Nonuniqueness of physical solutions to the Lorentz-Dirac equation*

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The Lorentz-Dirac equation, with the usual constraint ($a \to 0$ as $\tau \to \infty$) for physical solutions, is shown to permit two or more solutions to some problems. In particular, we consider a point charge initially at rest a finite distance s from a bounded region of an attractive, uniform electric field. In one physical solution the particle remains at rest; in another, it is "pulled" into the field region by preacceleration. The maximum distance s for which multiple solutions occur is, however, extremely small for reasonable field strengths.

I. INTRODUCTION

The classical motion of point charges in electromagnetic fields $F^{\mu\nu}$ is usually described by the Lorentz-Dirac (LD) equation[1] (we use units with $m = e = c = 1$)

$$a^\mu = F^{\mu\nu} u_\nu + \tfrac{2}{3}(\dot{a}^\mu + a^2 u^\mu), \tag{1}$$

where u^μ is the proper-time derivative of the position x^μ: $u^\mu = dx^\mu/d\tau \equiv \dot{x}^\mu$, and $a^\mu = \dot{u}^\mu$. Here a^2 is the Lorentz scalar $a^\mu a_\mu \equiv a_0^2 - \vec{a}^2$. For given initial values of x^μ and u^μ, Eq. (1) has infinitely many solutions, most of which are unphysical "runaways" in which the acceleration increases without bound as $\tau \to \infty$. To ensure a physical solution, one normally imposes the additional constraint[2]

$$a^\mu \to 0 \text{ as } \tau \to \infty \tag{2}$$

and implicitly assumes that one and only one such physical solution exists.

The LD equation has been the subject of numerous studies,[3] partly because of the unusual properties of some solutions and partly in the hope that a better understanding of the classical interaction of particles and fields would illuminate related problems in quantum electrodynamics. The equation has been derived in several ways,[4] and various alternative equations of motion have been investigated.[5]

The unusual properties of some solutions arises from nonlocal effects in the Lorentz-Dirac equation, as discussed by Rohrlich.[6] One such property is the existence of preacceleration, whereby a charged particle can be accelerated by a field before the field has reached the position of the particle. Nonlocal effects are generally only important within a distance of roughly one unit (i.e., one classical charge radius) of the particle.

The LD equation has nevertheless been shown to give reasonable and apparently reliable results for many problems. In particular, for head-on collisions of like-charged point particles, the equation, although difficult to solve, has been probed to very high energies and found to give reasonable solutions.[7] On the other hand, problems have also been found for which no physical solutions exists.[8] These principally involve situations in which a singularity lies on the expected classical trajectory.[9] Nonlocality of the LD equation apparently causes effects of the singularity to be propagated along the trajectory to much earlier times, with disastrous effects.

The significance of cases involving singularities on the trajectory can be questioned by noting that quantum-mechanical uncertainty forbids such a precise specification of the trajectory. Failure of Eq. (1) is then seen as arising from the application of a classical theory beyond its range of validity. Furthermore, the singularity is in itself unphysical and can be blamed for the difficulties.

It is generally accepted that the LD equation (1), together with its physical constraint [Eq. (2)], gives accurate results providing the energy scale is sufficiently small ($\lesssim mc^2$) and the distance scale sufficiently large ($\gtrsim \hbar m^{-1} c^{-1}$). Moniz and Sharp[10] have, in fact, derived a nonrelativistic quantum-mechanical operator equation which, although it reduces to the nonrelativistic LD equation in the classical limit, admits no runaway solutions and no significant preacceleration. It is not particularly surprising that classical electrodynamics is no longer accurate when relatively small dimensions or high energies play an essential role. Nevertheless, a study of how and where the theory fails may be worthwhile. The scope of the present paper is limited to such a study.

We point out here that in addition to cases for which *no* physical solution of Eq. (1) exists, there are also cases for which *two* or *more* physical solutions exist. Two such solutions are found explicitly and analytically for the problem of a point charge initially at rest outside a region of attractive, uniform electric field. Significantly, the field here does not need to be singular. Nevertheless, we show that for reasonable field strengths

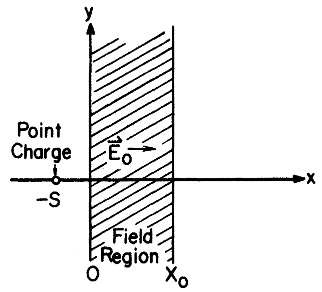

FIG. 1. A problem for which two physical solutions exist: A charge is initially at rest a small distance s from a region of strong uniform field.

the initial separation of the charge from the field region must be extremely small: It must be less than a classical charge radius if the field strength is less than a rest-mass energy per unit charge per classical charge radius.

II. PROBLEM

Consider a point charge $e=1$ initially at rest at $\tau_i<0$, $x_i=-s$, $y_i=z_i=0$, a distance s from the region of a uniform field $E_0\hat{x}$ (see Fig. 1):

$$\vec{E}=\begin{cases}E_0\hat{x}, & 0<x<x_0\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

The obvious physical solution of Eqs. (1) and (2) for the y and z components of the motion is $y=z=0$. The equation for motion in the x direction can then be simplified[8] to

$$\dot{\xi}=E+\tfrac{2}{3}\ddot{\xi}, \tag{4}$$

where

$$\xi=\ln(\gamma+u), \tag{5}$$

with

$$u=\dot{x}\equiv dx/d\tau, \tag{6}$$

$$\gamma=(1+u^2)^{1/2}. \tag{7}$$

The physical constraint [Eq. (2)] is

$$\dot{\xi}\to 0 \text{ as } \tau\to\infty. \tag{8}$$

The trivial solution to Eqs. (3)–(8) is

$$x(\tau)=-s=\text{const}. \tag{9}$$

For sufficiently small s, however, there is an additional physical solution in which the charged particle is accelerated along the x axis. It moves into the field region by preacceleration. Let us call $\tau=0$ the time at which $x=0$. Then for $\tau<0$, $E=0$, so that Eq. (4) gives

$$\dot{\xi}(\tau)=\dot{\xi}(0)e^{3\tau/2}, \quad \tau<0. \tag{10}$$

A further integration gives

$$\xi(\tau)=\xi(0)+\tfrac{2}{3}[\dot{\xi}(\tau)-\dot{\xi}(0)], \quad \tau<0. \tag{11}$$

The values $\xi(0)$ and $\dot{\xi}(0)$ of $\xi(\tau)$ and $\dot{\xi}(\tau)$ as the particle enters the field region depend on the initial separation s as well as on the strength and extent of the field. Once the charge is in the field region, it is accelerated along the x axis, only to emerge at x_0 at a later time, say τ_0.

For still later times, $\tau>\tau_0$, the solution has the form of Eq. (10):

$$\dot{\xi}(\tau)=\dot{\xi}(\tau_0)e^{3(\tau-\tau_0)/2}, \quad \tau>\tau_0. \tag{12}$$

In order to guarantee a physical solution, we must thus have

$$\dot{\xi}(\tau_0)=0 \tag{13}$$

in order to satisfy the constraint of Eq. (8).

While the charge is in the field region $(0<\tau<\tau_0)$ the solution to Eqs. (3) and (4) which yields Eq. (13) is readily seen to be

$$\dot{\xi}(\tau)=E_0(1-e^{-3(\tau_0-\tau)/2}), \quad 0<\tau<\tau_0. \tag{14}$$

A further integration gives

$$\xi(\tau)=\xi(0)+E_0\tau-\tfrac{2}{3}E_0e^{-3\tau_0/2}(e^{3\tau/2}-1), \quad 0<\tau<\tau_0. \tag{15}$$

The net increase in ξ due to acceleration in the field region is thus

$$\xi(\tau_0)-\xi(0)=E_0[\tau_0-\tfrac{2}{3}(1-e^{-3\tau_0/2})]. \tag{16}$$

The value of $\dot{\xi}$ as the charge enters the field is [see Eq. (14)]

$$\dot{\xi}(0)=E_0(1-e^{-3\tau_0/2}), \tag{17}$$

which is essentially E_0 for any field of macroscopic extent. The distance s is related to $\xi(0)$ by

$$s=\int_{\tau_i}^{0}d\tau\, u = \int_0^{\xi(0)}d\xi(\sinh\xi)/\dot{\xi}. \tag{18}$$

Application of Eq. (11) thus gives

$$s=\tfrac{2}{3}\int_0^{\xi(0)}d\xi\,\frac{\sinh\xi}{\xi+a}, \quad a\equiv\tfrac{2}{3}\dot{\xi}(0)-\xi(0)\geq 0. \tag{19}$$

The largest value of s for which a physical solution exists through the field region occurs when $\xi(0)=\frac{2}{3}\dot{\xi}(0)$ and is

$$s=\frac{2}{3}\int_0^{\xi(0)}d\xi\,\frac{\sinh\xi}{\xi}\equiv\frac{2}{3}\,\mathrm{Shi}\,\xi(0) = \frac{2}{3}\sum_{n=0}^{\infty}\frac{[\xi(0)]^{2n+1}}{(2n+1)(2n+1)!}, \tag{20}$$

where $\mathrm{Shi}\,\xi$ is the hyperbolic sine integral.[11]

The time $-\tau_i$ which the particle takes in going from $x=-s$ to $x=0$ is found from Eqs. (10) and (11) with $\xi(\tau_i)=0$:

$$\tau_i = \tfrac{2}{3}\ln[1-\tfrac{3}{2}\,\xi(0)/\dot{\xi}(0)]\,. \qquad (21)$$

When $\xi(0)=\frac{2}{3}\,\dot{\xi}(0)$, $\tau_i=-\infty$.

Equations (10), (11), (14), and (15), together with

$$\xi(\tau)=\xi(\tau_0), \quad \tau>\tau_0 \qquad (22)$$

$$\dot{\xi}(\tau)=0, \quad \tau>\tau_0 \qquad (23)$$

[see Eqs. (12) and (13)], constitute a second physical solution to the LD equation [Eqs. (4) and (8)].

III. DISCUSSION AND CONCLUSIONS

We have seen above that there are two physical solutions for the problem of a point charge initially at rest a small distance from a region of uniform electric field. There must be many other situations in which more than one physical solution of the LD equation exists. For example, there was no special reason to take a uniform electric field, beyond the simplicity of the resulting solution.

Furthermore, there must be problems for which three or more physical solutions exist. Consider, for example, a point charge initially at rest in a field-free corner bounded by three regions of uniform fields in the x, y, and z directions, respectively. For initial positions sufficiently close to the corner, there must be at least four physical solutions. There appears no rational method of choosing one solution in preference to the others.

It is important, however, to investigate the range of s values for which multiple solutions exist. From Eqs. (20) and (17)

$$s \leq \tfrac{2}{3}\,\mathrm{Shi}\,\xi(0) < \tfrac{2}{3}\,\mathrm{Shi}(\tfrac{2}{3}E_0)\,. \qquad (24)$$

For electric fields $E_0 \gtrsim 10$, s can be many de Broglie wavelengths, but for $E_0 \lesssim 1$, $s<1$ classical-charge radius. An electric field of one unit is tremendously strong. It corresponds to one rest mass per unit charge per classical charge radius (roughly 10^{18} V/cm for an electron).

It is not too surprising that the classical theory of particle-field interaction breaks down at very high energies (where pair production must be important) or at very small separations (where quantum waves are significant). It is nevertheless instructive to see how the theory falters. Whereas in some instances *no* physical solutions exist, we have here demonstrated that in others, *two* or *more* may be present.

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