_{k'}\tfrac{1}{2}\omega_k'. \tag{B7}$$

From (B1), (B6), and (B7) we then find[14] for the one-loop correction to the soliton mass ($k=\nu m$)

$$\Delta M=\tfrac{3}{2}m-\int\frac{d\omega}{2\pi}\delta+\frac{3m}{\pi}\int\frac{d\nu}{(\nu^2+4)^{1/2}}, \tag{B8}$$

where δ is the phase shift to be found from the scattering solutions of (B2). It can be written in the form (ϵ is the sign function)

$$\delta=\pi\epsilon(\nu)-\tan^{-1}\nu-\tan^{-1}\frac{\nu}{2}, \tag{B9}$$

with the limiting value $3/\nu$ for large ν. Integrating the second term in (B8) by parts we find

$$\begin{aligned}\Delta M&=\frac{\sqrt{3}}{2}m-\frac{3m}{\pi}+\int\frac{d\nu}{2\pi}\omega\frac{d\delta}{d\nu}+\frac{3m}{\pi}\int\frac{d\nu}{(\nu^2+4)^{1/2}}\\&=\frac{\sqrt{3}}{2}m-\frac{3m}{\pi}-\frac{3m}{2\pi}\int\frac{d\nu}{(\nu^2+1)(\nu^2+4)^{1/2}}\\&=\frac{m}{2\sqrt{3}}-\frac{3m}{\pi}.\end{aligned} \tag{B10}$$

The calculation for the sine-Gordon case proceeds along the same lines. The mass, including the self-energy, is

$$M_0=8\left(\frac{m^2}{\lambda}\right)m_0=\frac{8m^3}{\lambda}-\frac{4m}{\lambda}\delta m^2, \tag{B11}$$

where

$$\delta m^2=-\frac{\lambda}{4}\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\frac{1}{(\nu^2+1)^{1/2}}. \tag{B12}$$

The excited states to be included at the one-loop level satisfy the equation

$$\left\{-\frac{d^2}{dx^2}+m^2\cos\left[\frac{\sqrt{\lambda}}{m}\phi_s(x)\right]\right\}\psi_k(x)=\omega_k^{\,2}\psi_k(x), \tag{B13}$$

where $\phi_s(x)$ is given by (1.5). From the solutions of (B13) one finds the phase shift

$$\delta=\frac{\pi}{2}\epsilon(\nu)-\tan^{-1}\nu. \tag{B14}$$

The analogous expression (B8) for the one-loop mass correction is

$$\Delta M=-\int\frac{d\omega}{2\pi}\delta+m\int\frac{d\nu}{2\pi}\frac{1}{(\nu^2+1)^{1/2}}, \tag{B15}$$

where (B11) and (B12) were used for the self-energy part. Integrating by parts the remaining integral cancels the last term in (B15) and we obtain

$$\Delta M=-\frac{m}{\pi}. \tag{B16}$$

The total mass can then be written as

$$M=\frac{8m^3}{\lambda}-\frac{m}{\pi}=\frac{8m^3}{\lambda'}, \tag{B17}$$

where λ' is the renormalized coupling constant

$$\lambda'=\lambda\left(1-\frac{\lambda}{8\pi m^2}\right)^{-1}. \tag{B18}$$

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†John S. Guggenheim Fellow on leave from the University of Pennsylvania.

[1]A. Klein and F. Krejs, Phys. Rev. D 12, 3112 (1975), referred to as I.

[2]A. Klein and F. Krejs, this issue, Phys. Rev. D 13, 3282 (1976), referred to as II.

[3]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[4]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 11, 3424 (1975); referred to as DHN.

[5]See also N. H. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).

[6]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4130 (1974).

[7]This will be evident once more in our next paper on the Gross-Neveu model.

[8]Another advantage *to* rather than *of* the present method was the availability of the results of DHN as a crucial control and check of our results.

[9]We note also that our methods suggest that we may indeed be able to do relativistic quantum field theory in general by never considering other than a given instant of time. This is a conjecture to which we hope to return.

[10]We should replace n by $n+c$, where c is an unknown integration constant of order unity, which we have simply chosen to be zero so that our solution is sensible for $n=0$, 1. This choice is arbitrary, since the theory only dictates that c be of order unity. See the discussion in Sec. IV where the inclusion of the first quantum correction forces a different choice of c.

[11]P. Carruthers and M. M. Nieto, Rev. Mod. Phys. 40, 44 (1968).

[12]This method is the quantum analog of the classical calculations carried out by DHN.

[13]For a truly one-dimensional quantum system the corrections $\Delta E^{(1)}$ and $\Delta E^{(2)}$ are the *only* quantum corrections and can be used to derive a suitable version of the WKB formula.

[14]The occurrence of the term $(-m)$ has been explained by DHN, using a mode-counting argument. Such terms always cancel, upon integration by parts, against a contribution arising from the discontinuity of $\delta(\nu)$ at $\nu=0$ when there are bound-state solutions to (4.8) and (4.9). This is the expression of Levinson's theorem. In the calculations in Appendix B, we have omitted the terms which cancel.

[15]The same result is achieved by dropping the term $-2m\cos\alpha$ or its equivalent in (4.24) and then choosing $c=1$. This is what DHN do without considering that any discussion is necessary. However, this imposition of a constraint on the result for small n is decisive in making the calculation so impressively accurate (perhaps even exact).