Covariant Heisenberg picture of a relativistic positive-energy theory: The operator algebra of the rigid string

L. P. Staunton
Department of Physics, Drake University, Des Moines, Iowa 50311
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The covariant operator Heisenberg equations of motion and commutation relations following from positive-energy wave equations are obtained. The resulting theory is identical to that of a dual string model restricted to excitations of only the lowest normal modes. It is suggested that recent classical Dirac-bracket formulations of the full dual string are subject to reinterpretation, and are apparently Poincaré covariant in four dimensions. The nucleus of the complete set of covariant quantum string relations is obtained from the restricted model, and it is shown that covariant normal-mode operators and those of the null plane cannot both have simple creation-operator character.

I. INTRODUCTION

The proper quantization of the relativistic dual string model of Nambu[1] has been a very interesting problem to which considerable effort has been applied.[2] The result,[3] following from a null-plane quantization procedure, that the Lorentz algebra is satisfied only in 26 space-time dimensions is physically unacceptable, while the model itself is nonetheless useful. Several authors[4] have recently suggested that a more physical result might perhaps be obtained should the quantization be completed in other than the null-plane formalism, and quite recently Marnelius[5] presented a classical Dirac-bracket algebra of the relativistic string in an arbitrary, orthonormal gauge.

These results are only partially correct; however, our analysis, although restricted, is sufficient to indicate that at the classical bracket level, and within the "proper-time" gauge, covariance in four dimensions is immediate. At the quantum level, the restricted nature of the present model causes the question to remain open, but indicates that the same result is highly probable.

The model which we shall consider represents a generalization of the spinless, relativistic positive-energy theory of Dirac.[6] In Ref. 7, hereafter designated I, a spin-$\frac{1}{2}$, positive-energy relativistic wave equation was presented. This equation had the property, lacking in the spinless theory, that a minimal coupling to an Abelian gauge field, such as the electromagnetic field, could be formally accommodated. More importantly, for our present purpose, the wave-function solution to this equation has a null-plane character and is one of the eigenstates of a null-plane Hamiltonian theory of composite elementary particles advanced by Biedenharn, Han, and van Dam.[8] A null-plane Poincaré algebra for this model has been reported.[9]

Subsequently, in Ref. 10, hereafter designated II, the classical relativistic limit of the theory proposed in I was obtained in terms of a Poisson-bracket algebra, and it was noticed that the result could be interpreted as describing the end point of a classical string whose motion was restricted to translations and rotations only, i.e., one having only the lowest covariant normal modes.

Now, such a model represents a severe restriction of a true string. However, it has the considerable advantage of being obtained via a limit procedure from a theory which is already consistent, fully covariant, and quantized. We find that the results of an analysis of the quantum Heisenberg picture, while restricted, nevertheless contain much of the nucleus of the proper quantization rules for an unrestricted theory.

In Sec. II, we recall the positive-energy theory of I on which our results are based. In Sec. III, a complete covariant Heisenberg operator algebra based on a generalized Lorentz-scalar Hamiltonian is obtained. In Sec. IV, we obtain the indispensable set of operator and state conditions following from the complete model, and recall, as well, the corresponding conditions which obtain in the classical limit of II. Section V contains the results of our analysis recast into the language of the dual string model. There we discuss also the effects of the restricted nature of our model upon the generality of our results and indicate our conclusions about a complete model. A very important result about the relationship between null-plane and manifestly covariant boson operators is developed in the Appendix.

II. POSITIVE-ENERGY WAVE EQUATIONS

The model which we shall consider was presented and discussed at length in I. For completeness, and in order to fix the notation, we shall review here some of the essential elements of that discussion.

Our model is based on covariant wave equations involving operators Γ_μ and $S_{\mu\nu}$, which are realizations of the Lie algebra of SO(3, 2):

$$[\Gamma_\mu, \Gamma_\nu] = iS_{\mu\nu},$$
$$[\Gamma_\mu, S_{\alpha\beta}] = i(g_{\mu\beta}\Gamma_\alpha - g_{\mu\alpha}\Gamma_\beta), \qquad (2.1)$$
$$[S_{\mu\nu}, S_{\alpha\beta}] = i(g_{\mu\alpha}S_{\nu\beta} - g_{\mu\beta}S_{\nu\alpha} + g_{\nu\beta}S_{\mu\alpha} - g_{\nu\alpha}S_{\mu\beta}).$$

The representation content of realizations of this Lie algebra may be conveniently specified in terms of the Lorentz scalar operators

$$F \equiv \tfrac{1}{4}S_{\mu\nu}S^{\mu\nu},$$
$$G \equiv \tfrac{1}{8}\epsilon^{\mu\nu\alpha\beta}S_{\mu\nu}S_{\alpha\beta},$$
and $\qquad (2.2)$
$$D \equiv \Gamma^\mu\Gamma_\mu,$$

the first two of which are the Casimir operators of the Lorentz group.

We shall be concerned with the Majorana representation[11] of the Lorentz group, one which may be defined by the statements

$$F = -\tfrac{3}{8},$$
$$G = 0,$$
and $\qquad (2.3)$
$$D = -\tfrac{1}{2}.$$

Majorana's representation is sometimes realized in terms of infinite-dimensional Hermitian matrices. However, we choose to consider a realization[12] as differential operators on the space of L_2 functions of two dimensionless variables, q_1 and q_2. We define the quantum conjugates $\eta_j = (-i)\partial/\partial q_j$, so that $(j, k = 1, 2)$

$$[q_j, \eta_k] = i\delta_{jk}. \qquad (2.4)$$

Then the Hermitian operators Γ_μ and $S_{\mu\nu}$ may be given the realization

$$\Gamma_0 = \tfrac{1}{4}(q_1^2 + q_2^2 + \eta_1^2 + \eta_2^2),$$
$$\Gamma_1 = \tfrac{1}{2}(-q_1\eta_1 + q_2\eta_2),$$
$$\Gamma_2 = \tfrac{1}{2}(q_1\eta_2 + q_2\eta_1),$$
$$\Gamma_3 = \tfrac{1}{4}(q_1^2 + q_2^2 - \eta_1^2 - \eta_2^2),$$

$$S_{10} = \tfrac{1}{4}(q_1^2 - \eta_1^2 - q_2^2 + \eta_2^2),$$
$$S_{20} = \tfrac{1}{2}(\eta_1\eta_2 - q_1q_2), \qquad (2.5)$$
$$S_{30} = \tfrac{1}{2}(q_1\eta_1 + \eta_2 q_2),$$

$$S_{12} = \tfrac{1}{2}(q_1\eta_2 - q_2\eta_1),$$
$$S_{31} = \tfrac{1}{4}(q_2^2 + \eta_2^2 - q_1^2 - \eta_1^2),$$
$$S_{23} = -\tfrac{1}{2}(q_1q_2 + \eta_1\eta_2).$$

When this realization is used, the representation conditions (2.3) are operator identities. It follows then that[7,10]

$$\Gamma^\mu S_{\mu\nu} = S_{\nu\mu}\Gamma^\mu = -\tfrac{3}{2}i\Gamma_\nu,$$
$$S_{\alpha\mu}S^{\alpha}{}_{\nu} = \tfrac{3}{2}iS_{\mu\nu} - \Gamma_\mu\Gamma_\nu - \tfrac{1}{2}g_{\mu\nu},$$
and $\qquad (2.6)$
$$S_{\mu\alpha}\Gamma_\nu - S_{\mu\nu}\Gamma_\alpha + S_{\alpha\nu}\Gamma_\mu = 0.$$

The wave equation considered in (I) is

$$(m\Gamma_\mu - \kappa P_\mu + iS_{\mu\nu}P^\nu)\psi = 0, \qquad (2.7)$$

where κ and m are constants, with $m > 0$. Use of the realization (2.5) implies that the wave function is $\psi = \psi(x^\mu, q_1, q_2)$, a single component function of the indicated arguments. The wave equation (2.7) is not Lorentz-invariant; rather it is Lorentz-covariant. If it is satisfied in any one Lorentz frame, then it is satisfied in every Lorentz frame. The wave function ψ transforms under the Lorentz group generated by the operators

$$M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu}, \qquad (2.8)$$

where $L_{\mu\nu} \equiv x_\nu P_\mu - x_\mu P_\nu$ denotes the usual space-time generators, and the operators $S_{\mu\nu}$ account for the spin degrees of freedom of the states.

The covariant equation (2.7) comprises four conditions upon the single function ψ. Let $(\kappa \neq 0)$

$$T_\mu \equiv \frac{1}{\kappa}(m\Gamma_\mu + iS_{\mu\nu}P^\nu). \qquad (2.9)$$

Then (2.7) becomes

$$T_\mu\psi = P_\mu\psi, \qquad (2.10)$$

and the necessary condition that there exist a wave function which is simultaneously a solution to the four equations (2.10) is that

$$[T_\mu, T_\nu]\psi = 0. \qquad (2.11)$$

Now,

$$[T_\mu, T_\nu] = \frac{i}{\kappa^2}(m^2 - P^2)S_{\mu\nu} - \frac{1}{\kappa}(P_\mu T_\nu - P_\nu T_\mu), \qquad (2.12)$$

so that the six conditions (2.11) reduce via (2.2), (2.3), and (2.10) to the single condition

$$(P^2 - m^2)\psi = 0. \qquad (2.13)$$

Any solution to Eq. (2.7) is thus a timelike state of fixed mass, m.

Other consequences of Eq. (2.7) following directly from the use of Eqs. (2.3) and (2.6) are the equations

$$(\Gamma^{\mu}P_{\mu} - \kappa m)\psi = 0$$

and (2.14)

$$W^2\psi = -m^2(\kappa^2 - \tfrac{1}{4})\psi ,$$

where W_μ is the Pauli-Lubanski operator following from Eq. (2.8). Further, it was shown in (I) that solutions to Eq. (2.7) exist *if and only if* the constant κ has one of the fixed values $\kappa = \frac{1}{2}$ or $\kappa = 1$.

The first of Eqs. (2.14) is just the usual form of Majorana's equation.[11] It follows, for the case $\kappa = \frac{1}{2}$, that the solution to Eqs. (2.7) is just the (timelike) massive, spinless solution of Majorana's equation. As is well known, timelike solutions of Majorana's equation have strictly positive energies, so that Eq. (2.7) defines a relativistic positive-energy system. In the only other consistent case, that of $\kappa = 1$, Eq. (2.7) describes the (timelike) massive spin-$\frac{1}{2}$ solution of Majorana's equation, and the positive-energy character is again present.

The spinless, positive-energy equation recently presented[6] by Dirac is considerably different in form from Eq. (2.7), in the spinless case of $\kappa = \frac{1}{2}$. However, it was shown in I that the two spinless theories are completely equivalent. Dirac's new equation simply defines a projection upon the infinite solution spectrum of Majorana's equation, selecting the spinless, timelike state. Dirac has given the general momentum-eigenstate solution of his new equation, a function which is also the solution, for $\kappa = \frac{1}{2}$, to Eq. (2.7),

$$\psi_D = N\psi_0(q,p)\exp(-ip^{\mu}x_{\mu}) , \tag{2.15}$$

where N is a normalization factor, and where[13]

$$\psi_0(q,p) = \exp\left\{ -\frac{1}{2(p_0+p_3)}\left[m(q_1{}^2+q_2{}^2) + ip_1(q_1{}^2 - q_2{}^2) - 2ip_2(q_1q_2)\right] \right\} . \tag{2.16}$$

The general momentum-eigenfunction solution to the spin-$\frac{1}{2}$ equation of I, i.e., Eq. (2.7) with $\kappa = 1$, is given by

$$\psi = (Aq_1 + Bq_2)\psi_0(q,p)\exp(-ip^{\mu}x_{\mu}) , \tag{2.17}$$

with A and B arbitrary. The spin-$\frac{1}{2}$ character is reflected in the two degrees of freedom A and B transforming[9] one into the other under the action of the Lorentz group in the manner appropriate to a spin-$\frac{1}{2}$ system.

The wave function (2.15) has been interpreted by Biedenharn, Han, and van Dam[8] as a description of the ground state of a relativistic composite particle bound by a harmonic-oscillator interaction specified in the null plane. The spin-$\frac{1}{2}$ wave function (2.17) is the first excited state of this null-plane relativistic dynamical system. The Hamiltonian of which these two wave functions are eigenfunctions has the null-plane Galilean form[8]

$$H_G \equiv \frac{(P_1{}^2+P_2{}^2)}{2P_+} + \omega\left[\frac{\epsilon}{P_+}(\Gamma_0+\Gamma_3) + \frac{P_+}{\epsilon}(\Gamma_0-\Gamma_3) - 2\frac{P_1}{m}\Gamma_1 - 2\frac{P_2}{m}\Gamma_2 + \frac{\epsilon}{P_+}\,\frac{(P_1{}^2+P_2{}^2)}{m^2}(\Gamma_0+\Gamma_3)\right] , \tag{2.18}$$

where $P_+ \equiv (P_0 + P_3)$ plays the role of a Galilean "mass," and the constant ϵ represents a freedom of choice of scale present due to the Galilean structure.[9] [The results of Ref. 9 indicate that the choice $\epsilon = m$ is required in order to complement the definition of the $M_{\mu\nu}$ chosen in Eq. (2.8).]

The harmonic-oscillator nature of the interaction is reflected in the appearance of the Γ_μ within the bracketed interaction term of the Hamiltonian. The first two terms within the bracket involving $(\Gamma_0 \pm \Gamma_3)$ are just the usual harmonic-oscillator terms when the particular Majorana representation (2.5), chosen for its null-plane form, is used. The remaining terms within the bracket are present in order to ensure[9] the full covariance of the theory. The Hamiltonian, of course, involves the boson operators $a_i, \bar{a}_j$ constructed in the usual way from the q_i, η_j; however, when expressed in terms of operators $a_i(p), \bar{a}_j(p)$ obtained from the $a_i, \bar{a}_j$ via a p^{μ}-dependent Lorentz transformation,[9,14] then the interaction part assumes the form of a sum over simple number operators, $N_i(p)$. It follows that the number of excited oscillator quanta present in a given eigenstate of the properly constructed operator (2.18) is an invariant to all observers, so that eigenfunctions of H_G may simultaneously satisfy covariant wave equations, such as (2.7).

The particular generator $P_- \equiv 2H_G$ of Biedenharn, Han, and van Dam has been integrated[9] into an (interacting) Lie algebra of the complete Poincaré group, but one which closes properly only when applied to the states. At the Poincaré level, the full set of eigenfunctions of H_G define a Chew-Frautschi mass-spin spectrum ($m^2 \sim s$) (ωP_+ becomes a Poincaré-invariant constant), but only within an infinite-direct-sum Hilbert space. A unified Poincaré Hilbert space incorporating these eigenfunctions as other than a direct sum is, of course, impossible.[15] The full set of states is a unified entity only when analyzed in the submanifold of the null-plane, and not at the full Poincaré four-space level.[16]

The implication of these results for the dual string model is that unless one wishes to obtain the trivial case,[16] then any set of null-plane generators incorporating a number operator Hamiltonian and realizing the Poincaré algebra in four dimensions may close their commutation relations only upon the states of the theory.

Returning the discussion to the case at hand, the Majorana representation (2.5), it has so far been possible to write single mass, single spin covariant wave equations for the eigenfunctions of H_G only for the cases[17] of no quanta (spin zero) and one quantum (spin $\frac{1}{2}$), i.e., Eq. (2.7). The Regge behavior of these cases may, of course, be extracted, subject to the remarks above, via the Hamiltonian H_G in a null-plane analysis.[14]

We shall turn, in the next section, to the covariant four-space Heisenberg picture of the theory defined by Eq. (2.7). The classical (nonquantum) relativistic limit theory has been reported in II. There it was found that the variables x^μ exhibit a type of *Zitterbewegung*, at the classical level, and that this motion was a planar rotation at the velocity of light, hence our interpretation of this structure as a rigid string, with end point x^μ. We shall obtain a similar *Zitterbewegung* at the quantum level, along with a complete operator algebra whose direct relation to the dual string operator algebra will be shown in Sec. V.

III. A HEISENBERG PICTURE

In a nonrelativistic theory, one has a single Hamiltonian operator, and the Heisenberg picture is obtained via a time-dependent unitary transformation generated by the Hamiltonian. In a relativistic theory, there exist, in general, several operators which may be considered to be generalized Hamiltonians and which may be used to generate alternative Heisenberg pictures.

In particular, in the present case, one may focus upon the operators T_μ of Eq. (2.9), for $\kappa=\frac{1}{2}$ or $\kappa=1$. The theory defined by Eq. (2.7) may then be written

$$T_\mu \approx P_\mu , \tag{3.1}$$

where the symbol $\approx$ denotes a condition holding only when the operators are applied to the states of the theory, and not as an operator identity.

In view of Eq. (3.1), the operator T_0 of Eq. (2.9) is the "usual" Hamiltonian operator. Now the four operators T_μ, each of which is a Hamiltonian in the relativistic sense, do not commute, as demonstrated in Eq. (2.12). Of course, according to Eq. (2.11),

$$[T_\mu, T_\nu] \approx 0 . \tag{3.2}$$

Nevertheless, the Heisenberg pictures generated by each of the individual operators T_μ are alternative and exclusive. Each such picture must be developed and viewed singly, and therefore presents a picture which is not *manifestly* covariant. The Heisenberg picture generated by T_0, for instance, singles out x_0 as a parameter, and necessarily results in a three-vector formulation equivalent to the usual Heisenberg picture.

Following Dirac,[6] we shall consider here the Heisenberg picture generated by a Lorentz scalar operator suggested by Eq. (2.14):

$$\Phi \equiv \kappa m - \Gamma^\mu P_\mu . \tag{3.3}$$

We view Φ as a generalized Hamiltonian[18] with an associated c-number parameter τ. The theory is then defined by Eq. (3.3), along with the state constraints

$$\Phi \approx 0 , \qquad M^2 \approx m^2 , \tag{3.4a}$$

and, of course,

$$T_\mu \approx P_\mu , \tag{3.4b}$$

where we have defined the operator $M^2 \equiv P^\mu P_\mu$ and expressed Eq. (2.13) in constraint form.

The operator equations of motion in this picture are

$$i\frac{dP^\mu}{d\tau} = [P^\mu, \Phi] , \qquad i\frac{dx^\mu}{d\tau} = [x^\mu, \Phi] , \qquad i\frac{d\Gamma^\mu}{d\tau} = [\Gamma^\mu, \Phi] , \tag{3.5a}$$

and

$$i\frac{dS^{\mu\nu}}{d\tau} = [S^{\mu\nu}, \Phi] ,$$

which may be evaluated, via (2.1), to yield

$$\frac{dP^\mu}{d\tau} = 0 , \qquad \frac{dx^\mu}{d\tau} = \Gamma^\mu , \qquad \frac{d\Gamma^\mu}{d\tau} = -S^{\mu\nu}P_\nu , \tag{3.5b}$$

and

$$\frac{dS^{\mu\nu}}{d\tau} = \Gamma^\mu P^\nu - \Gamma^\nu P^\mu .$$

The linear equations (3.5b) may be iterated to obtain

$$\frac{d^2\Gamma^\mu}{d\tau^2}+M^2\Gamma^\mu=P^\mu(\kappa m-\Phi)\,, \qquad (3.6)$$

where the combination $(\kappa m-\Phi)$ has been substituted for the operator $\Gamma^\mu P_\mu$.

Dirac has repeatedly emphasized[18] that constraint equations such as (3.4) may not be employed until all commutators, such as those of Eq. (3.5a), have been evaluated. In his partial discussion[6] of some of the Heisenberg equations of the Majorana theory, Dirac chose to make the substitutions $\Phi\approx 0$ and $M^2\approx m^2$ at this point, and consequently obtained operator solutions valid only upon states. Our purpose is to study the complete operator solutions, especially the commutation relations, so that we shall retain Eq. (3.6) in its present form.

The mathematical problem presented in Eq. (3.6) is a simple one, since M^2, P^μ, and Φ are all τ independent, mutually commuting operators. The solution to (3.6) may be obtained by inspection, and the set (3.5b) then solved serially to obtain, finally,

$$\begin{aligned}\Gamma^\mu&=A^\mu\cos M\tau+B^\mu\sin M\tau+M^{-2}P^\mu(\kappa m-\Phi)\,,\\ S^{\mu\nu}&=(A^\mu P^\nu-A^\nu P^\mu)M^{-1}\sin M\tau\\ &\quad-(B^\mu P^\nu-B^\nu P^\mu)M^{-1}\cos M\tau+D^{\mu\nu}\,,\\ &\text{and}\\ x^\mu&=M^{-2}P^\mu\tau(\kappa m-\Phi)\\ &\quad+A^\mu M^{-1}\sin M\tau-B^\mu M^{-1}\cos M\tau+C^\mu\,,\end{aligned} \qquad (3.7)$$

where the operators A^μ, B^μ, C^μ, and $D^{\mu\nu}$ are τ-independent integration "constants." The commutators of the operators of Eq. (3.7) with P^μ yield the information

$$\begin{aligned}&[A^\mu,P^\nu]=[B^\mu,P^\nu]=0\,,\\ &[D^{\mu\nu},P^\alpha]=0\,,\\ &\text{and}\\ &[C^\mu,P^\nu]=-ig^{\mu\nu}\,.\end{aligned} \qquad (3.8)$$

The last result above indicates that the operator C^μ is that part of the x^μ affected by a translation of axes, and therefore describes an origin for that motion of the x^μ which is parameterized by τ.

The equations (3.5), since they are linear, impose conditions upon the $A^\mu, B^\mu, D^{\mu\nu}$ which read

$$A^\mu P_\mu=B^\mu P_\mu=D^{\mu\nu}P_\nu=0\,, \qquad (3.9)$$

and which are to be understood as operator equations, not as conditions upon the states. Consequently, the operators A^μ, B^μ, and $D^{\mu\nu}$ must explicitly contain the P^μ in such a way that (3.9) are algebraic identities. Explicit expressions demonstrating this fact are given in the Appendix.

Our use of the operator M, and also M^{-1}, is valid so long as the spectrum of the operator M^2 is strictly positive, and M is defined to be the positive square root operator. Therefore, the solutions represented in Eq. (3.7) retain their validity only so long as the state conditions preclude zero mass and negative energy. In a general SO(3, 2) theory, such as Majorana's, or the usual Dirac equation, the results (3.7) are limited to the positive-energy timelike sector. For the positive-energy theory defined by Eq. (2.7), the results are general.

In terms of the solutions (3.7), and using the conditions (3.9), we obtain Eq. (3.3) as an identity, while the operators T_μ of Eq. (2.9) assume the form

$$\begin{aligned}T^\mu&=\frac{1}{\kappa}(mA^\mu-iB^\mu M)\cos M\tau\\ &\quad+\frac{1}{\kappa}(mB^\mu+iA^\mu M)\sin M\tau\\ &\quad+\frac{m}{\kappa}M^{-2}P^\mu(\kappa m-\Phi)\,.\end{aligned} \qquad (3.10)$$

Then the state conditions (3.4) yield the results

$$A^\mu\approx iB^\mu \qquad (3.11)$$

so that, from (3.7),

$$x^\mu\approx\frac{\kappa}{m}P^\mu\tau+\frac{B^\mu}{m}e^{-im\tau}+C^\mu\,. \qquad (3.12)$$

The positive-energy nature of the theory is reflected here in the absence of the corresponding negative-frequency *Zitterbewegung* term in Eq. (3.12). Nevertheless, since *Zitterbewegung* behavior is present, we must interpret its origin in terms of the composite-particle nature of the states. If expectation values are taken, then, of course, these terms vanish.

Turning our attention now to the commutation relations among the operators on the right-hand side of Eq. (3.7), we may evaluate the commutator of, say, Γ^μ with Φ and impose the requirement given by Eq. (3.5a) to obtain the results

$$\begin{aligned}&[A^\mu,\Phi]=iMB^\mu\\ &\text{and}\\ &[B^\mu,\Phi]=-iMA^\mu\,.\end{aligned} \qquad (3.13)$$

The same procedure applied sequentially to $S^{\mu\nu}$ and x^μ yields

$$\begin{aligned}&[D^{\mu\nu},\Phi]=0\\ &\text{and}\\ &[C^\mu,\Phi]=iM^{-2}P^\mu(\kappa m-\Phi)\,.\end{aligned} \qquad (3.14)$$

Further requirements on the operators A^μ, B^μ, C^μ, and $D^{\mu\nu}$ are obtained from the fact that the operators of Eq. (3.7) must obey the SO(3, 2) Lie algebra. The commutation relations between Γ^μ and Γ^ν yield the results

$$[A^\mu, A^\nu] = [B^\mu, B^\nu] = iD^{\mu\nu}$$

and (3.15)

$$[A^\mu, B^\nu] = [A^\nu, B^\mu].$$

Then the commutation relations between Γ^μ and $S^{\alpha\beta}$ yield, in addition,

$$[A^\mu, B^\nu] = i(g^{\mu\nu} - M^{-2}P^\mu P^\nu)M^{-1}(\kappa m - \Phi),$$

$$[A^\mu, D^{\alpha\beta}] = i(g^{\mu\beta} - M^{-2}P^\mu P^\beta)A^\alpha - i(g^{\mu\alpha} - M^{-2}P^\mu P^\alpha)A^\beta, \quad (3.16)$$

and

$$[B^\mu, D^{\alpha\beta}] = i(g^{\mu\beta} - M^{-2}P^\mu P^\beta)B^\alpha - i(g^{\mu\alpha} - M^{-2}P^\mu P^\alpha)B^\beta,$$

while those among the $S^{\mu\nu}$ yield, then,

$$[D^{\mu\nu}, D^{\alpha\beta}] = i(g^{\mu\alpha} - M^{-2}P^\mu P^\alpha)D^{\nu\beta} - i(g^{\mu\beta} - M^{-2}P^\mu P^\beta)D^{\nu\alpha} + i(g^{\nu\beta} - M^{-2}P^\nu P^\beta)D^{\mu\alpha} - i(g^{\nu\alpha} - M^{-2}P^\nu P^\alpha)D^{\mu\beta}. \quad (3.17)$$

The requirement that x^μ commute with Γ^ν yields the new information that

$$[C^\mu, A^\nu] = iM^{-2}A^\mu P^\nu$$

and (3.18)

$$[C^\mu, B^\nu] = iM^{-2}B^\mu P^\nu.$$

Then the vanishing of the commutator of x^μ with $S^{\alpha\beta}$ yields

$$[C^\mu, D^{\alpha\beta}] = -iM^{-2}(D^{\mu\alpha}P^\beta - D^{\mu\beta}P^\alpha), \quad (3.19)$$

while the fact that the x^μ are self-commuting yields, finally,

$$[C^\mu, C^\nu] = -iM^{-2}D^{\mu\nu}. \quad (3.20)$$

Thus, the operators C^μ which fix the origin of that motion of x^μ which is parameterized by τ, and which might be termed the "origin" operators, are noncommuting.

Other operators of interest are the Lorentz group generators $M_{\mu\nu}$ and the Pauli-Lubanski operators W^μ. We have, from Eq. (2.8),

$$M^{\mu\nu} = x^\nu P^\mu - x^\mu P^\nu + S^{\mu\nu}, \quad (3.21)$$

so that, with (3.7),

$$M^{\mu\nu} = C^\nu P^\mu - C^\mu P^\nu + D^{\mu\nu}. \quad (3.22)$$

The Pauli-Lubanski operator is

$$W^\mu \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}M_{\nu\alpha}P_\beta, \quad (3.23)$$

so that, with (3.22),

$$W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}D_{\nu\alpha}P_\beta \quad (3.24)$$

and, via (3.8) and (3.9),

$$W^2 = -\tfrac{1}{2}D_{\mu\nu}D^{\mu\nu}M^2. \quad (3.25)$$

The operators $M^{\mu\nu}$ of Eq. (3.22) and W^μ of Eq. (3.24) are thus manifestly τ-independent constants of the motion, as expected. Further, it is the operator $D^{\mu\nu}$ which should be understood as the fundamental intrinsic spin tensor, not only because of Eqs. (3.22) and (3.25), but in particular because of the additional property from Eq. (3.9) that $D^{\mu\nu}P_\nu = 0$. It follows that in the proper Lorentz frame, the space-time components of $D^{\mu\nu}$ vanish when acting upon the states, as is proper for an intrinsic spin tensor.

This same property permits the definition

$$D^{\mu\nu} \equiv -M^{-2}\epsilon^{\mu\nu\alpha\beta}W_\alpha P_\beta, \quad (3.26)$$

so that the $D^{\mu\nu}$ may be uniquely extracted from the Pauli-Lubanski operators. Alternatively, since we are viewing the x^μ, P^μ, Γ^μ, and $S^{\mu\nu}$ as the primary operators of this theory, according to Eq. (3.5), we may express the $D^{\mu\nu}$ in terms of the primary set as

$$D^{\mu\nu} = S^{\mu\nu} + M^{-2}P^\mu S^{\nu\alpha}P_\alpha - M^{-2}P^\nu S^{\mu\alpha}P_\alpha, \quad (3.27)$$

from which the properties developed above may be directly obtained.

The covariant intrinsic spin operators $D^{\mu\nu}$ do not obey the usual commutation relations of the $M^{\mu\nu}$, but rather those of Eq. (3.17), as first pointed out by Finkelstein.[19]

Before closing this section, we would like to consider one further operator,

$$X^\mu \equiv C^\mu - \tfrac{1}{2}M^{-2}P^\mu P^\nu C_\nu - \tfrac{1}{2}C_\nu P^\nu P^\mu M^{-2}, \quad (3.28)$$

in terms of which the Lorentz generators may also be expressed in the fundamental form

$$M^{\mu\nu} = X^\nu P^\mu - X^\mu P^\nu + D^{\mu\nu}, \quad (3.29)$$

involving the $D^{\mu\nu}$. Using (3.28) and the properties of the C^μ, we obtain the results

$$X^\mu P_\mu + P_\mu X^\mu = 0 \quad (3.30a)$$

and

$$P_\mu X^\mu = \tfrac{3}{2}i, \quad (3.30b)$$

so that X^μ has a particularly interesting representation in terms of the Poincaré generators:

$$X^\mu = -\tfrac{1}{2}M^{-2}(M^{\mu\nu}P_\nu + P_\nu M^{\mu\nu}). \quad (3.31)$$

This representation identifies X^μ as the postulated "center" operator of Finkelstein and also of

Bacry.[20] The results

$$[X^\mu, P^\nu] = -i(g^{\mu\nu} - M^{-2}P^\mu P^\nu), \tag{3.32a}$$

$$[X^\mu, M^2] = 0, \tag{3.32b}$$

$$[X^\mu, W^2] = 0, \tag{3.32c}$$

and

$$[X^\mu, X^\nu] = -iM^{-2}M^{\mu\nu} \tag{3.32d}$$

follow[19] from (3.30) and (3.31).

Now, the result (3.28) indicates that X^μ, since it is independent of τ, is properly to be considered an operator of origin type, rather than one of "center" type. Further, the quantity $P^\mu C_\mu$ cannot, for convenience, be set to zero in a quantum theory, as demonstrated, for instance, by the essential difference between Eqs. (3.20) and (3.32d). However, *after a transition to the classical level*, the result (3.30a) indicates that the right-hand side of (3.30b) vanishes, and one may then set $P^\mu C_\mu$, or even C^μ itself, to zero (see II), provided that no classical bracket relations are *subsequently* computed. These remarks will have a bearing on our discussion of the classical string model results in Sec. V.

IV. MAGNITUDES OF THE INVARIANTS

In obtaining the results of the last section, we have made use of the SO(3, 2) Lie algebra, Eq. (2.1), but not of the fact that we are considering a particular realization of the operators $\Gamma_\mu, S_{\mu\nu}$ for which the Lorentz scalar operators of Eq. (2.2) have the values (2.3).

The representation condition $D = -\frac{1}{2}$, applied to the operators Γ_μ of Eq. (3.7), yields the information

$$A^\mu A_\mu = B^\mu B_\mu = -\tfrac{1}{2} - M^{-2}(\kappa m - \Phi)^2 \tag{4.1a}$$

and

$$A^\mu B_\mu + B^\mu A_\mu = 0. \tag{4.1b}$$

The first of the identities (2.6) then yields the relations

$$A^\mu B_\mu = \tfrac{3}{2} i M^{-1}(\kappa m - \Phi), \tag{4.2a}$$

$$A^\mu D_{\mu\nu} = -\tfrac{1}{2} iA_\nu - M^{-1}B_\nu(\kappa m - \Phi), \tag{4.2b}$$

and

$$B^\mu D_{\mu\nu} = -\tfrac{1}{2} i B_\nu + M^{-1}A_\nu(\kappa m - \Phi). \tag{4.2c}$$

Subsequently, the representation condition $F = -\frac{3}{8}$ may be shown to imply that

$$D^{\mu\nu}D_{\mu\nu} = -\tfrac{1}{2} + 2M^{-2}(\kappa m - \Phi)^2. \tag{4.3}$$

Finally, the condition $G = 0$, in the form of the last of identities (2.6), implies the results

$$\epsilon^{\mu\nu\alpha\beta}D_{\nu\alpha}A_\beta = 0, \tag{4.4a}$$

$$\epsilon^{\mu\nu\alpha\beta}D_{\nu\alpha}B_\beta = 0, \tag{4.4b}$$

and

$$D^{\mu\nu}(\kappa m - \Phi) = (A^\mu B^\nu - A^\nu B^\mu)M. \tag{4.4c}$$

Quite properly, no conditions on the magnitude of the "origin" operators C^μ may be obtained.[21]

It is instructive at this point to compare the conditions following from (4.1a) and (4.2a) with that of Eq. (3.11), i.e.,

$$A^\mu A_\mu = B^\mu B_\mu \approx -(\kappa^2 + \tfrac{1}{2}), \tag{4.5a}$$

and

$$A^\mu B_\mu \approx \tfrac{3}{2} i\kappa, \tag{4.5b}$$

with

$$A^\mu \approx iB^\mu. \tag{4.5c}$$

The fact that the operators A^μ and B^μ do not commute precludes any contradiction from arising. Applying A_μ on the left to both sides of Eq. (4.5c), we obtain the result

$$A^\mu A_\mu \approx iA^\mu B_\mu, \tag{4.6}$$

so that (4.5a) and (4.5b) yield the constraint

$$\kappa^2 + \tfrac{1}{2} = \tfrac{3}{2}\kappa. \tag{4.7}$$

Thus, we recover the result of (I) that the set of equations (2.7) is consistent if and only if the constant κ has one of the fixed values $\kappa = \frac{1}{2}$ or $\kappa = 1$. We are also reminded that we may replace operators with their values upon states only when they have been commuted to the right of an expression.

We may also obtain the magnitude of the operator W^2 of Eq. (3.25). The result (4.3) implies that

$$W^2 = \tfrac{1}{4}M^2 - (\kappa m - \Phi)^2, \tag{4.8}$$

so that

$$W^2 \approx -m^2(\kappa^2 - \tfrac{1}{4}), \tag{4.9}$$

in agreement with (2.14). It follows again that the states are spinless if $\kappa = \frac{1}{2}$ and have spin $\frac{1}{2}$ if $\kappa = 1$.

It is convenient at this point to recall the results developed in II. A completely defined nonquantum relativistic theory may be obtained from Eqs. (3.3), (3.4a), and (3.5) via the familiar technique of replacing commutators with Poisson brackets. This possibility exists for the current theory because the choice of the (unitary) Majorana representation of SO(3, 2) permits the definition of the operators $\Gamma_\mu, S_{\mu\nu}$ as differential operators, as in Eq. (2.5). These definitions may be taken directly to the classical level as definitions of classical functions of conjugate real variables. Poisson brackets may then be unambigously defined in terms of classical partial derivatives.

One obtains then, in a classical fashion, exactly the same Eqs. (3.5b). Now, however, the Eqs. (3.4a) must be applied as exact equations of motion, so that we arrive instead at the classical solutions

$$\Gamma^{\mu} \overset{\text{cl}}{=} \left(\frac{\kappa}{m}\right) p^{\mu} + A^{\mu} \cos m\tau + B^{\mu} \sin m\tau ,$$

$$S^{\mu\nu} \overset{\text{cl}}{=} (A^{\mu} p^{\nu} - A^{\nu} p^{\mu}) m^{-1} \sin m\tau - (B^{\mu} p^{\nu} - B^{\nu} p^{\mu}) m^{-1} \cos m\tau + D^{\mu\nu} , \tag{4.10}$$

and

$$x^{\mu} \overset{\text{cl}}{=} \left(\frac{\kappa}{m}\right) p^{\mu} \tau + A^{\mu} m^{-1} \sin m\tau - B^{\mu} m^{-1} \cos m\tau + C^{\mu} ,$$

where the operator P^{μ} has been replaced by the classical vector p^{μ}, and the conditions (3.9) are again applicable. The condition (3.4b) may not be applied classically,[22] but is replaced by the condition $p^0 \geqslant m$.

One next confronts the problem of the magnitudes of the invariants F, D, and G. [Use of the quantum values (2.3) results in an inconsistent theory.] As shown in II, *the correct conditions upon these quantities follows from the fact that they are c-number quantum operators, and implies that, as classical functions, they must vanish identically*. Of course, these conditions are met when the particular functional forms (2.5) are used in a classical evaluation.

We obtain, then, results differing from those above:

$$A^{\mu} A_{\mu} = B^{\mu} B_{\mu} \overset{\text{cl}}{=} -\kappa^2 , \tag{4.11a}$$

$$A^{\mu} B_{\mu} \overset{\text{cl}}{=} 0 , \tag{4.11b}$$

$$D^{\mu\nu} D_{\mu\nu} \overset{\text{cl}}{=} 2\kappa^2 , \tag{4.12}$$

$$W^2 \overset{\text{cl}}{=} -\kappa^2 m^2 , \tag{4.13}$$

and

$$\kappa D^{\mu\nu} \overset{\text{cl}}{=} A^{\mu} B^{\nu} - A^{\nu} B^{\mu} . \tag{4.14}$$

It was shown in II to follow that, in the proper Lorentz frame, where the time variable is $\kappa\tau$, the motion of the classical three-vector $\vec{x}$ is a rotation in the plane defined by the orthogonal three-vectors $\vec{A}$ and $\vec{B}$, with a velocity c.

These results were interpreted as being descriptive of a massive, extended particle composed of two massless constituents, where the particle mass, m, is dynamical in origin and the spin is due to the co-rotation of the constituents. Alternatively, an interpretation in terms of a rigid strong was suggested.[23] In the following section we shall recast these classical results and their quantum counterparts into the notation customarily employed[2] in discussions of the dual string, and shall draw conclusions about the proper covariant quantization of the string model.

V. THE RIGID STRING

The classical, relativistic string, in the orthonormal gauge, may be characterized by the form[2,5]

$$x^{\mu}(\sigma, \tau') = (2\alpha')^{1/2} \left(\tilde{q}^{\mu} + a_0^{\mu} \tau' + i \sum_{n \neq 0} \frac{1}{n} a_n^{\mu} \cos n\sigma e^{-in\tau'} \right) , \tag{5.1}$$

where α' is a constant to be identified with the slope of the leading trajectory, the integers n are positive, negative, or zero, $(a_n^{\mu})^* = a_{-n}^{\mu}$, τ' is an unconstrained dimensionless parameter, and $\sigma \in [0, \pi]$ is a parameter specifying a point along the string. The $x^{\mu}(\sigma, \tau')$ of Eq. (5.1) obeys the equations of motion following from the Nambu action[1] provided that

$$\frac{\partial x^{\mu}}{\partial \tau'} \frac{\partial x_{\mu}}{\partial \sigma} = 0$$

and (5.2)

$$\frac{\partial x^{\mu}}{\partial \tau'} \frac{\partial x_{\mu}}{\partial \tau'} + \frac{\partial x^{\mu}}{\partial \sigma} \frac{\partial x_{\mu}}{\partial \sigma} = 0 ,$$

relations which are taken to represent[5] the classical counterparts of the Virasoro[24] constraints.

Once the form (5.1) is specified, then the properties of the string may be completely determined in terms of those of either of its end points. Taking, say, $\sigma = 0$, we may then focus upon the quantity

$$x^{\mu}(\tau') = (2\alpha')^{1/2} \left(\tilde{q}^{\mu} + a_0^{\mu} \tau' + i \sum_{n \neq 0} \frac{1}{n} a_n^{\mu} e^{-in\tau'} \right) , \tag{5.3}$$

subject to the single condition

$$V^{\mu} V_{\mu} = 0 , \tag{5.4a}$$

where

$$V^{\mu} \equiv \frac{dx^{\mu}}{d\tau'} = (2\alpha')^{1/2} \left(a_0^{\mu} + \sum_{n \neq 0} a_n^{\mu} e^{-in\tau'} \right) . \tag{5.4b}$$

The opinion is usually expressed[2] that, after quantization, the quantities a_n^{μ}, $n \neq 0$, should acquire the status of simple creation and annihilation operators. As we shall see, this opinion is incorrect. Another classical result, that a_0^{μ} is, up to a constant, the four-momentum, is also technically incorrect at the quantum level. In order to establish these results, it is necessary to make identifications of the quantities appearing in Eqs. (5.3) and (5.4) with those of the model of the previous sections. This may be accomplished by setting $a_n^\mu = 0$, $|n| \geq 2$, so that the string considered is one which is restricted to excitations of only the lowest normal modes, i.e., to translations and rigid rotations. Clearly, this represents a mutilation of the concept of the string. Nevertheless, considerable information may be obtained from such a restricted model. Dimensional considerations shall also be important in what follows, so that we shall absorb the factor $(2\alpha')^{1/2}$ and replace τ' by a dimensional parameter τ.

We replace, then, the model (5.3) and (5.4) by the restricted model

$$x^\mu(\tau) = q^\mu + a_0^\mu \tau + \frac{i}{m} a_{(1)}^\mu e^{-im\tau} - \frac{i}{m} a_{(-1)}^\mu e^{im\tau}, \quad (5.5)$$

subject to

$$\Gamma^\mu \Gamma_\mu \overset{\text{cl}}{=} 0, \quad (5.6a)$$

where

$$\Gamma^\mu \equiv \frac{dx^\mu}{d\tau} = a_0^\mu + a_{(1)}^\mu e^{-im\tau} + a_{(-1)}^\mu e^{im\tau}, \quad (5.6b)$$

and m has the dimensions of mass in order that τ should have the dimensions of reciprocal mass (time).

A direct correspondence may now be established with the classical limit results (4.10) via the identifications

$$q^\mu \overset{\text{cl}}{=} C^\mu, \quad (5.7a)$$

$$a_0^\mu \overset{\text{cl}}{=} \left(\frac{\kappa}{m}\right) p^\mu, \quad (5.7b)$$

$$a_{(1)}^\mu \overset{\text{cl}}{=} \tfrac{1}{2}(A^\mu + iB^\mu), \quad (5.7c)$$

and

$$a_{(-1)}^\mu \overset{\text{cl}}{=} \tfrac{1}{2}(A^\mu - iB^\mu). \quad (5.7d)$$

In this restricted model, there are only three conditions following from Eq. (5.6a), namely,

$$L_n = 0, \quad n = 1, 0, -1, \quad (5.8)$$

where

$$L_0 = \tfrac{1}{2}(a_0 \cdot a_0 + 2a_{(1)} \cdot a_{(-1)}), \quad (5.9a)$$

$$L_1 = a_0 \cdot a_{(1)}, \quad (5.9b)$$

and

$$L_{-1} = a_0 \cdot a_{(-1)}. \quad (5.9c)$$

The classical results, from Eq. (3.9),

$$A^\mu p_\mu = B^\mu p_\mu = 0, \quad (5.10)$$

then imply that two of the conditions are identically satisfied. Moreover, (5.7b) yields

$$a_0 \cdot a_0 \overset{\text{cl}}{=} \kappa^2 \quad (5.11a)$$

while the results (4.11) yield

$$a_{(1)} \cdot a_{(-1)} \overset{\text{cl}}{=} -\tfrac{1}{2}\kappa^2, \quad (5.11b)$$

so that the condition $L_0 = 0$ is also satisfied. We should emphasize that the results (5.10) follow from the Lie algebra, while those of (5.11) follow from the requirement, discussed in Sec. IV, that the invariant function D of Eq. (2.2) must vanish at the classical level.

The final classical result of interest may be obtained from Eq. (4.14),

$$\kappa D^{\mu\nu} \overset{\text{cl}}{=} 2i(a_{(1)}^\mu a_{(-1)}^\nu - a_{(-1)}^\mu a_{(1)}^\nu), \quad (5.12)$$

so that, from (3.22),

$$\kappa M^{\mu\nu} \overset{\text{cl}}{=} m q^\nu a_0^\mu - m q^\mu a_0^\nu + \kappa D^{\mu\nu}, \quad (5.13)$$

which, at least for the spin-$\frac{1}{2}$ case, which might accommodate couplings, is just the usual result[2,5] in dimensional form.

We are now prepared to turn our attention to the quantum problem, and we focus on the results (3.7)

$$\Gamma^\mu = M^{-2} P^\mu (\kappa m - \Phi) + A^\mu \cos M\tau + B^\mu \sin M\tau \quad (5.14a)$$

and

$$x^\mu = C^\mu + M^{-2} P^\mu (\kappa m - \Phi)\tau + A^\mu M^{-1} \sin M\tau - B^\mu M^{-1} \cos M\tau. \quad (5.14b)$$

The same type of correspondence may be established with a quantum version of Eqs. (5.5) and (5.6b) via

$$q^\mu = C^\mu, \quad (5.15a)$$

$$a_0^\mu = M^{-2} P^\mu (\kappa m - \Phi), \quad (5.15b)$$

$$a_{(1)}^\mu = \tfrac{1}{2}(A^\mu + iB^\mu), \quad (5.15c)$$

and

$$a_{(-1)}^\mu = \tfrac{1}{2}(A^\mu - iB^\mu). \quad (5.15d)$$

The reader is urged to take particular note of the difference between Eqs. (5.7b) and (5.15b). The other forms (5.15) are the same as those of (5.7) due to the fact that proper dimensional factors were introduced[25] into (5.5) and (5.6b).

The quantum relations developed from the model in Sec. III may now be translated into the notation of the string model. We obtain, from (3.8) ($n = 1, 0, -1$),

$$[q^\mu, P^\nu] = -i g^{\mu\nu} \quad (5.16a)$$

and

$$[a_n^\mu, P^\nu] = 0, \text{ all } n \quad (5.16b)$$

while, from (3.13) and (3.14), we obtain

$$[a_0^\mu, q^\nu] = i(g^{\mu\nu} - M^{-2} P^\mu P^\nu) M^{-2} (\kappa m - \Phi) \quad (5.17a)$$

and

$$[a_0^\mu, a_n^\nu] = nM^{-1}P^\mu a_n^\nu\ , \tag{5.17b}$$

and, from (3.18) and (3.20),

$$[q^\mu, a_n^\nu] = iM^{-2}P^\nu a_n^\mu\ , \quad n \neq 0 \tag{5.18a}$$

and

$$[q^\mu, q^\nu] = -iM^{-2}D^{\mu\nu}\ . \tag{5.18b}$$

Before listing the remaining relations, it is worthwhile to consider, at this point, a comparison with recent classical results. Applying Eqs. (5.15b), (5.17a), and (5.17b) to the states of the theory, we obtain

$$a_0^\mu \approx \left(\frac{\kappa}{m}\right) P^\mu\ , \tag{5.19a}$$

$$\left(\frac{m}{\kappa}\right)[a_0^\mu, q^\nu] \approx i(g^{\mu\nu} - m^{-2}P^\mu P^\nu)\ , \tag{5.19b}$$

and

$$\left(\frac{m}{\kappa}\right)[a_0^\mu, a_n^\nu] \approx \frac{n}{\kappa} a_n^\nu P^\mu\ . \tag{5.19c}$$

In a recent paper, Marnelius[5] has developed the classical Dirac brackets of a string model for the general orthonormal gauge. Now, our result (3.9) reads

$$P_\mu a_n^\mu = 0\ , \quad n \neq 0 \tag{5.20}$$

so that certain of his results in what he has called the "proper time gauge" may be compared[26] with ours. Making a classical transition from our relations (5.19), we would obtain the results

$$\left(\frac{m}{\kappa}\right) a_0^\mu \stackrel{\text{cl}}{=} p^\mu\ , \tag{5.21a}$$

$$\left\{\left(\frac{m}{\kappa}\right) a_0^\mu, q^\nu\right\} \stackrel{\text{cl}}{=} g^{\mu\nu} - \frac{1}{\kappa^2} a_0^\mu a_0^\nu\ , \tag{5.21b}$$

and

$$\left\{\left(\frac{m}{\kappa}\right) a_0^\mu, a_n^\nu\right\} \stackrel{\text{cl}}{=} -i\, \frac{n}{\kappa} a_n^\nu \left(\frac{m}{\kappa}\right) a_0^\mu\ . \tag{5.21c}$$

Marnelius has concluded, on the basis of results similar to these, that these operators are not transforming covariantly under translations, a result which he attributes to a fixed origin in τ.[27]

Our conclusion, however, is that Poincaré covariance is not in doubt; rather it is the fact that Eq. (5.19a) represents a state condition which causes Eqs. (5.21) to imply *apparently* inconsistent results. Stated differently, Dirac has emphasized[18] that classical constraint equations may not be employed until after all bracket relations have been calculated, and we would like to append the remark that constraint equations may also not be employed in the interpretation of bracket relations, particularly in any theory in which the Hamiltonian vanishes as a result of the equations of motion. While Eqs. (5.21b) and (5.21c) are correct as they stand, one may not use Eq. (5.21a) in the interpretation of these results vis-à-vis translations. The proper quantum relations from which to make a classical transition with subsequent interpretation of the effects of translations are the Eqs. (5.16).

Continuing now with the quantum relations following from the analysis of Sec. III, we may express the generator of homogeneous Lorentz transformations as

$$M^{\mu\nu} = q^\nu P^\mu - q^\mu P^\nu + D^{\mu\nu}\ , \tag{5.22}$$

and we have, from (3.8) and (3.14),

$$[D^{\mu\nu}, a_0^\alpha] = 0\ , \tag{5.23}$$

and, from (3.16), for $n \neq 0$,

$$[D^{\mu\nu}, a_n^\alpha] = i(g^{\mu\alpha} - M^{-2}P^\mu P^\alpha) a_n^\nu - i(g^{\nu\alpha} - M^{-2}P^\nu P^\alpha) a_n^\mu\ . \tag{5.24}$$

The last two results may also be obtained directly from Eqs. (5.16), (5.17), and (5.18). It is then straightforward to verify that the quantities of Eq. (5.15) transform correctly under the action of the $M^{\mu\nu}$. The required relations among the $M^{\mu\nu}$ may be obtained by making use of the additional result, Eq. (3.17), for the commutators among the $D^{\mu\nu}$.

In our opinion, the results developed to this point are not likely to change in a model which is not restricted as to the number of modes present.[28] However, those given below follow from the restrictions imposed and are very likely to change. Using (3.15) and (3.16), we may obtain the relations

$$[a_{(1)}^\mu, a_{(1)}^\nu] = [a_{(-1)}^\mu, a_{(-1)}^\nu] = 0 \tag{5.25}$$

and

$$[a_{(1)}^\mu, a_{(-1)}^\nu] = \tfrac{1}{2}(g^{\mu\nu} - M^{-2}P^\mu P^\nu) M^{-1}(\kappa m - \Phi) + \tfrac{1}{2} iD^{\mu\nu}\ , \tag{5.26}$$

so that

$$D^{\mu\nu} = i(a_{(1)}^\nu a_{(-1)}^\mu + a_{(-1)}^\nu a_{(1)}^\mu - a_{(1)}^\mu a_{(-1)}^\nu - a_{(-1)}^\mu a_{(1)}^\nu)\ , \tag{5.27}$$

in a form which shows the expected symmetrization in n and antisymmetrization in μ and ν. The result analogous to Eq. (5.12) may be obtained from Eq. (4.4c), and reads

$$D^{\mu\nu}(\kappa m - \Phi) = iM(a_{(1)}^\mu a_{(-1)}^\nu + a_{(-1)}^\nu a_{(1)}^\mu - a_{(-1)}^\mu a_{(1)}^\nu - a_{(1)}^\nu a_{(-1)}^\mu)\ , \tag{5.28}$$

and, from (3.9), or from (5.20),

$$D^{\mu\nu}P_\nu = 0 . \qquad (5.29)$$

Even within this restricted model, it should be clear, from Eq. (5.26), that the a_n^μ, $n \neq 0$, are not a complete set of simple boson operators. While the symmetric part of the commutator might suggest such an identification, the antisymmetric part, involving the operator $D^{\mu\nu}$, is an *independent* and necessary part required in a covariant theory. The underlying set of simple boson operators is the q_i, η_j of Eqs. (2.4) and (2.5). Creation and annihilation operators constructed from the q_i, η_j, and discussed briefly[29] in the context of a null-plane theory in Sec. II, are of course linear in the q_i, η_j.

Now, within our restricted context, there exist 10 independent bilinear operators which may be formed from the set of q_i, η_j, and these have been arranged into the convenient set of operators of Eq. (2.5). (These of course, commute with the x^μ.) On the other hand, in the Heisenberg picture the operators of interest are Φ, A^μ, B^μ, and $D^{\mu\nu}$. Since the P^μ are independent of q_i, η_j, then the conditions (5.20) and (5.29) reduce the number of independent operators of this set to exactly 10. Therefore, we expect that the Φ, A^μ, B^μ, and $D^{\mu\nu}$ are intrinsically bilinear operators, i.e., not of creation-annihilation type, and, in particular, that the $D^{\mu\nu}$ must be regarded as fundamental entities, essentially independent from the a_n^μ, Eq. (5.27) nonwithstanding.[30] Explicit expressions for these Heisenberg picture quantities in terms of the q_i, η_j are obtained in the Appendix, where these remarks are verified.

In our restricted model, we have three operator conditions which follow from the Majorana representation conditions (2.3), and which were obtained in Sec. IV. These we shall *label* as Virasoro conditions, although with severe reservations as outlined in the preceding paragraph. From Eqs. (5.20), (4.1), and (4.2a) we obtain, then,

$$L_0 = -\tfrac{1}{4} \qquad (5.30a)$$

and

$$L_1 = L_{-1} = 0 , \qquad (5.30b)$$

where

$$L_0 \equiv \tfrac{1}{2}(a_0 \cdot a_0 + a_{(1)} \cdot a_{(-1)} + a_{(-1)} \cdot a_{(1)}) ,$$
$$L_1 \equiv \tfrac{1}{2}(a_0 \cdot a_{(1)} + a_{(1)} \cdot a_0) , \qquad (5.31)$$

and

$$L_{-1} \equiv \tfrac{1}{2}(a_0 \cdot a_{(-1)} + a_{(-1)} \cdot a_0) .$$

The results (5.30b) are operator identities, following from Eq. (5.20), and, as discussed in the Appendix, follow directly from the form of the a_n^μ, $n \neq 0$. As such, they are not bona fide restrictions. On the other hand, the particular magnitude of the invariant (5.30a), a genuine condition, is due to the value $D = -\frac{1}{2}$ obtained for the realization (2.5). (In no sense is the operator L_0 a Hamiltonian in this model.) We should also remark that the *mere* inclusion of a larger number[31] of primitive oscillator variables q_i, η_j, even with the attendant modification of the null-plane Hamiltonian (2.18) to accommodate an arbitrary number of modes,[32] will have an effect upon the magnitudes of the invariants, but not upon the restricted form of the result (5.5). Rather more fundamental generalizations are required in order to realize a full string model, and these are under investigation.

We would finally like to discuss the implications of the state conditions (3.4b) in the string model notation. We have, from Eq. (3.11), the result

$$a^\mu_{(-1)} \approx 0 , \qquad (5.32)$$

which, of course, only reflects the positive-energy nature of the original, unitary theory. Now, inferences drawn from a single sample are very likely to prove incorrect; however, if we may be permitted to speculate, then we shall point out that in a complete theory with an infinite number of *covariant* normal modes, an *ad hoc* infinite set of on-shell gauge conditions may certainly be quite simply imposed via the generalization of Eq. (5.32) to arbitrary negative n. Inasmuch as the operators a_n^μ are bilinear in terms of the simple, null-plane boson operators q_i, η_j, then such a set of relations does have the proper algebraic status of "number operator" relations, while those involving operators of the type (5.31) are rather quartic in the primitive operators. This is an attractive possibility, due to its simplicity, but one whose physical utility cannot be judged until the development of a less restricted theory is completed.

APPENDIX

In order to include a treatment of all of the quantities of the theory, to explicitly demonstrate the bilinear nature of the string operators a_n^μ, and to afford the reader an opportunity for comparison with the null-plane analysis of (II), we detail here the Heisenberg picture of the quantities q_j, η_k of Eqs. (2.4) and (2.5).

The equations analogous to (3.5) are $(j, k = 1, 2)$

$$i\frac{dq_j}{d\tau}=[q_j,\Phi],$$

and (A1)

$$i\frac{d\eta_k}{d\tau}=[\eta_k,\Phi].$$

In terms of the null-plane quantities $P_\pm\equiv(P_0\pm P_3)$, we obtain via Eqs. (2.4) and (2.5)

$$\frac{dq_1}{d\tau}=-\tfrac{1}{2}(P_+\eta_1+P_1q_1-P_2q_2),$$

$$\frac{dq_2}{d\tau}=-\tfrac{1}{2}(P_+\eta_2-P_2q_1-P_1q_2),$$

$$\frac{d\eta_1}{d\tau}=\tfrac{1}{2}(P_-q_1+P_1\eta_1-P_2\eta_2), \qquad \text{(A2)}$$

and

$$\frac{d\eta_2}{d\tau}=\tfrac{1}{2}(P_-q_2-P_2\eta_1-P_1\eta_2).$$

The linear equations (A2) may be iterated to obtain the results

$$\frac{d^2q_j}{d\tau^2}+\tfrac{1}{4}M^2q_j=0$$

and (A3)

$$\frac{d^2\eta_k}{d\tau^2}+\tfrac{1}{4}M^2\eta_k=0,$$

so that

$$q_j=\alpha_j\cos\tfrac{1}{2}M\tau+\beta_j\sin\tfrac{1}{2}M\tau$$

and (A4)

$$\eta_k=\lambda_k\cos\tfrac{1}{2}M\tau+\sigma_k\sin\tfrac{1}{2}M\tau,$$

where the α_j, β_j, λ_k, and σ_k are τ-independent operator constants of integration.

The linear equations (A2) then imply four relations among the eight operator "constants," which read

$$\begin{aligned}P_-\alpha_1&=M\sigma_1-P_1\lambda_1+P_2\lambda_2,\\ P_-\alpha_2&=M\sigma_2+P_2\lambda_1+P_1\lambda_2,\\ P_-\beta_1&=-M\lambda_1-P_1\sigma_1+P_2\sigma_2,\\ P_-\beta_2&=-M\lambda_2+P_2\sigma_1+P_1\sigma_2,\end{aligned} \qquad \text{(A5)}$$

relations which imply that the λ_j,σ_j may be considered the independent operators. Equivalently, one may obtain instead the relations

$$\begin{aligned}P_+\lambda_1&=-M\beta_1-P_1\alpha_1+P_2\alpha_2,\\ P_+\lambda_2&=-M\beta_2+P_2\alpha_1+P_1\alpha_2,\\ P_+\sigma_1&=M\alpha_1-P_1\beta_1+P_2\beta_2,\\ P_+\sigma_2&=M\alpha_2+P_2\beta_1+P_1\beta_2,\end{aligned} \qquad \text{(A6)}$$

which imply the consideration of the α_j,β_j as independent.

The commutation relations among the q_j,η_k and also the relations (A5) or (A6) may then be employed to determine the sets of relations

$$\begin{aligned}&[\alpha_j,\alpha_k]=[\beta_j,\beta_k]=0,\\ &[\alpha_j,\beta_k]=-iM^{-1}P_+\delta_{jk},\end{aligned} \qquad \text{(A7a)}$$

and

$$\begin{aligned}&[\lambda_j,\lambda_k]=[\sigma_j,\sigma_k]=0,\\ &[\lambda_j,\sigma_k]=-iM^{-1}P_-\delta_{jk}.\end{aligned} \qquad \text{(A7b)}$$

Relations across the independent sets may be obtained from (A5) or (A6).

Now the particular results (3.3) and (3.7) must hold, while at the same time the quantities Γ^μ and $S^{\mu\nu}$ are expressed, according to (2.5), in terms of the q_i,η_k of (A4). Taking, say, the α_j,β_j as independent, then a great deal of straightforward but tedious algebra yields the result

$$\alpha_1{}^2+\alpha_2{}^2+\beta_1{}^2+\beta_2{}^2=4M^{-2}P_+(\kappa m-\Phi), \qquad \text{(A8)}$$

as well as the explicit expressions for the A_μ,β_μ of Eq. (3.7)

$$\begin{aligned}A_0&=(P_+)^{-2}[\tfrac{1}{4}(P_+P_3+P_1{}^2+P_2{}^2)\gamma_1+\tfrac{1}{2}MP_1\gamma_2-\tfrac{1}{2}MP_2\gamma_3],\\ A_1&=(P_+)^{-1}[\tfrac{1}{2}M\gamma_2+\tfrac{1}{4}P_1\gamma_1],\\ A_2&=(P_+)^{-1}[-\tfrac{1}{2}M\gamma_3+\tfrac{1}{4}P_2\gamma_1],\\ A_3&=(P_+)^{-2}[\tfrac{1}{4}(P_+P_0-P_1{}^2-P_2{}^2)\gamma_1-\tfrac{1}{2}MP_1\gamma_2+\tfrac{1}{2}MP_2\gamma_3],\\ B_0&=(P_+)^{-2}[\tfrac{1}{4}(P_+P_3+P_1{}^2+P_2{}^2)\gamma_4-\tfrac{1}{4}MP_1\gamma_5+\tfrac{1}{2}MP_2\gamma_6],\\ B_1&=(P_+)^{-1}[-\tfrac{1}{4}M\gamma_5+\tfrac{1}{4}P_1\gamma_4],\\ B_2&=(P_+)^{-1}[\tfrac{1}{2}M\gamma_6+\tfrac{1}{4}P_2\gamma_4],\end{aligned} \qquad \text{(A9)}$$

and

$$B_3=(P_+)^{-2}[\tfrac{1}{4}(P_+P_0-P_1{}^2-P_2{}^2)\gamma_4+\tfrac{1}{4}MP_1\gamma_5-\tfrac{1}{2}MP_2\gamma_6],$$

where

$$\begin{aligned}\gamma_1&\equiv\alpha_1{}^2+\alpha_2{}^2-\beta_1{}^2-\beta_2{}^2,\\ \gamma_2&\equiv\alpha_1\beta_1-\alpha_2\beta_2\\ \gamma_3&\equiv\alpha_1\beta_2+\alpha_2\beta_1\\ \gamma_4&\equiv\alpha_1\beta_1+\beta_1\alpha_1+\alpha_2\beta_2+\beta_2\alpha_2\\ \gamma_5&\equiv\alpha_1{}^2-\alpha_2{}^2-\beta_1{}^2+\beta_2{}^2,\\ \gamma_6&\equiv\alpha_1\alpha_2-\beta_1\beta_2.\end{aligned} \qquad \text{(A10)}$$

The commutation relations (3.15) may then be employed to obtain explicit, P^μ dependent expressions for the $D^{\mu\nu}$. These expressions involve the remaining three independent bilinear quantities which may be constructed from the α_i,β_j. Direct calculation then yields the results $A^\mu P_\mu=B^\mu P_\mu=D^{\mu\nu}P_\nu=0$, as algebraic operator identities.

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[12]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[13]The appearance of the null-plane combination (p_0+p_3) is due to the choice of the special null-plane form of the operators Γ_μ, $S_{\mu\nu}$. See Refs. 6 and 8.

[14]L. C. Biedenharn and H. van Dam, Phys. Rev. D 9, 471 (1974).

[15]This follows from the general analysis of E. P. Wigner, Ann. Math. 40, 149 (1939), and the various theorems of L. O'Raifeartaigh, Phys. Rev. Lett. 14, 575 (1965); Phys. Rev. 139, B1052 (1965). See Ref. 9.

[16]It is possible to consider a null-plane Poincaré algebra which realizes exact operator closure, but in this case the states are all spinless. See Ref. 9. The theory is then physically isomorphic to the spinless theory of Dirac, Ref. 6, which fails to couple. This result is not restricted to the case of two oscillators, but holds for any number.

[17]The representation of Majorana is reducible, containing both the principal series with lowest spin $\frac{1}{2}$ and the complementary series with lowest spin zero. The spinless equation of Dirac defines a projection onto the ground state of the complementary series, while that of Ref. 7 defines a projection onto the ground state of the principal series. Any equation selecting a higher state of either series will probably have to include also the ground states, and cannot then be a projection.

[18]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, 1930); *Lectures on Quantum Mechanics* (Yeshiva Univ. Press, New York, 1964).

[19]R. J. Finkelstein, Phys. Rev. 75, 1079 (1949).

[20]H. Bacry, J. Math. Phys. 5, 109 (1964). Note, however, that Eq. (3.31) serves as a definition of Bacry's operator. The result (3.32d) follows from this relation, and, inasmuch as Bacry's operator is claimed to be self-commuting, his analysis is apparently inconsistent. The correct relations are given in Ref. 19.

[21]The condition, used in Ref. 5, that $P^\mu C_\mu = 0$ cannot be imposed at the quantum level. See our discussion of the operator X^μ in Sec. III.

[22]Quantum equations which are not manifestly Hermitian, but are rather self-adjoint within the quantum inner product, cannot be applied at the classical level. See Ref. 6.

[23]It is amusing to note that the two interpretations are apparently not incompatible.

[24]M. A. Virasoro, Phys. Rev. D 1, 2933 (1970).

[25]There is no unique way to introduce the factors m into the classical expressions (5.5) and (5.6). We have chosen to keep the expression for Γ^μ simple. The choice becomes important at the quantum level, since these factors become the operator M, and we have the result (5.16a) with which to contend.

[26]At the classical level, this gauge is also specified by the condition that $p^\mu C_\mu = 0$. In this case, one has that $C^\mu \overset{\text{cl}}{=} X^\mu$, where the operator X^μ was discussed in Sec. III. This condition may not be applied prior to the calculation of bracket conditions, and not at all at the quantum level. This accounts for part of the discrepancy between the results.

[27]At the classical level, the origin of the parameter τ is fixed by the requirement $p^\mu C_\mu \overset{\text{cl}}{=} 0$. See Ref. 26. Removing this requirement in the Dirac bracket construction forces also the removal of the Virasoro string condition $L_0 \overset{\text{cl}}{=} 0$, resulting in a physically different model, but one with a fully covariant bracket algebra. [R. Marnelius (private communication)].

[28]We base this opinion on the fact that these relations closely resemble those of Ref. 5. The remaining results of Ref. 5 suggest, as well, that changes are required in those to follow.

[29]The reader is cautioned against notational confusion between the operators a_i of Sec. II and the a_n^μ of this section. The two are (nearly) unrelated.

[30]We mean here that, at the classical level, the $D^{\mu\nu}$ may be *defined*, as in Eq. (5.12), to be certain combinations of the a_n^μ, suggesting that the a_n^μ comprise a complete set. At the quantum level, these combinations become commutators, and the $D^{\mu\nu}$ acquire independent status.

[31]See the Appendixes of II, Ref. 10.

[32]An arbitrary number of null-plane modes may be accommodated as a direct-sum representation of SO(3, 2). This technique has been used to incorporate multiplets, giving symmetric SU(6) Young diagrams, while avoiding the hypotheses of the no-go theorems: At the full Poincaré level, an infinite-direct-sum Hilbert space results. See L. P. Staunton and H. van Dam, Lett. Nuovo Cimento 7, 371 (1973).