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Static sourceless gauge field

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A static sourceless gauge field, coupled only to itself, is exhibited for the group SL(2, C). Its field strength is $O(r^{-2})$ as $r\to\infty$. It is also a static sourceless gauge field for SO(3,1), the orthochronous Lorentz group without inversion. A conjecture is made that such solution does not exist for compact groups.

I. INTRODUCTION

A number of years ago we searched[1] for a static sourceless gauge field for SU(2). All the solutions obtained at that time had, however, a singularity at the origin $\vec{r}=0$ at which the field is not a gauge field.[2] We want now to report on an explicit solution of a static sourceless gauge field for the group SL(2, C) with field strengths that go to zero as $O(1/r^2)$ as $r\to\infty$ and are everywhere analytic.

The solution is obtained in the following way. Hsu has found[3] a static sourceless *complex* solution for the group SU(2) by taking the following ansatz[4]:

$$\begin{aligned}&b_i^\alpha=\epsilon_{i\alpha\tau}x_\tau f(r)/r,\\&b_4^\alpha=ix_\alpha g(r)/r, \qquad [r=(x_1^2+x_2^2+x_3^2)^{1/2}],\\&\phi\equiv 1+rf,\quad G\equiv rg.\end{aligned}\tag{1}$$

The condition of sourcelessness is

$$\begin{aligned}&r^2\phi''=\phi(\phi^2-1-G^2),\\&r^2G''=2\phi^2G\end{aligned}\tag{2}$$

(where a prime denotes differentiation with respect to r), as can be seen by direct substitution of (1) into the equations for sourcelessness. Hsu pointed out[5] that (2) is satisfied by

$$\underline{\phi}=\frac{\beta r}{\sinh\beta r},\quad \underline{G}=i(\beta r\coth\beta r-1),\tag{3}$$

where

$$\beta \text{ is any complex number with real part} > 0.\tag{4}$$

We have underlined ϕ and G to indicate that they are not real.[6]

The "electric" and "magnetic" fields for (3) are complex. However, if we follow the rules discussed in Ref. 6, these complex gauge fields for the group SU(2) can be converted into real gauge fields for the group SL(2,C), which has six generators which can be represented as X_1, X_2, X_3, iX_1, iX_2, iX_3, where X_1, X_2, X_3 are the generators of SU(2). According to a theorem of Ref. 6, *this (real) gauge field for* SL(2,C) *is static and sourceless. It will be called gauge field A.*

The Lagrangian and Hamiltonian of gauge field A will be evaluated in Sec. IV.

II. FIELD STRENGTHS

The field strengths for the SU(2) group if ansatz (1) is satisfied (for f and g complex or real) are

$$\begin{aligned}&\underline{E}_j^\alpha\equiv i\underline{f}_{j4}^\alpha=\delta_{j\alpha}\underline{\phi}\underline{G}r^{-2}+x_jx_\alpha(r\underline{G}'-\underline{G}-\underline{\phi}\underline{G})r^{-4},\\&\underline{H}_j^\alpha\equiv\tfrac{1}{2}\epsilon_{ijk}\underline{f}_{ik}^\alpha=\delta_{j\alpha}\underline{\phi}'r^{-1}-x_jx_\alpha(r\underline{\phi}'-\underline{\phi}^2+1)r^{-4}.\end{aligned}\tag{5}$$

As $r\to\infty$, the complex sourceless solution (3) approaches

$$\underline{\phi}\to 2\beta re^{-\beta r},\quad \underline{G}\to i[\beta r-1+O(re^{-2\beta r})].\tag{6}$$

Substitution into (5) leads to

$$\underline{E}_j^\alpha\to ix_jx_\alpha r^{-4},\quad \underline{H}_j^\alpha\to -x_jx_\alpha r^{-4}\qquad\text{(independently of }\beta).\tag{7}$$

As $r\to 0^+$,

$$\begin{aligned}&\underline{b}_i^\alpha=-\epsilon_{i\alpha\tau}\beta^2x_\tau/6+\cdots,\\&\underline{b}_4^\alpha=-\beta^2x_\alpha/3+\cdots,\\&\underline{H}_1^1=\underline{H}_2^2=\underline{H}_3^3=-\tfrac{1}{3}\beta^2+O(r),\\&\underline{E}_1^1=\underline{E}_2^2=\underline{E}_3^3=-\tfrac{1}{3}i\beta^2+O(r),\end{aligned}\tag{8}$$

and other components of $\underline{E}$ and $\underline{H}$ go to zero.

These field strengths for the group SU(2) are complex. Their real and imaginary parts are[6] the six (real) components of the field strengths of gauge field A for SL(2,C):

$$\underline{E}_j^1 = (E_j^1)_A + i(E_j^4)_A \quad \text{etc.} \tag{9}$$

It is obvious from the above that gauge field A for SL(2,C) is everywhere analytic and has field strengths that are $O(r^{-2})$ as $r \to \infty$.

III. LAGRANGIAN AND HAMILTONIAN

Given a Lie group, the definitions of $f_{\mu\nu}^k$ and the current J_μ^k in terms of b_μ^k are unambiguous. Hence the definition[7] of sourcelessness is unambiguous. However, the definitions of the Lagrangian and the Hamiltonian require the definition of a group metric[8] $\mathcal{G}_{ij}$ since contraction of the group indices i, j, etc. is necessary. For example, the metric must be chosen so that such expressions as

$$f_{\mu\nu}^i \mathcal{G}_{ij} f_{\alpha\beta}^j \tag{10}$$

are gauge invariant. Under an infinitesimal gauge transformation, $f_{\mu\nu}^i$ transforms according to the adjoint representation with generators C_i so that

$$\langle j | C_i | k \rangle = C_{ik}^j \tag{11}$$

are the structure constants. The invariance of (10) thus means[8]

$$\tilde{C}_i \mathcal{G} + \mathcal{G} C_i = 0. \tag{12}$$

Thus, conditions on the group metric $\mathcal{G}$ are (12), together with

$$\mathcal{G} = \text{real symmetrical and } \det \mathcal{G} \neq 0 . \tag{13}$$

Not all Lie groups[8] allow the existence of such a metric. For a semisimple Lie group, such a $\mathcal{G}$ always exists. However, for the choice of $\mathcal{G}$, besides the trivial freedom of a multiplicative constant, there are often-times additional freedoms. Such is the case for SL(2,C). By examining the structure constants one can prove without difficulty the following.

Theorem 1. For SU(2), choosing $C_{ij}^k = \epsilon_{ijk}$, (12) and (13) determine $\mathcal{G}$ uniquely, to a real multiplicative constant, as the unit matrix.

Theorem 2. For SL(2,C), take the structure constants so that

$$C_i = \begin{pmatrix} C_i^0 & 0 \\ 0 & C_i^0 \end{pmatrix}, \quad C_{i+3} = \begin{pmatrix} 0 & -C_i^0 \\ C_i^0 & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{14}$$

where

$$C_i^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_3^0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

There are exactly two linearly independent solutions for $\mathcal{G}$ from (12) and (13):

$$\mathcal{G}^a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathcal{G}^b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

($\mathcal{G}^a$ is proportional to the Cartan-Killing form which is often the choice of $\mathcal{G}$.)

For SU(2), we shall thus choose

$$\mathcal{G} = \text{unit matrix}. \tag{17}$$

For SL(2,C), we shall choose (θ = real)

$$\mathcal{G} = (\cos\theta)\mathcal{G}^a + (\sin\theta)\mathcal{G}^b . \tag{18}$$

The Lagrangian density is

$$L = -\tfrac{1}{4} f_{\mu\nu}^i \mathcal{G}_{ij} f^{j\mu\nu} . \tag{19}$$

For SU(2) and SL(2,C) this is

$$L = -\tfrac{1}{4} f_{\mu\nu}^i f^{i\mu\nu} = \tfrac{1}{2} \sum_{ij} [(E_j^i)^2 - (H_j^i)^2] \text{ for SU(2)}, \tag{20}$$

$$L = \tfrac{1}{2}\cos\theta \left\{ \sum_{i,j=1}^{3} [(E_j^i)^2 - (H_j^i)^2] - \sum_{i,j=1}^{3} [(E_j^{i+3})^2 - (H_j^{i+3})^2] \right\} + \tfrac{1}{2}\sin\theta \left\{ 2 \sum_{i,j=1}^{3} [E_j^i E_j^{i+3} - H_j^i H_j^{i+3}] \right\} \text{ for SL(2,}C\text{)}. \tag{21}$$

Theorem 3. Consider a complex gauge potential $\underline{b}$ for SU(2). Let its Lagrangian (20) be $\underline{L}$. It is, in general, complex. It has a corresponding[6] real gauge potential for SL(2,C), with Lagrangian L' given by (21). Then

$$L' = \text{real part of } (e^{-i\theta} \underline{L}) . \tag{22}$$

Proof. Using (9) in (20), one obtains (22) immediately.

For the Hamiltonian density we use Eqs. (39), (17), and (18):

$$\mathcal{H}=\tfrac{1}{2}\sum_{i,j}[(E_j^i)^2+(H_j^i)^2] \text{ for SU(2)}, \tag{23}$$

$\mathcal{H}$ = same as right-hand side of (21) but with signs of all terms quadratic in H changed, for SL(2,C). (24)

Thus we have the following.

Theorem 4. Using similar notations as in Theorem 3,

$$\mathcal{H}' = \text{real part of } (e^{-i\theta}\underline{\mathcal{H}})\,. \tag{25}$$

IV. EVALUATION OF LAGRANGIAN AND HAMILTONIAN FOR STATIC SOURCELESS GAUGE FIELD FOR SL(2,C)

We shall first evaluate $\int \underline{L}\,d^3x$ and $\int \underline{\mathcal{H}}\,d^3x$ for the complex gauge field (5). Integrating over a sphere first, we obtain by substituting (5) into (20) and (23)

$$\int \underline{L}\,d^3x = -2\pi\int_0^\infty [-(r\underline{G}'-\underline{G})^2 -2\underline{\phi}^2\underline{G}^2+2\underline{\phi}'^2r^2+(1-\underline{\phi}^2)^2]r^{-2}dr\,, \tag{26}$$

$$\int \underline{\mathcal{H}}\,d^3x = 2\pi\int_0^\infty [+(r\underline{G}'-\underline{G})^2 +2\underline{\phi}^2\underline{G}^2+2\underline{\phi}'^2r^2+(1-\underline{\phi}^2)^2]r^{-2}dr\,. \tag{27}$$

Now (3) leads to the following identities:

$$r\underline{G}'-G=i(1-\underline{\phi}^2),\quad r\underline{\phi}'=i\underline{\phi}\underline{G}\,. \tag{28}$$

Thus

$$\int \underline{\mathcal{H}}\,d^3x=0\,, \tag{29}$$

$$-\int \underline{L}\,d^3x = 4\pi\int [2\underline{\phi}'^2r^2+(1-\underline{\phi}^2)^2]r^{-2}dr = 4\pi\beta\,. \tag{30}$$

Equations (22) and (25) now show that *for the sourceless static gauge field A for SL(2, C) the total Lagrangian and Hamiltonian are*

$$-\int L_A\,d^3x = 4\pi[\text{real part of } (e^{-i\theta}\beta)]\,, \tag{31}$$

$$\int \mathcal{H}_A\,d^3x = 0\,. \tag{32}$$

The former can have any value between $4\pi|\beta|$ and $-4\pi|\beta|$ by a suitable choice of θ. *The total Hamiltonian is, however, zero for any values of the parameters β and θ.*

V. REMARKS

(A) The group SL(2,C) is locally the same as the orthochronous inversion-free Lorentz group SO(3,1). Globally, two elements of SL(2,C) are mappable into one element of the latter group. According to the definition of gauge fields,[7] gauge field A is also a gauge field for SO(3,1).

(B) The metric (18) is indefinite for any value of θ, that is, it has both positive and negative eigenvalues. This is related to the fact that SL(2,C) is semisimple and noncompact, and is an illustration of a general theorem.

Theorem 5. For a semisimple group, i.e., for a group for which

$$\det \mathcal{G}'\neq 0,\quad \mathcal{G}'=\|-\text{Trace}\,C_iC_j\|\,,$$

if any choice of metric $\mathcal{G}$ satisfying (12) and (13) is positive (or negative) definite, then the group is compact.

Proof. If $\mathcal{G}$ is positive definite, there exists a real matrix M so that

$$M\mathcal{G}M^{-1}=I=\text{unit matrix}.$$

Equation (12) then shows that MC_iM^{-1} is antisymmetrical. Any linear combination of C_i is therefore also antisymmetrical. Use M as a tensor transformation on the generators X_i of the group. Then the new structure constants form matrices $(C_i)^{\text{new}}$ that are antisymmetrical. This antisymmetrization shows that $(\mathcal{G}')^{\text{new}}=\|-\text{Trace}\,(C_i)^{\text{new}}(C_j)^{\text{new}}\|$ is positive definite. Hence the group is compact.

(C) Gauge field A is static, sourceless, and $O(r^{-2})$ as $r\to\infty$. We conjecture that static sourceless gauge fields that vanish as $r\to\infty$ exists only for noncompact groups. If this conjecture is correct, theorem 5 shows that for any semisimple group for which the Hamiltonian can be chosen positive definite there does *not* exist a static sourceless gauge field that vanishes as $r\to\infty$.

(D) For any value of β, the total Lagrangian $\int \underline{L}\,d^3x$ is stationary with respect to any variation of the gauge potential. This seems to suggest that $\int \underline{L}\,d^3x$ is independent of β, which contradicts (30). To resolve this puzzle, we emphasize that $\int \underline{L}\,d^3x$ is stationary if $\delta b_i^k=0$ at large r. Since $(\partial b_i^k/\partial\beta)\delta\beta$ does not vanish at large r, $\int \underline{L}\,d^3x$ is not necessarily stationary against such variations. In fact, $\delta\int^R \underline{L}\,d^3x$ has a "surface term":

$$\delta\int^R \underline{L}\,d^3x = -2\pi[-2(\delta\underline{G})(r\underline{G}'-\underline{G})r^{-1}+4\underline{\phi}'\delta\underline{\phi}]_0^R\,.$$

Using (3), this goes to $4\pi\delta\beta$ as $R\to\infty$, confirming (30).

(E) Is there a possible physical meaning to a gauge theory with a group that is semisimple and noncompact, since the energy for such a theory is necessarily not positive definite according to theorem 5? We do not know the answer to this question. A simple negative answer suggests itself, but we do not believe such an answer is necessarily right.

APPENDIX: ENERGY-MOMENTUM TENSOR

Starting from the Lagrnagina density (19), we want to find the Hamiltonian density for any group. We shall in this appendix use the notation of Ref. 7. In particular, $x^\mu = (x_0, x_1, x_2, x_3)$ and $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$. Latin indices refer to the group, and Greek indices refer to space-time. Equation (19) can be written as

$$L = -\tfrac{1}{4} f^i_{\mu\nu} f^{\mu\nu}_i . \tag{33}$$

According to Ref. 9, the energy-momentum tensor is

$$T^\alpha_\beta = b^k_{\mu,\beta} \frac{\partial L}{\partial b^k_{\mu,\alpha}} - \delta^\alpha_\beta L$$

$$= -b^k_{\mu,\beta} f^{\mu\alpha}_k - \delta^\alpha_\beta L ,$$

where we have used the notation $F_{,\beta}$ for $\partial F / \partial x^\beta$. We write

$$T^\alpha_\beta = T^{\alpha\prime}_\beta + A^\alpha_\beta , \tag{34}$$

where

$$T^{\alpha\prime}_\beta = -f^k_{\mu\beta} f^{\mu\alpha}_k - \delta^\alpha_\beta L , \tag{35}$$

and

$$-A^\alpha_\beta = (b^k_{\beta,\mu} + C^k_{ij} b^i_\mu b^j_\beta) f^{\mu\alpha}_k = (b^k_\beta f^{\mu\alpha}_k)_{,\mu} , \tag{36}$$

where we have used the condition of sourcelessness. It is easy to see that

$$A^\alpha_{\beta,\alpha} = 0$$

and A^0_β is the 3-divergence (37)

$$A^0_\beta = -(b^k_\beta f^{\mu 0}_k)_{,\mu} .$$

Thus

$$\int A^0_\beta \, d^3x = \text{surface integral.} \tag{38}$$

Equations (37) and (38) justify taking $T^{\alpha\prime}_\beta$ to be the energy-momentum tensor. Notice that

$$T'_{\alpha\beta} = T'_{\beta\alpha} .$$

The Hamiltonian density is

$$\mathcal{H} = T^{0\prime}_0$$

$$= -f^k_{\mu 0} f^{\mu 0}_k - L$$

$$= \tfrac{1}{2} \sum_{\mu=1}^{3} (E^k_\mu g_{kj} E^j_\mu + H^k_\mu g_{kj} H^j_\mu) . \tag{39}$$

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[2]Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D 12, 3845 (1975).

[3]J. P. Hsu, Univ. of Texas at Austin report (unpublished).

[4]We follow the notation of Ref. 1. (In b^α_i, i refers to space coordinates and α to isospin coordinates.)

[5]Hsu's solution (3) is very much similar to the solution of M. K. Prasad and C. M. Sommerfield [Phys. Rev. Lett. 35, 760 (1975)], which has a scalar field coupled to the SU(2) gauge field. Prasad and Sommerfield were the first to introduce hyperbolic functions into attempts to find classical solutions.

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