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Numerical solutions to two-body problems in classical electrodynamics: Head-on collisions with retarded fields and radiation reaction. II. Attractive case*

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Head-on collisions of oppositely charged particles obeying the Lorentz-Dirac equation with retarded fields have been investigated both numerically and analytically. We show, in agreement with Eliezer for this case, that no physical solutions exist with finite initial values of position, energy, and acceleration and that Clavier's contention to the contrary is flawed. If the electric field is everywhere finite, a physical solution does exist, but in the limit as the field becomes singular, one or more of the initial values of the physcial solution must become infinite. Thus, difficulties with the "physical solution" of the Lorentz-Dirac equation for this problem occur not just when the particles are close together, but as soon as they are released from rest at large separations.

I. INTRODUCTION

It is surprising that some of the simplest and most basic problems of classical electrodynamics remain unsolved. In particular, the interaction of two charged particles in head-on collision has not been treated adequately in the past using the Lorentz-Dirac equation and retarded fields. In a previous paper[1] we have presented solutions for head-on collisions of particles with like charges. Here we consider the attractive interaction of oppositely charged particles.

Both the nature of the third-order Lorentz-Dirac equation and the use of retarded fields have probably hindered progress on problems of two interacting charges. There has also been the notion[2,3] that such problems are of no intrinsic value if they require the particles to come within a Compton wavelength of each other, since at such separations the classical theory is not applicable and must be replaced by quantum electrodynamics. Nevertheless, neither classical nor quantum electrodynamics gives a completely satisfactory picture of elementary interactions, and they share some of the same difficulties. (Both require mass renormalization in order to subtract away divergent energy terms, for example.)

If studies of the classical Lorentz-Dirac equation are to suggest origins of difficulties associated with both classical and quantum theories of electrodynamics, problems must not be restricted by somewhat artificial size and energy limitations. Instead, one needs to investigate the range of validity of the theory within a classical context. In particular, are there problems for which no physically reasonable solutions of the Lorentz-Dirac equation exist? If so, what are these problems and what is the nature of their unphysical solutions? One of the aims of this paper and of paper I has been to answer these questions for head-on collisions of charged point particles.

In paper I, physical trajectories were calculated for repulsive interactions by integration backward in time. The Lorentz-Dirac equation gave physically reasonable results in all cases considered, including collisions at high energy in which the particles come well within a classical charge radius of each other (less than 0.005 of a Compton wavelength for electrons). Here we consider attractive interactions and let the particles of opposite charges fall directly toward one another. No longer can we use backward integration to avoid unphysical runaway solutions since presumably any physical trajectory ends at or at least passes through the singular point at the origin. (However, we can use backward integration for *finite* fields and take the limit as the singular field is approached. See Sec. V.) Instead, we must find a way of choosing that initial acceleration which gives a physical solution. We must then limit the trajectory pathlength sufficiently to prevent round-off error from growing into a runaway solution.

Head-on collisions of oppositely charged point particles have been studied previously, both with and without radiation reaction. In numerical studies of the Lorentz equation with retarded fields[4] it was shown that without radiation reaction, two particles of equal mass fall together with an acceleration which approaches $\sqrt{2}$ as the separation goes to zero (in natural units, $e=m=c=1$). Similarly, one can show that if one of the particles is infinitely massive, the other falls in with an acceleration which approaches zero. In both cases the velocity $\dot{x}$ approaches unity.

In analytic studies of the Lorentz-Dirac equation, on the other hand, Eliezer[5] showed that in all solutions with radiation reaction, particles falling in an attractive Coulomb field never reach the origin.

Instead, they always turn around at finite separation and apparently follow runaway trajectories outward. Unfortunately, Eliezer also made less well-founded assertions of the nonexistence of physical solutions to the Lorentz-Dirac equation.[6] Plass[3] pointed out errors in some of Eliezer's work and Clavier[7] even claimed to have found physical solutions for a particle falling toward the origin of an attractive Coulomb field.

In this paper we first describe the numerical technique and show how an expansion is derived for finding the initial acceleration which appears best able to avoid runaway solutions (Sec. II). Then (Sec. III) we present and discuss features of some of the numerical solutions. An analysis (Sec. IV) confirms Eliezer's conclusions for a charge falling directly toward the center of an attractive Coulomb field and extends his results by calculating lower bounds on the point of closest approach for any initial conditions. An essential error in Clavier's derivation of a physical solution is pointed out. Conclusions and a brief discussion of the significance of these results follow (Sec. V).

II. NUMERICAL METHOD

In order to investigate with the Lorentz-Dirac equation how two point particles of opposite charge approach one another, we integrate forward in time since there is no well-defined final state from which to integrate backward. Consequently, we must contend with rapidly growing runaway solutions.[1] In any physically reasonable solution to the problem, the two particles presumably meet together at the origin where the electric field of the other particle is singular. The usual equation of motion consists of the third-order Lorentz-Dirac equation plus the imposed boundary condition

$$\ddot{x}(t\to\infty)\to 0 . \tag{1}$$

(Notation is the same as in I.) However, because of the singularity, it does not appear sensible to demand Eq. (1), and we are left with only the third-order equation. Given any three initial values $x(0)$, $\dot{x}(0)$, and $\ddot{x}(0)$, the trajectory is uniquely determined. However, physically, only two values, usually $x(0)$ and $\dot{x}(0)$, can be chosen independently. Each choice of $\ddot{x}(0)$ then determines a different trajectory, and all but perhaps one such trajectory are runaway solutions. An obvious strategy is to find that $\ddot{x}(0)$ value which minimizes the runaway component of the solution.

The question naturally arises: How do we determine the size of the runaway component? The Lorentz-Dirac equation for motion along a straight line can be put in the simple form

$$\frac{d}{d\tau}(\gamma' e^{-\tau/\epsilon}) = -\tfrac{3}{2}E(\tau)e^{-\tau/\epsilon} , \tag{2}$$

where $\gamma' = d\gamma/dx$, $\epsilon=\frac{2}{3}$, and γ is the dimensionless energy, $\gamma=(1-\dot{x}^2)^{-1/2}$. Integration of Eq. (2) with respect to the proper time τ gives the integro-differential equation

$$\gamma'(\tau)=\gamma'(\tau_0)e^{-(\tau_0-\tau)/\epsilon} + \int_0^{(\tau_0-\tau)/\epsilon} dy\, e^{-y}E(\tau+\epsilon y) , \tag{3}$$

where τ_0 is any reference time. Usually one takes $\tau_0=\infty$ and $\gamma'(\infty)=0$ to ensure no runaway component is present, but as mentioned above, this choice is not appropriate here since we expect a singularity from the second term on the right of Eq. (3) whenever τ_0 is larger than or equal to the proper time at which the particles reach the origin. Instead, we set $\tau_0=0$ and take $\gamma_0'\equiv\gamma'(\tau_0)$ to be the initial value of $\gamma'(\tau)$. Any error in γ_0' leads for $\tau>\tau_0$ to an exponentially growing term which in just a few time units swamps the physical component. If $\gamma'(\tau_0)$ is too small, a runaway inward, i.e., toward the origin, apparently results; if $\gamma'(\tau_0)$ is too large, we obtain a runaway outward.

Next we need a method of finding the desired value of $\gamma'(\tau_0)$. In principle, one can find $\gamma'(\tau_0)$ by trial and error, but this can be tedious since we need $\gamma'(\tau_0)$ as accurate as possible, namely to about 15 significant figures for double precision runs on an IBM 360-65 computer. To find $\gamma'(\tau_0)$, we use the suggestion[8] that the physical solution for $\gamma'(\tau)$ should be an analytic function of ϵ. We rewrite Eq. (2) as

$$\gamma' = E+\epsilon u y'' , \tag{4}$$

where $\gamma'\equiv d\gamma/dx=\gamma^3\ddot{x}$, $\gamma''\equiv d^2\gamma/dx^2=\gamma^3\dddot{x}/\dot{x}+3\gamma^5\ddot{x}^2$, and $u\equiv dx/d\tau=\gamma\dot{x}$. In the limit $u\to 0$, $\gamma'=E$ is *not* a solution of Eq. (4) because unless $\ddot{x}=0$, $\gamma''\to\infty$ in this limit. However, by iterating Eq. (4) γ' can be written as a power series in ϵ,

$$\gamma' = E+\epsilon uE' + \epsilon^2(u^2E''+\gamma EE') + \epsilon^3 u(2\gamma E'^2+3\gamma EE''+u^2E'''+E^2E') + O(\epsilon^4) , \tag{5}$$

where primes indicate derivatives with respect to x and $O(\epsilon^4)$ means terms of order ϵ^4. With $\epsilon=0$, the Lorentz (no radiation reaction) result is obtained, whereas with $\epsilon=\frac{2}{3}$, the power series seems to be a useful asymptotic expansion for γ' when E and its spatial derivatives are small. In the static case (one particle very massive)

$$E=\begin{cases}-x^{-2}, & x>0\\ x^{-2}, & x<0 .\end{cases} \tag{6}$$

For the results presented in the following section, the particles were taken to be at rest at $t=-\infty$, and the field is "turned on" at $t=0$. The value of γ in terms of γ' can be found by integrating Eq. (4) with $E=0$. One finds

$$\gamma = \cosh\epsilon\gamma' , \quad t \leq 0 . \tag{7}$$

Equations (5) and (7) can be solved simultaneously by iteration [start with $\gamma=1$ in Eq. (5), substitute the resulting γ' into Eq. (7), etc.] to obtain γ_0 and γ_0'.

III. NUMERICAL RESULTS

The expansion (5) for the initial value of γ' did indeed seem to be the best value for avoiding runaway solutions. Higher values of $\gamma_0' = \gamma_0^3 \ddot{x}(0)$ gave trajectories in which the particle, after falling a short distance toward the attractive Coulomb center, then turned around and headed outward with exponentially increasing energy (i.e., an outward runaway), whereas lower values of γ_0' appeared to give inward runaways.

With γ_0' as given by Eq. (5), trajectories were calculated until they seemed to become obvious runaways. Minor adjustments were made in γ_0' even into the 15th decimal place so as to extend what we took to be the physical solution as far as possible inward. We were limited to trajectories which lasted less than about 22 time units since for longer times, the round-off error itself, in the 16th significant figure, could give rise to runaways. Separations at $t=0$, when the field was "switched on," were usually 6–10 natural units.

Several characteristics were common to the solutions which we considered physical. For one, the initial inward acceleration was greater in magnitude than when radiation reaction was omitted. This would mean, of course, that shortly after release, the particles gain more kinetic energy when radiation reaction is included than when it is omitted. The excess energy presumably comes through the Schott ($\dddot{x}$) term from the interference of radiative and bound fields. However, curiously, the acceleration did not continue to increase in magnitude as the particles approached one another. Rather, $-\ddot{x}$ reached a maximum and then decreased to zero well before the particles came together.

In Fig. 1, the time and position at which $\ddot{x}=0$ is plotted as a function of the acceleration ratio $\ddot{x}(0)/\ddot{x}_{\text{Lorentz}}(0)$ where $\ddot{x}_{\text{Lorentz}} = \gamma^{-3}E$ is the value of the initial acceleration when radiation reaction is omitted. For the plots of Fig. 1(a), one particle is taken to be extremely massive and hence static. For Fig. 1(b), both particles have the same mass and the retarded fields include effects of the preacceleration for $t<0$. The behavior of particle 1 is seen to be qualitatively the same for $m_2=m_1$ as for $m_2=\infty$.

A trajectory in which $\ddot{x}=0$ at some $x=r_0$ is a physical solution of certain—albeit artificial—problems. For simplicity let us concentrate on the static ($m_2=\infty$) case. If the Coulomb field of particle 2 is due to a thin spherical shell of charge of radius r_0, and if, once particle 1 has penetrated the shell and before it reemerges, the charge is neutralized, then particle 1 experiences the Coulomb attraction only until x decreases to r_0; afterward it is a free uncharged particle. A trajectory in which the acceleration vanishes at $x=r_0$ is thus a physical solution. It is evident from Fig. 1(a) that as r_0 becomes smaller the preacceleration must become larger. In fact, it takes particle 1 less time to reach the boundaries of very small charged shells than those of some larger shells, as is seen by the maximum in the plot of $t(\ddot{x}=0)$ as a function of $\ddot{x}(0)/\ddot{x}_{\text{Lorentz}}(0)$.

Judging from Fig. 1, it is conceivable that an infinite amount of preacceleration is required to obtain a physical solution for $r_0=0$. We investigate this question analytically in the next section and

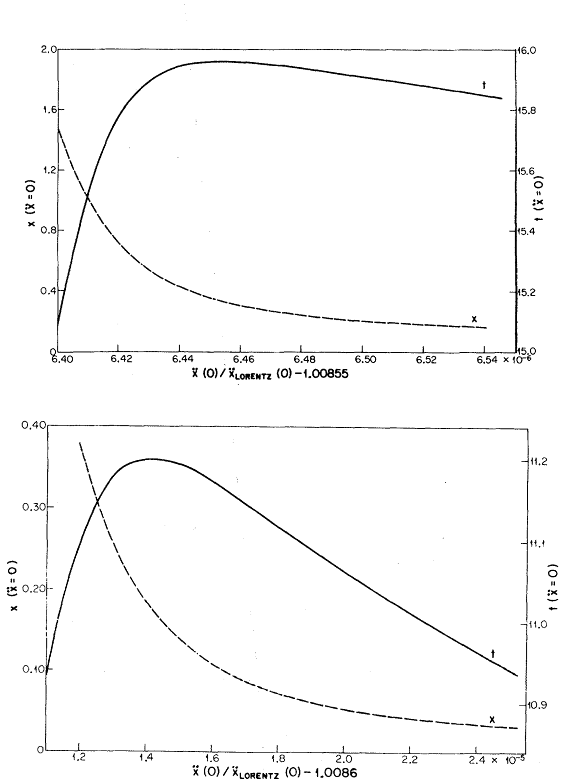

FIG. 1. Time (solid line) and postion (dashed line) at which the acceleration is zero, as a function of the initial acceleration ratio $\ddot{x}(0)/\ddot{x}_{\text{Lorentz}}(0)$ for an initial separation of six units: (a) One particle is fixed (very massive) at the origin; (b) the particles have equal masses.

find an even stronger result: No solution with a finite preacceleration ever reaches the origin. If the initial acceleration is finite, the solution turns around at some finite $x_{min} > 0$ and becomes a runaway outward.

IV. SOME INEQUALITIES

The Lorentz-Dirac equation for a charge falling directly toward the fixed point source of an attractive Coulomb field is given by Eq. (4) with $u = -(\gamma^2 - 1)^{1/2}$, $E = -x^{-2}$, and $\epsilon = \frac{2}{3}$. Expressing the equation in terms of spatial derivatives of γ, we have

$$-\gamma'' = \tfrac{3}{2}(x^{-2} + \gamma')/(\gamma^2 - 1)^{1/2} . \tag{8}$$

Let the initial values be $x = x_0$, $\gamma = \gamma_0$, and $\gamma' = \gamma_0'$. A value $x = x_1 > 0$ can always be found for which $x_1^{-2} + \gamma' \geq 0$ and, therefore, $-\gamma'' > 0$. If $\gamma_0' \geq -x_0^{-2}$ then of course x_0 is such a point. Otherwise, as shown in detail in the Appendix, such a point is

$$x_1 = (-\gamma_1')^{-1/2} , \tag{9}$$

where

$$-\gamma_1' \equiv -\gamma_0' + \tfrac{3}{2}\ln 2 + (\tfrac{3}{2}\ln 2 - \gamma_0')[\exp(\tfrac{3}{2}x_0) - 1] > 0 . \tag{10}$$

As x becomes smaller than x_1, $-\gamma''$ and hence γ' increase until γ drops to unity. At this point the velocity is zero and, since $\dot{x}$ is continuous at finite x, the particle turns around. In the Appendix we show that the turn-around position, x_{min}, obeys the inequality

$$x_{min} > x_0/\exp[1 + 2\gamma_0(\gamma_0 - 1)/3] \quad \text{if } \gamma_0' \geq 0 \tag{11}$$

and

$$x_{min} > x_1/\exp[2 + 2(\gamma_1 - \gamma_1' x_1)(\gamma_1 - \gamma_1' x_1 - 1)/3] \quad \text{otherwise} , \tag{12}$$

where

$$\gamma_1 = \gamma_0 \exp[(\ln 2 - 2\gamma_0'/3)(e^{3x_0/2} - 1)] . \tag{13}$$

We have thus established a lower positive limit for the turning point of any solution of the Lorentz-Dirac equation [Eq. (8)], given that the initial values x_0, γ_0, and γ_0' are themselves not infinite.

Once the particle turns around and starts to move outward, the proper velocity $u = (\gamma^2 - 1)^{1/2}$ so that the sign of γ'' in Eq. (8) must be changed. For outward motion after the turning point then γ'' and γ' remain positive, and consequently γ grows without limit. We conclude that all trajectories with finite initial values x_0, γ_0, and γ_0' are "runaways" and hence unphysical solutions. There are *no* physical solutions to the Lorentz-Dirac equation for this problem which have finite initial values.

Similar conclusions were drawn by Eliezer[5] about 30 years ago. Although he did not establish limits for x_{min}, his arguments appear valid and his conclusions are essentially the same as ours. It is necessary to rederive the results here because some of Eliezer's work has been discredited.[3] In particular, Clavier[7] asserted that by adding a pointlike distribution *at the origin* to normal integrable functions, he could obtain a physical solution to the problem. Since none of the solutions actually ever reach the singularity at the origin, it is difficult to understand how Clavier's modification can help. Indeed, it appears that an inequality which Clavier used to establish the existence of solutions is in error. In Eq. (50) of Ref. 7, the left-hand side should be compared with $(u_0 + \gamma_0)^2$ and not with zero, and Clavier's "solution" is not acceptable.

The results of this section are easily extended to the head-on collision of two particles with equal mass. In Eq. (8) the electric field $E = -x^{-2}$ must be replaced by

$$E = -(x + x_R)^{-2}(1 - V_R)/(1 + V_R) , \tag{14}$$

where the subscript R refers to retarded quantities, satisfying

$$x_R \equiv x(t_R) , \quad V_R \equiv \dot{x}(t_R) , \qquad x + x_R = t - t_R . \tag{15}$$

While the particles are coming together, $-1 < V_R < 0$ and x_R satisfies

$$x_R < x + |V_1|(t - t_R) , \tag{16}$$

where $|V_1|$ is the largest magnitude obtained by $\dot{x}$ during the collision. Combining Eqs. (15) and (16) we find

$$x_R < x(1 + |V_1|)/(1 - |V_1|) . \tag{17}$$

An upper bound γ_1 to $\gamma \equiv (1 - V^2)^{-1/2}$ for $0 \leq x \leq x_0$ is derived in the Appendix [Eq. (13)] and is true for any electric field $E \leq 0$. Therefore,

$$|V_1| = (1 - \gamma_1^{-2})^{1/2} . \tag{18}$$

Equations (14) and (15) give a lower bound for $-E$:

$$-E \geq \left(\frac{1 - |V_1|}{2x}\right)^2 . \tag{19}$$

The derivation of a lower bound on x_{min} is similar to that above. In the results [Eqs. (11) and (12)], we need merely replace the factors of $\frac{2}{3}$ as follows:

$$\frac{2}{3} \to \frac{2}{3}\left[\frac{2}{(1 - |V_1|)}\right]^2 . \tag{20}$$

V. DISCUSSION AND CONCLUSIONS

From the inequalities of Sec. IV we are forced to conclude that for head-on collisions of oppositely charged point particles, all solutions of the Lorentz-Dirac equation with finite initial values x_0, γ_0, and γ_0' are runaways. Since, it seems, the singularity in E at the origin is to blame for difficulties in finding a physical solution, we might well ask: What happens to the physical solution of problems with *finite* E, in the limit that E becomes singular?

The numerical results of Sec. III indicate an answer for at least one such limit. One particle was represented not as a point, but as a thin spherical shell of charge which is neutralized when penetrated by the opposite charge. As the radius of the shell was made smaller, the preacceleration, hence $-\gamma_0'$, required by the physical solution became larger. Evidently as the radius goes to zero, the physical solution with any finite x_0 and γ_0 must have $-\gamma_0' \to \infty$. We emphasize that this conclusion holds for any finite separation x_0: x_0 may be half a classical-charge radius or it may be thousands of deBroglie wavelengths.

To further test the limiting behavior of physical solutions as the electric field becomes singular, we have computed solutions by backward numerical integration for the attractive field

$$E = -x/(x^2 + a_0^2)^{3/2} \tag{21}$$

with a range of a_0 values. The backward integration ensures that any runaway components initially present are quickly damped to zero.[1] The final values (with which the integration was started) were $x_F = -50$, $\gamma_F = 1$, and $\gamma_F' = 0$. The initial values were taken both at $x_0 = 500$ and $x_0 = 50$. Values of γ_0 and γ_0' are shown as a function of a_0 in Fig. 2. In the limit $a_0 \to 0$, the physical solution apparently has $\gamma_0 \to \infty$. Similar results were obtained with other x_F values, including $|x_F| \ll a_0$.

In this last example, as the field is made singular *at the origin*, the initial conditions *at large separations* x_0 become infinite. This result is consistent with the conclusion above that all solutions with finite initial conditions are runaways, but it gives new insights into the limitations of the Lorentz-Dirac equation. If the problem were simply that there are no physical solutions, it would be tempting to blame the close approach of the two particles for somehow changing the physical solution into a runaway. However, the problem seems more serious: For any *bounded* field E there *is* a physical solution, but as the field approaches that of a point charge, initial values (γ_0 and/or γ_0') for the physical solution become infinite. The difficulty thus occurs *before* the particles have come close to each other; they may even be separated by "classical" distances. It is contrary to our instincts of causality to blame the difficulty with initial values on the collision that occurs much later. Nevertheless, the presence of a singular field is in itself not sufficient to cause trouble. (Consider, for example, head-on collisions of like point charges[1] for which no difficulty arises.) Evidently, in accordance with discussions of the existence of physical solutions,[2,3] difficulty occurs when a singularity actually lies on the true trajectory. What we have shown here is that, due presumably to effects of nonlocality in the Lorentz-Dirac equation,[9] the difficulty can manifest itself noncausally at much earlier points in the trajectory.

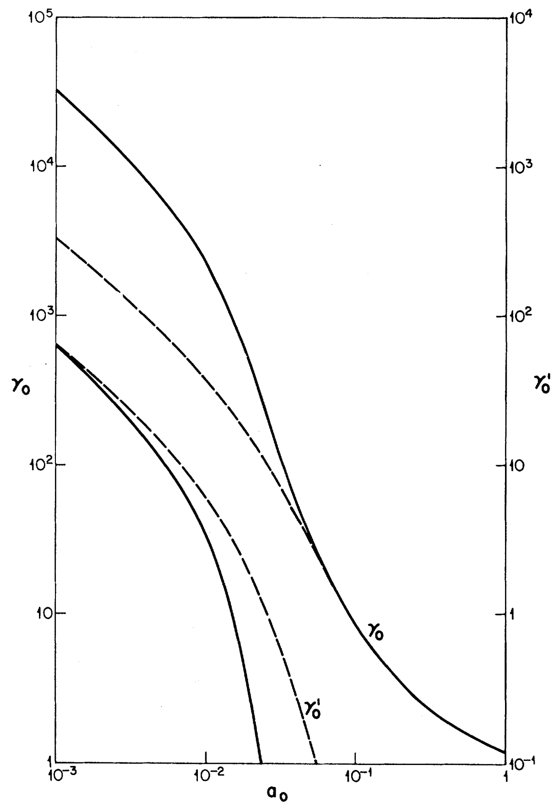

FIG. 2. Initial energy γ_0 and its spatial derivative γ_0' as a function of the size parameter a_0 of the static electric field. Solid lines are used for an initial separation $x_0 = 500$ natural units, dashed lines for $x_0 = 50$.

APPENDIX

Lower bounds on the distance of closest approach $x_{\min}$ are derived here for use in Sec. IV. One can easily find higher lower bounds than those found here. We have accepted more modest bounds in order to keep the derivation simple.

Head-on collisions of a charge with the Coulomb field of a fixed point charge are described by the Lorentz-Dirac equation in the form

$$-\gamma'' = \tfrac{3}{2}(\alpha x^{-2} + \gamma')/(\gamma^2 - 1)^{1/2}\,, \qquad \text{(A1)}$$

where $\alpha > 0$ measures the strength of coupling to the electric field. Consider two ranges of initial values:

Case 1: $\gamma_0' > -\alpha x_0^{-2}$

and

Case 2: $\gamma_0' < -\alpha x_0^{-2}$.

Case 1. γ'' is initially negative so that γ' increases as x is decreased below x_0. Consequently γ'' becomes more negative; indeed

$$-\gamma'' \geq \tfrac{3}{2}(\alpha x^{-2} + \gamma_0')/(\gamma^2 - 1)^{1/2}\,, \quad x \leq x_0 \qquad \text{(A2)}$$

and

$$\gamma' \geq \gamma_0'\,, \quad x \leq x_0\,. \qquad \text{(A3)}$$

Case 1a. $\gamma_0' \geq 0$. In this case, $(\gamma^2 - 1)^{1/2} < \gamma \leq \gamma_0$ so that

$$-\gamma'' > \tfrac{3}{2}\alpha\gamma_0^{-1}x^{-2}\,, \quad x \leq x_0 \qquad \text{(A4)}$$

and integration from x to x_0 gives a stronger result than (A3):

$$\gamma' - \gamma_0' \geq \tfrac{3}{2}\alpha\gamma_0^{-1}\left(\frac{1}{x} - \frac{1}{x_0}\right). \qquad \text{(A5)}$$

A further integration gives an upper bound on γ,

$$\gamma < \gamma_0 + \tfrac{3}{2}\alpha\gamma_0^{-1}[1 - \ln(x_0/x)]\,, \quad x \leq x_0 \qquad \text{(A6)}$$

which with $\gamma \geq 1$ gives as a lower bound on x

$$x_{\min} > x_0/\exp[1 + 2\gamma_0(\gamma_0 - 1)/3\alpha]\,. \qquad \text{(A7)}$$

Case 1b. $\gamma_0' < 0$. In this case we can use $(\gamma^2 - 1)^{1/2} < \gamma \leq \gamma_0 - \gamma_0'(x_0 - x) < \gamma_0 - \gamma_0' x_0$ and integrating twice as above we find

$$\gamma < \gamma_0 - \gamma_0' x_0 + \tfrac{3}{2}\alpha(\gamma_0 - \gamma_0' x_0)^{-1}[2 - \ln(x_0/x)]\,, \qquad \text{(A8)}$$

which gives as a lower bound on x

$$x_{\min} > x_0/\exp[2 + 2(\gamma_0 - \gamma_0' x_0)(\gamma_0 - \gamma_0' x_0 - 1)/3\alpha]\,. \qquad \text{(A9)}$$

Case 2. Now γ'' is initially positive. By Eq. (A1) it is bounded from above by

$$\gamma'' < -\frac{3\gamma'}{2(\gamma^2 - 1)^{1/2}}\,. \qquad \text{(A10)}$$

As long as γ' remains negative, then since $\gamma'' = \gamma' d\gamma'/d\gamma$

$$-\frac{d\gamma'}{d\gamma} < \frac{3}{2(\gamma^2 - 1)^{1/2}}\,. \qquad \text{(A11)}$$

Integration gives

$$\gamma_0' - \gamma' \leq \tfrac{3}{2}\ln\left|\frac{\gamma + (\gamma^2 - 1)^{1/2}}{\gamma_0 + (\gamma_0^2 - 1)^{1/2}}\right| < \tfrac{3}{2}\ln(2\gamma/\gamma_0)\,, \qquad \text{(A12)}$$

which is easily rewritten

$$(-\gamma'/\gamma)[\tfrac{3}{2}\ln(2\gamma/\gamma_0) - \gamma_0']^{-1} < 1/\gamma \leq 1\,. \qquad \text{(A13)}$$

A further integration gives

$$\ln(\gamma/\gamma_0) < \ln(\gamma_1/\gamma_0) \equiv (\ln 2 - 2\gamma_0'/3)[\exp(3x_0/2) - 1]\,. \qquad \text{(A14)}$$

Combining inequalities (A12) and (A14) we find that $-\gamma'$ is bounded from above by

$$-\gamma' < -\gamma_1' \equiv -\gamma_0' + \tfrac{3}{2}\ln 2 + (\tfrac{3}{2}\ln 2 - \gamma_0')[\exp(3x_0/2) - 1] \qquad \text{(A15)}$$

Define

$$x_1 \equiv (-\alpha/\gamma_1')^{1/2}\,. \qquad \text{(A16)}$$

By Eq. (A15), at $x = x_1$,

$$\gamma' > -\alpha x_1^{-2} \qquad \text{(A17)}$$

and we now apply the results for case 1 to obtain the lower bound

$$x_{\min} > x_1/\exp[2 + 2(\gamma_1 - \gamma_1' x_1)(\gamma_1 - \gamma_1' x_1 - 1)/3\alpha]\,, \qquad \text{(A18)}$$

where γ_1 and γ_1' are given by Eqs. (A14) and (A15). This completes the derivation. When the field is that of a fixed (very massive) point particle, we set $\alpha = 1$ in natural units. If we consider the head-on collision of two point charges of like mass, then we can replace α in equation (A18) by its lower bound,

$$\alpha \geq \left[\frac{(1 - |V_1|)}{2}\right]^2\,, \qquad \text{(A19)}$$

where $|V_1| = (1 - \gamma_1^{-2})^{1/2}$ is an upper limit on the magnitude of the inward velocity (see Sec. IV).

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