Effective Lagrangian and energy-momentum tensor in de Sitter space

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The effective Lagrangian and vacuum energy-momentum tensor $\langle T^{\mu\nu}\rangle$ due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $\langle T^{\mu\nu}\rangle$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper[1] (to be referred to as I) the effective Lagrangian $\mathcal{L}^{(1)}$ due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger[2] and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,[3] who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.[4] We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for $\mathcal{L}^{(1)}$ in terms of the quantum-mechanical propagator, $K(x'', x', \tau)$,

$$\mathcal{L}^{(1)}(x') = -\tfrac{1}{2} i \lim_{x''\to x'} \int_0^\infty d\tau\, \tau^{-1} K(x'', x', \tau) e^{-im^2\tau} + X(x')\,. \tag{1}$$

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the τ integration, which is the final operation, is performed, then we can take the coincidence limit, $x'' = x'$, through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let x'' approach x'. Secondly, the term X does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, $W^{(1)}$.

The Schwinger-DeWitt procedure is to derive an expression for K, either in closed form or as a general expansion to powers of τ, then to effect the coincidence limit in (1), and finally to perform the τ integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, $\mathfrak{M}$, of dimension d. The coincidence limit $K(x, x, \tau)$ can be expanded,[5]

$$K(x, x, \tau) = i(4\pi i\tau)^{-d/2} \sum_{n=0}^{\infty} a_n(x)(i\tau)^n\,, \tag{2}$$

where the a_n are scalars constructed from the curvature tensor on $\mathfrak{M}$ and whose functional form is independent of d. The manifold $\mathfrak{M}$ must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

$$\mathcal{L}^{(1)}(x) = \tfrac{1}{2} i(4\pi)^{-d/2} \sum_n a_n(x) \int_0^\infty (i\tau)^{n-d/2-1} e^{-im^2\tau} d\tau\,. \tag{3}$$

The infinite terms are those for which $n \leq d/2$ (for d even) or $n \leq (d-1)/2$ (for d odd). For $d=4$, e.g. space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.[4]

Another popular regularization technique is dimensional regularization.[6] In this method the dimension, d, is considered to be complex and all expressions are defined in a region of the d plane where they converge. The infinities appear when an analytic continuation to $d=4$ is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum representation. A natural generalization to curved spaces would be to take the series (2) as our definition of the propagator $K(x'', x', \tau, \omega)$ for $d=2\omega$ complex. Then the integral in (3) can be evaluated for $\omega<1$ to give

$$\mathcal{L}^{(1)}(\omega)=\tfrac{1}{2}(4\pi)^{-\omega}\sum_n a_n(m^2)^{\omega-n}\Gamma(n-\omega)\,, \tag{4}$$

which has a pole at $\omega=2$ whose residue involves a_0, a_1, and a_2, in agreement with the proper-time results.

Since this particular variant of dimensional regularization virtually coincides with the zeta-function regularization described later, we shall not pursue it further.

It should be noted that this method of dimensional regularization differs from that employed by Candelas and Raine. If the manifold $\mathfrak{M}$ has some symmetry this may suggest a different procedure from that outlined above. Thus for de Sitter space, which is essentially a 4-sphere, S^1_4, the natural thing would be to generalize to $S^1_{2\omega}$ so that the dimension, 2ω, is explicitly displayed in closed formulas. This is what Candelas and Raine do in their interesting paper.[3]

The difference is that whereas in (2) and (4) the coefficients a_n are taken to be specific dimension-independent functions of the curvature, if we expand the propagator K on a sphere $S^1_{2\omega}$ in powers of τ, the coefficients will be those functions of 2ω obtained by substituting the curvature expression of the sphere into the a_n of (2). We would expect the two methods to produce the same renormalized theory after continuation to $d=2\omega=4$.

We now turn to another regularization method—the zeta-function method. We start from the Feynman Green's function $G_\infty(x'', x')$ expressed in proper-time parametric form

$$\underline{G}_\infty=i\int_0^\infty d\tau\, e^{-im^2\tau}\underline{K}(\tau)\,, \tag{5}$$

with

$$G_\infty(x'',x')=\langle x''|\underline{G}_\infty|x'\rangle$$

and

$$K(x'',x',\tau)=\langle x''|\underline{K}(\tau)|x'\rangle\,,$$

and construct the space-time matrix power $\underline{G}_\infty{}^\nu$. Use of the semigroup property, $\underline{K}(\sigma)\underline{K}(\tau)=\underline{K}(\sigma+\tau)$, rapidly gives

$$(-i\underline{G}_\infty)^\nu=[\Gamma(\nu)]^{-1}\int_0^\infty d\tau\,\tau^{\nu-1}e^{-im^2\tau}\underline{K}(\tau)\,, \tag{6}$$

where we now consider ν to be a complex variable.

If we compare Eq. (6) with one generalization of the Riemann zeta function[5] we are led to call $\underline{G}_\infty{}^\nu$ the zeta function for the manifold $\mathfrak{M}$,

$$\underline{\zeta}_{\mathfrak{M}}(\nu,m^2)=\underline{G}_\infty{}^\nu\,. \tag{7}$$

Usually, zeta functions are defined for elliptic operators on compact manifolds, and then only for the heat equation, not Schrödinger's equation.[7] Thus if λ_l are the eigenvalues and Φ_l the eigenfunctions of the operator in question, the zeta function is defined by

$$\underline{\zeta}_{\mathfrak{M}}(\nu,w)=\sum_l\frac{|\Phi_l\rangle\langle\Phi_l|}{(\lambda_l+w)^\nu}\,,$$

in dyadic notation. The behavior of $\underline{\zeta}_{\mathfrak{M}}(\nu,w)$ as a function of ν is different for the diagonal and off-diagonal elements. We are more interested in the coincidence limit, $\mathrm{diag}\underline{\zeta}_{\mathfrak{M}}(\nu,w)$. Conventional theory[5,7] usually concentrates on the functional trace, $\mathrm{Tr}\underline{\zeta}_{\mathfrak{M}}=\int_{\mathfrak{M}}\mathrm{diag}\underline{\zeta}_{\mathfrak{M}}\equiv\zeta_{\mathfrak{M}}$, and the term zeta function often refers to this object.

The regularization consists of using $G_\infty{}^\nu$ in place of G_∞. If ν is chosen appropriately everything will converge. The regularization is then relaxed by letting ν tend to unity.

To relate $\mathcal{L}^{(1)}$ and G_∞ we note, with Schwinger,[2,8] that from (1) and (5)

$$\frac{\partial\mathcal{L}^{(1)}(x,m^2)}{\partial m^2}=\tfrac{1}{2}i\lim_{\substack{x''\to x\\ x'\to x}}G_\infty(x'',x',m^2)\,. \tag{8}$$

Then we have generally

$$\mathcal{L}^{(1)}_{\mathrm{reg}}(x,m^2)=-\tfrac{1}{2}i\int_{m^2}^\infty G^{\mathrm{reg}}_\infty(x,x,\mu^2)d\mu^2\,, \tag{9}$$

since we assume that $\mathcal{L}^{(1)}_{\mathrm{reg}}(x,\infty)$ is zero.

The zeta-function regularization is effected in (9) by replacing G^{reg}_∞ by $G_\infty{}^\nu$ and defining

$$\mathcal{L}^{(1)}_{\mathrm{reg}}=\mathcal{L}^{(\nu)},\quad \mathcal{L}^{(1)}=\lim_{\nu\to1}\mathcal{L}^{(\nu)}\,,$$

with

$$\begin{aligned}\mathcal{L}^{(\nu)}&=-\tfrac{1}{2}i\int_{m^2}^\infty \mathrm{diag}\underline{\zeta}_{\mathfrak{M}}(\nu,\mu^2)d\mu^2\\&=\tfrac{1}{2}i(\nu-1)^{-1}\mathrm{diag}\underline{\zeta}_{\mathfrak{M}}(\nu-1,m^2)\,.\end{aligned} \tag{10}$$

Then we have

$$\begin{aligned}\mathcal{L}^{(1)}=&-\tfrac{1}{2}i\lim_{\nu\to1}(\nu-1)^{-1}\mathrm{diag}\underline{\zeta}_{\mathfrak{M}}(0,m^2)\\&-\tfrac{1}{2}i\,\mathrm{diag}\underline{\zeta}'_{\mathfrak{M}}(0,m^2)\,,\end{aligned} \tag{11}$$

where $\underline{\zeta}'(\nu,w)=(d/d\nu)\underline{\zeta}(\nu,w)$. The first term in (11) will have to be removed by an infinite renormalization. There may still be finite renormalizations from the ζ' term. It is this term only that is yielded by the method of Salam and Strathdee,[9] which consists of noting that $\ln G=dG^\nu/d\nu|_{\nu=0}$ and then using the formal result $\mathcal{L}^{(1)}=-\frac{1}{2}i\,\mathrm{diag}\ln\underline{G}_\infty$.

In a general space-time we will not know ζ in closed form and we must have recourse to the proper-time expansion (2). In this way we shall make contact with the dimensional regularization given earlier. If (2) is substituted into (6) we find

$$\mathrm{diag}\underline{\zeta}(\nu, m^2) = i(4\pi)^{-d/2} \sum_{n=0}^{\infty} a_n (m^2)^{d/2-n-\nu} \times \frac{\Gamma(\nu - \frac{1}{2}d + n)}{\Gamma(\nu)}. \quad (12)$$

This gives us some of the analytic structure of $\mathrm{diag}\underline{\zeta}(\nu, m^2)$ (cf. Minakshishundaram and Pleijel[5]). Thus there are poles at $\nu = d/2, d/2-1, \ldots, 1$ if d is even (we ignore the odd-d case from now on). The residue at $\nu = d/2 - p$ is

$$i(4\pi)^{-d/2}[(\tfrac{1}{2}d - p)!]^{-1} \sum_{n=0}^{p} \frac{(-1)^{p-n}}{(p-n)!} (m^2)^{p-n} a_n . \quad (13)$$

From (10) we find

$$\mathfrak{L}^{(\nu)} = \frac{\pi^{-d/2}}{2\Gamma(\nu)} \sum_{0}^{\infty} a_n (m^2)^{d/2-n-\nu+1} \Gamma(\nu - \tfrac{1}{2}d + n - 1), \quad (14)$$

and we may compare this expression with the dimensional-regularization result, (4), for $\mathfrak{L}^{(1)}(\omega)$. We see that the two expressions are basically the same (ω is equivalent to $d/2 - \nu + 1$, so that the terms of the summation are the same).

From (11) we see that we need

$$\mathrm{diag}\underline{\zeta}(0, m^2) = i(4\pi)^{-d/2} \sum_{n=0}^{d/2} \frac{(-m^2)^{d/2-n}}{(\frac{1}{2}d - n)!} a_n . \quad (15)$$

Thus, for $d = 4$ we have

$$\mathrm{diag}\underline{\zeta}(0, m^2) = i(4\pi)^{-2}(\tfrac{1}{2}a_0 m^4 - a_1 m^2 + a_2), \quad (16)$$

the standard result.

This is probably a good place to discuss some remarks of Candelas and Raine on the form of the infinities. They deal with the specific case of de Sitter space and use dimensional regularization, but the points are of general significance. Essentially, Candelas and Raine start from Eq. (8), with a regularized G_∞, which is next expanded about the physical point $\omega = 2$, or, for us, $\nu = 1$. G_∞ has a pole at this point with residue depending on a_0 and a_1 but *not* on a_2. This apparently leads Candelas and Raine to say that the exact $\mathfrak{L}^{(1)}$, obtained from G_∞ by an (indefinite) integration with respect to m^2, also has a pole at the physical point with residue depending on only a_0 and a_1. However, they then notice that the perturbation expansion, essentially the expansion in powers of τ, produces a pole in $\mathfrak{L}^{(1)}$ with residue involving also a_2. Since this is independent of m^2 they can cancel it with a constant of integration. Or they can say that the a_2 pole term in the perturbation form of $\mathfrak{L}^{(1)}$ arises as a constant of integration when integrating the exact form of G_∞.

Our only quarrel with this is that it implies that the a_2 pole should not occur in $\mathfrak{L}^{(1)}$. In fact, if we expand G_∞ about the physical point first and then integrate over m^2 (the Candelas and Raine procedure), it is not correct to throw away all those terms in G_∞ of order $(\omega - 2)$, or $(\nu - 1)$, and higher. The Taylor series does not converge after integrating over m^2, which just reflects the existence of the a_2 pole term in $\mathfrak{L}^{(1)}$. The residue at the physical point, $\nu = 1$, is then given by (15) in general and by (16) for space-time.

The particular value $\zeta(0, m^2)$ of the zeta function appears to be an important one in zeta-function theory[7,10] in that it is related to the Atiyah-Singer index. It is just the constant term in the expansion of $e^{-im^2\tau}K(\tau)$.

In a sense, expression (11) is an exact solution for the (unrenormalized) effective Lagrangian, which is thus known insofar as the zeta function on $\mathfrak{M}$ is known. The difficulty is the evaluation of $\mathrm{diag}\underline{\zeta}'(0, m^2)$. The expansion (12) yields only an asymptotic series which we write in the form

$$\mathrm{diag}\underline{\zeta}'(0, m^2) = i(4\pi)^{-d/2}\left[\sum_{n=0}^{d/2} \frac{(-m^2)^{d/2-n}}{(\frac{1}{2}d - n)!} a_n \big(\psi(\tfrac{1}{2}d - n + 1) + \gamma - \ln m^2\big) + \sum_{n=d/2+1}^{\infty} a_n \Gamma(n - \tfrac{1}{2}d)(m^2)^{d/2-n}\right], \quad (17)$$

with

$$\psi(z) = \Gamma'(z)/\Gamma(z) \text{ and } \gamma = -\psi(1).$$

The first term can be interpreted as a finite renormalization, although in the absence of *any* infinite part to $\mathfrak{L}^{(1)}$, as in the Salam-Strathdee method, we would prefer to take the a_2 term (for $d = 4$) as contributing to the radiative corrections.

The asymptotic series (17) is not much use if we are interested in investigating what happens in a strong gravitational field. One would like to have a closed expression for $\mathfrak{L}^{(1)}$, and this is one reason for choosing $\mathfrak{M}$ to be de Sitter space. However, before getting into this, our main topic, we must discuss the energy-momentum tensor, which is the central physical quantity.

III. THE ENERGY-MOMENTUM TENSOR

The total action will be $S+W^{(1)}$, where S is the bare action of the gravitational field ignoring any matter contribution

$$S=\int d^4x(-g)^{1/2}[(16\pi G_0)^{-1}R-\lambda_0+\alpha_0 R^2+\beta_0 R_{\mu\nu}R^{\mu\nu}+\gamma_0 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}].$$

The R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ terms are included in order to counter similar, infinite terms in $W^{(1)}$. The field equations are

$$(8\pi G_0)^{-1}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu})+\lambda_0 g_{\mu\nu}+2\alpha_0\,{}^{(1)}H_{\mu\nu}+2\beta_0\,{}^{(2)}H_{\mu\nu}+2\gamma_0\,{}^{(3)}H_{\mu\nu}=-\langle T_{\mu\nu}\rangle\,, \tag{18}$$

where ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$, and ${}^{(3)}H_{\mu\nu}$ are tensors constructed from the curvature and its derivatives.[11] On the right-hand side of (18), the effective energy-momentum tensor $\langle T_{\mu\nu}\rangle$ is given by

$$\langle T_{\mu\nu}\rangle=(-g)^{-1/2}\,\frac{\delta W^{(1)}}{\delta g^{\mu\nu}}=\frac{\langle 0\text{ out}|\,T_{\mu\nu}(\varphi,\varphi)|\,0\text{ in}\rangle}{\langle 0\text{ out}|\,0\text{ in}\rangle}, \tag{19}$$

where $T_{\mu\nu}(\varphi,\varphi)$ is the energy-momentum tensor of the scalar field, φ,

$$T_{\mu\nu}=2(-g)^{1/2}\,\frac{\delta S_\varphi}{\delta g^{\mu\nu}}\,.$$

S_φ is the φ-field action. In the case that the φ-field equations take the form

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu+\xi R+m^2)\varphi=0\,, \tag{20}$$

$T_{\mu\nu}$ is given by[4]

$$\begin{aligned}T_{\mu\nu}={}&(1-2\xi)\nabla_\mu\varphi\nabla_\nu\varphi+(2\xi-\tfrac{1}{2})g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\varphi\nabla_\sigma\varphi+\tfrac{1}{2}g_{\mu\nu}m^2\varphi^2-2\xi\varphi\nabla_\mu\nabla_\nu\varphi\\&+2\xi g_{\mu\nu}g^{\lambda\sigma}\varphi\nabla_\lambda\nabla_\sigma\varphi-\xi(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\varphi^2\,.\end{aligned} \tag{21}$$

In (20) $\xi=0$ gives the minimal Klein-Gordon equation, while $\xi=(d-2)/(4d-4)$ corresponds to the "improved" equation.[12]

Following Schwinger[2] and Gibbons[13] [see also (18)] we can express $\langle T_{\mu\nu}\rangle$ in terms of G_∞,

$$G_\infty(x'',x')=i\langle 0\text{ out}|\,T\{\varphi(x'')\varphi(x')\}|\,0\text{ in}\rangle/\langle 0\text{ out}|\,0\text{ in}\rangle.$$

Thus,

$$\langle T_{\mu\nu}\rangle=-i\lim_{x'\to x}\;T_{\mu\nu'}G_\infty(x,x')\,, \tag{22}$$

where

$$\begin{aligned}T_{\mu\nu'}={}&(1-2\xi)\nabla_\mu\nabla_{\nu'}+g_{\mu\nu'}[(2\xi-\tfrac{1}{2})g^{\lambda\sigma'}\nabla_\lambda\nabla_{\sigma'}+\tfrac{1}{2}m^2]\\&-\xi(g_{\mu\rho'}\nabla^{\rho'}\nabla_{\nu'}+g_{\nu'\sigma}\nabla^\sigma\nabla_\mu)+\xi g_{\mu\nu'}(\nabla_\rho\nabla^\rho+\nabla_{\rho'}\nabla^{\rho'})-\tfrac{1}{2}\xi(R_\mu{}^\sigma g_{\sigma\nu'}+R^{\rho'}{}_{\nu'}g_{\mu\rho'})+\tfrac{1}{2}\xi Rg_{\mu\nu'}\,.\end{aligned}$$

Here $g_{\mu\nu'}(x,x')$ is the standard vector parallel propagator and the limit is understood to be the average of the expressions for x_0-x_0' positive and negative in turn.

$\langle T_{\mu\nu}\rangle$ will diverge. To investigate how, we can do several things. We can set $x'^\mu-x^\mu=\epsilon^\mu$ and investigate $\epsilon^\mu\to 0$. This is the (covariant) point-separation method.[4] Alternatively, G_∞ can be regularized, by our favorite method, so allowing the limit to be taken directly. In this case the direction of the limit is immaterial. The infinities are exhibited by relaxing the regularization. If we write, as usual, $G_\infty=\overline{G}+\frac{1}{2}iG^{(1)}$, where $\overline{G}$ is the average of the retarded and advanced Green functions, and is always real, then in place of (22) we have

$$\langle T_{\mu\nu}(x)\rangle=\tfrac{1}{2}\lim_{x'\to x}\;T_{\mu\nu'}G^{(1)}(x,x')\,,$$

in general agreement with DeWitt[4] [Eq. (254)].

In the general case, as DeWitt[4] indicates, we can use the proper-time expansion of $K(x'',x',\tau)$ to calculate (22) and, while there is no difficulty in principle, the result would be a little complicated and still not useful for strong fields. For this reason we turn to the special case when $\mathfrak{M}$ is a de Sitter space. Then, as we shall see, both (19) and (22) can be evaluated. They are trivially identical, as expected.

IV. de SITTER–SPACE FIELD THEORY

The basic equations have been given in I and so we start straightaway with dimensional regularization. We firstly rederive the results of Candelas and Raine[3] in, in our opinion, a preferable manner.

The Green's function on $S^1_{2\omega}$ is written in the form (5) so that we require the quantum-mechanical propagator, K, on $S^1_{2\omega}$. As explained in I we obtain this by continuation from that on the Euclidean sphere, $S_{2\omega}$. This is not necessary but is quite convenient.

On $S_{2\omega}$ the δ function is given by

$$\delta_{S_{2\omega}}(x'', x') = (2\pi a^2)^{-\omega}\left(\frac{d}{dp}\right)^{\omega-1}\delta(p-1)\,, \tag{23}$$

where

$$p(x'', x') = 1 + \frac{[\underline{\xi}(x'') - \underline{\xi}(x')]^2}{2a^2}$$

and $\underline{\xi}^2 = -a^2$ (remember that the metric on $S_{2\omega}$ is negative-definite).

Because of the symmetries of the sphere and the covariance of the equations of motion the propagator on $S_{2\omega}$, $K_E(x'', x', \tau)$, will be a function of x'' and x' through $p(x'', x')$ and therefore given by

$$K_E(x'', x', \tau) = K_E(p, \tau) = \exp\{-i[\Box_0{}^2 + a^{-2}2\xi\omega(2\omega-1)]\tau\}\,\delta_{S_{2\omega}}\,, \tag{24}$$

where $\Box_0{}^2$ is the "radial" part of $\Box^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$,

$$\Box_0{}^2 = a^{-2}\left[(p^2-1)\frac{d^2}{dp^2} + 2\omega p\frac{d}{dp}\right]. \tag{25}$$

The Legendre expansion,

$$\delta(p-1) = \sum_{n=0}^{\infty}(n+\tfrac{1}{2})P_n(p)\,,$$

is next used and (23) and (25) are substituted into (24) to yield

$$K_E(p, \tau) = (2\pi a^2)^{-\omega}\left(\frac{d}{dp}\right)^{\omega-1}\sum(n+\tfrac{1}{2})\exp\left\{i\frac{\tau}{a^2}[(\omega-\tfrac{1}{2})^2 - 2\xi\omega(2\omega-1) - (n+\tfrac{1}{2})^2]\right\}P_n(p)\,. \tag{26}$$

The most rapid way of deriving the Green's function is to insert (26) into (5), integrate over τ, and then use Dougall's formula to obtain the (Euclidean) Green's function G_E as a multiple derivative of $P_{-1/2+i\overline{m}a}(-p)$, where

$$\overline{m}^2a^2 \equiv m^2a^2 - (\omega-\tfrac{1}{2})^2 + 2\xi\omega(2\omega-1)\,. \tag{27}$$

This can then be rewritten as a hypergeometric function, and we find

$$G_E(x'', x', \omega) = a^2(4\pi a^2)^{-\omega}\frac{\Gamma(\omega-\frac{1}{2}+i\overline{m}a)\Gamma(\omega-\frac{1}{2}-i\overline{m}a)}{\Gamma(\omega)}\,{}_2F_1\left(\omega-\tfrac{1}{2}+i\overline{m}a,\,\omega-\tfrac{1}{2}-i\overline{m}a;\,\omega;\,\frac{1+p}{2}\right). \tag{28}$$

The Green's function, G_∞, on the pseudo-Euclidean sphere $S^1_{2\omega}$ is easily obtained from G_E by multiplying by i and allowing p to become larger than unity. If we now interpret ω and p as complex numbers, the hypergeometric function in (28) has a branch point at $p=1$. In this case we must set $p \to p - i0$ (see I), and thus we have derived the expression of Candelas and Raine. For $\omega = 2$ we get Tagirov's[14] result, which he obtained by eigenfunction summation.

In the region $\mathrm{Re}\,\omega < 1$ there is convergence at $p=1$, and the coincidence limit is

$$G_\infty(x, x, \omega) = ia^2(4\pi a^2)^{-\omega}\frac{\Gamma(\omega-\frac{1}{2}+i\overline{m}a)\Gamma(\omega-\frac{1}{2}-i\overline{m}a)}{\Gamma(\frac{1}{2}+i\overline{m}a)\Gamma(\frac{1}{2}-i\overline{m}a)}\times\Gamma(1-\omega)\,. \tag{29}$$

[This value can be derived even more rapidly without in fact going through the step of finding $G_\infty(x'', x', \omega)$.]

We may note immediately that $G_\infty(x, x, \omega)$ is pure imaginary (if ω is real), and hence, from (8), we see that $\mathcal{L}^{(1)}$ is real. Thus there is no pair creation, a fact that can be deduced in various ways (see I).

From (28) we can also calculate the real part, $\overline{G}_\infty$, of G_∞,

$$\overline{G}_\infty(x'', x', \omega) = \begin{cases} -\frac{1}{2}(2\pi a^2)^{1-\omega}\dfrac{(p-1)^{1-\omega}}{\Gamma(2-\omega)}\,{}_2F_1\left(\frac{1}{2}+i\overline{m}a,\,\frac{1}{2}-i\overline{m}a;\,2-\omega;\,\dfrac{1-p}{2}\right), & p>1 \\ 0, & p<1 \end{cases}$$

and the commutator function, G, then follows as $G=-\frac{1}{2}\epsilon(x'', x')\overline{G}$.

The further progress of the dimensional-regularization method, as pursued by Candelas and Raine, consists of expanding $G_\infty(x, x, \omega)$ about the pole at $\omega=2$, discarding terms of order $(\omega-2)$, and then integrating with respect to m^2.

As we have explained earlier this is not, in principle, a satisfactory procedure. One should integrate before expanding. However, in practice one can simply put back the missing terms by hand, since it can be shown that the only term lacking is the infinite a_2 pole.

Thus we do not follow the renormalization procedure of Candelas and Raine. Rather, our method corresponds to the more conventional view.[15] That is, we include the R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ terms in the bare gravitational Lagrangian and then renormalize the α_0, β_0, and γ_0 coefficients.

To do this consistently we define dimension-dependent renormalized coupling constants G, λ, α, β, and γ by

$$\begin{aligned}&\lambda_0=\lambda+C_0(\omega),\\&\omega(2\omega-1)(8\pi G_0)^{-1}=\omega(2\omega-1)(8\pi G)^{-1}-C_1(\omega),\\&4\omega^2(2\omega-1)\left[\alpha_0+\frac{1}{2\omega}\beta_0+\frac{1}{\omega(2\omega-1)}\gamma_0\right]\\&\qquad=4\omega^2(2\omega-1)\left[\alpha+\frac{1}{2\omega}\beta+\frac{1}{\omega(2\omega-1)}\gamma\right]-C_2(\omega),\end{aligned}\tag{30}$$

where $C_n(\omega)$ is the coefficient of the term in $\mathcal{L}^{(1)}(\omega)$ that goes like $(a^2)^{-n}$. The important thing to realize here is that $G(\omega)$, $\lambda(\omega)$, $\alpha(\omega)$, $\beta(\omega)$, and $\gamma(\omega)$ are analytic at $\omega=2$, which shows that the unrenormalized constants, $G_0^{-1}(\omega)$, etc., have to have poles at $\omega=2$ to cancel those coming from the $C_n(\omega)$. Equation (4) gives the first three $C_n(\omega)$ as

$$\begin{aligned}&C_0(\omega)=\tfrac{1}{2}(4\pi)^{-\omega}(m^2)^{\omega}\Gamma(-\omega),\\&C_1(\omega)=\tfrac{1}{2}(4\pi)^{-\omega}(m^2)^{\omega-1}\Gamma(1-\omega)2\omega(2\omega-1)(\tfrac{1}{6}-\xi),\\&C_2(\omega)=\tfrac{1}{6}(4\pi)^{-\omega}(m^2)^{\omega-2}\Gamma(2-\omega)[2\xi\omega(3\xi-1)(2\omega-1)+\tfrac{1}{3}(\omega^2-\tfrac{7}{10}\omega+\tfrac{3}{10})]\omega(2\omega-1).\end{aligned}\tag{31}$$

The expressions for the coefficients a_n can be found from the expansion of the propagator K, $=iK_E$ of (26), or from the exact formula (29) for $G_\infty(x, x, \omega)$, or by using the formula given by DeWitt[16] for a general Riemann space. This last formula provides a useful check of the algebra.

The total Lagrangian is given by

$$\begin{aligned}\mathcal{L}(\omega)=&(8\pi Ga^2)^{-1}\omega(2\omega-1)-\lambda+4a^{-4}\omega^2(2\omega-1)^2\left[\alpha+\frac{1}{2\omega}\beta+\frac{1}{\omega(2\omega-1)}\gamma\right]\\&+\mathcal{L}^{(1)}(\omega)-C_0(\omega)-a^{-2}C_1(\omega)-a^{-4}C_2(\omega),\end{aligned}\tag{32}$$

where $\partial\mathcal{L}^{(1)}/\partial m^2$ is given by (8) and $G_\infty(x, x)$ by (29). Everything is now expanded about $\omega=2$. For example, we have

$$\begin{aligned}G_\infty(x, x, \omega)=&\frac{i}{16\pi^2a^2}\{(m^2a^2+12\xi-2)[(\omega-2)^{-1}+2\,\mathrm{Re}\psi(\tfrac{3}{2}+i\overline{m}a)-\ln 4\pi a^2+\gamma-1]+12\xi'+14\xi-3\}\\&+O(\omega-2)\end{aligned}\tag{33}$$

[$\xi'\equiv d\xi/d\omega$ and $2\,\mathrm{Re}\psi(\frac{3}{2}+i\overline{m}a)$ is shorthand for $\psi(\frac{3}{2}+i\overline{m}a)+\psi(\frac{3}{2}-i\overline{m}a)$], with the asymptotic expansion

$$\mathrm{Re}\psi(\tfrac{3}{2}+i\overline{m}a)=\tfrac{1}{2}\ln m^2+(m^2a^2)^{-1}(6\xi-\tfrac{2}{3})+(m^2a^2)^{-2}(8\xi-36\xi^2-\tfrac{11}{30})-\cdots.\tag{34}$$

In these expansions all quantities that depend on ω such as ξ, ξ' are evaluated at the physical point $\omega=2$. As a check of the algebra we can, if we wish, use expressions (34) and (33) to find the values of the coefficients a_n and a_n' for $\omega=2$ by comparing it with the expansion of the series

$$G_\infty(x, x, \omega)=i(4\pi)^{-\omega}\sum_{n=0}^{\infty}a_n(\omega)(m^2)^{\omega-n-1}\Gamma(n-\omega+1).\tag{35}$$

Continuing $\mathcal{L}(\omega)$ to $\omega=2$ yields the renormalized Lagrangian

$$\mathcal{L} = 3(4\pi a^2 G)^{-1} - \lambda + 144a^{-4}(\alpha + \tfrac{1}{4}\beta + \tfrac{1}{6}\gamma) + (16\pi^2)^{-1}\Big\{[\tfrac{1}{4}m^4 + a^{-2}m^2(6\xi - 1)]\ln(m^2a^2) - \tfrac{1}{8}m^4 + \tfrac{1}{4}a^{-2}m^2 - a^{-2}\int dm^2(m^2a^2 + 12\xi - 2)\,\mathrm{Re}\,\psi(\tfrac{3}{2} + i\bar{m}a) + \tfrac{1}{2}a^{-4}(72\xi^2 - 12\xi + \tfrac{29}{15})\ln(m^2a^2)\Big\}, \tag{36}$$

where, in accordance with our previous remarks, we have included the effect of the a_2 pole.

Later, we shall want to set m equal to zero, and the last term in (36) produces a typical infrared divergence. We assume that this can be taken care of by introducing an upper cutoff to the proper-time integrals, for example. In any case the divergence does not carry through to the energy-momentum tensor.

V. ENERGY-MOMENTUM TENSOR IN de SITTER SPACE

We substitute the dimensionally regularized $G_\infty(x'', x', \omega)$ into Eq. (22) and take the limit directly. Because of the geometrical structure of de Sitter space all derivatives can be reduced to the operator d/dp in the following way. Take the typical quantity $\nabla_\mu f(p)$. This equals

$$\frac{df}{dp}\nabla_\mu p = \frac{df}{dp}\frac{\partial p}{\partial \xi^\alpha}\nabla_\mu \xi^\alpha = a^{-2}(\xi_\alpha - \xi'_\alpha)\nabla_\mu \xi^\alpha \frac{df}{dp}.$$

Then make use of the standard equations[17]

$$\xi_\alpha \nabla_\mu \xi^\alpha = 0,$$
$$\nabla_\mu \xi_\alpha \nabla_\nu \xi^\alpha = g_{\mu\nu},$$
$$\nabla_\mu \xi_\alpha \nabla'_\nu \xi'^\alpha = g_{\mu\nu'},$$
$$\nabla_\mu \nabla_\nu \xi^\alpha = a^{-2} g_{\mu\nu}\xi^\alpha$$

to give

$$\lim_{x'\to x} \nabla_\mu p = 0,$$
$$\lim_{x'\to x} \nabla_\mu \nabla_{\nu'} p = -a^{-2} g_{\mu\nu}\delta^\nu_{\nu'},$$
$$\lim_{x'\to x} \nabla_\mu \nabla_\nu p = a^{-2} g_{\mu\nu}.$$

Whence, from (21),

$$\langle T_{\mu\nu}(\omega)\rangle = -ia^{-2}g_{\mu\nu}\{(\omega - 1)G_\infty'(p) + [\tfrac{1}{2}m^2a^2 + \xi(2\omega^2 - 3\omega + 1)]G_\infty(p)\}_{p=1}, \tag{37}$$

with $G' = dG/dp$. It is trivial to calculate $G_\infty'(p=1)$. We find

$$G_\infty'(x, x, \omega) = \tfrac{1}{2} i a^2 (4\pi a^2)^{-\omega}\frac{\Gamma(\omega + \frac{1}{2} + i\bar{m}a)\Gamma(\omega + \frac{1}{2} - i\bar{m}a)}{\Gamma(\frac{1}{2} + i\bar{m}a)\Gamma(\frac{1}{2} - i\bar{m}a)}\Gamma(-\omega)$$
$$= -\frac{1}{2\omega}[(\omega - \tfrac{1}{2})^2 + \bar{m}^2a^2]G_\infty(x, x, \omega). \tag{38}$$

$\langle T^{\mu\nu}\rangle$ is then given by $\langle T^{\mu\nu}\rangle = g^{\mu\nu}T(\omega)$ with

$$T(\omega) = -\frac{im^2}{2\omega}G_\infty(x, x, \omega). \tag{39}$$

This also follows more directly by taking the trace of (22).

The intention now is to substitute this $\langle T^{\mu\nu}\rangle$ into the field equations, (18), which take the specific form for 2ω-dimensional de Sitter space

$$\left\{(8\pi a^2 G_0)^{-1}(2\omega - 1)(1 - \omega) + \lambda_0 - 4a^{-4}\omega(2\omega - 1)^2(\omega - 2)\left[\alpha_0 + \frac{1}{2\omega}\beta_0 + \frac{1}{\omega(2\omega - 1)}\gamma_0\right]\right\}g^{\mu\nu} = -\langle T^{\mu\nu}(\omega)\rangle. \tag{40}$$

We note that it is necessary to keep the contributions from ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$, and ${}^{(3)}H_{\mu\nu}$. It is true that these tensors vanish in four dimensions, but we recall that the unrenormalized α_0, β_0 and γ_0 have poles at $\omega = 2$ so that the last term on the left-hand side of (40) contributes a finite term of order a^{-4}.

The $\langle T^{\mu\nu}\rangle$ of (37) also contains a term of order a^{-4} which cancels the similar term on the left of (40). We give a few details of this mechanism.

Equation (40) can be obtained by varying the total action, the Lagrangian density for which takes the form

$$\mathcal{L}(\omega) = (8\pi G_0)^{-1}\omega(2\omega - 1)K - \lambda_0 + 4\omega^2(2\omega - 1)^2\left[\alpha_0 + \frac{1}{2\omega}\beta_0 + \frac{1}{\omega(2\omega - 1)}\gamma_0\right]K^2 + \mathcal{L}^{(1)}(\omega)$$

with $K \equiv a^{-2}$. The variational principle reduces to

$$\frac{K^{\omega+1}}{\omega}\frac{d}{dK}(\mathcal{L}K^{-\omega})=0\,,$$

so that we obtain Eq. (40) with (cf. Ref. 2)

$$\langle T^{\mu\nu}\rangle = g^{\mu\nu}T = g^{\mu\nu}\omega^{-1}K^{\omega+1}\frac{d}{dK}(\mathcal{L}^{(1)}K^{-\omega})\,. \tag{41}$$

The equivalence of this definition of $\langle T^{\mu\nu}\rangle$ [essentially Eq. (19)] with that of (21), i.e. (39), is easily established. Expansion (4) for $\mathcal{L}^{(1)}$ is substituted into (41) and the fact that a_n is proportional to K^n is used to perform the differentiation. The calculation is trivial and Eq. (39) emerges with $G_\infty(x,x,\omega)$ in the form of the series (35).

Equation (40) is next to be written in explicitly finite form. This is achieved by defining the renormalized energy-momentum tensor after subtracting "appropriate" terms from $\langle T^{\mu\nu}\rangle$,

$$\langle T^{\mu\nu}\rangle_{\text{ren}} = g^{\mu\nu}T_R(\omega)\,,$$
$$T_R(\omega)=T(\omega)-T_0(\omega)-a^{-2}T_1(\omega)-a^{-4}T_2(\omega)\,, \tag{42}$$

where T_n is the coefficient of $(a^{-2})^n$ in the expansion of $T(\omega)$. Specifically we have

$$T_0=-C_0\,,$$

$$T_1=\omega^{-1}(1-\omega)C_1\,,$$

$$T_2=\omega^{-1}(2-\omega)C_2\,.$$

The renormalized form of Eq. (40) is then

$$(8\pi a^2G)^{-1}(2\omega-1)(1-\omega)+\lambda-4a^{-4}\omega(2\omega-1)^2(\omega-2)\left[\alpha+\frac{1}{2\omega}\beta+\frac{1}{\omega(2\omega-1)}\gamma\right]=-T_R(\omega)\,, \tag{43}$$

with the *same* definition of renormalized coupling constants as before, Eq. (30).

This equivalence of renormalization can also be shown, of course, after the expansion about $\omega=2$ has been performed. A slight technical point arises here which may be useful to mention. If we try to expand the middle equation, say, in Eq. (30) we must assume that $1/G_0(\omega)$ has a pole at $\omega=2$ first. Then we find, near $\omega=2$,

$$3(4\pi G_0)^{-1}=3(4\pi G)^{-1}-A(\omega-2)^{-1}B+\tfrac{7}{12}A+O(\omega-2)\,,$$

where A and B are known constants defined by the expansion of $C_1(\omega)$,

$$C_1(\omega)=A(\omega-2)^{-1}+B+O(\omega-2)\,.$$

Now we give the expansions of $T(\omega)$ and $T_R(\omega)$:

$$T(2)=m^2(64\pi^2a^2)^{-1}\{(m^2a^2+12\xi-2)[(\omega-2)^{-1}+2\,\mathrm{Re}\,\psi(\tfrac{3}{2}+i\overline{m}a)-\ln 4\pi a^2+\gamma\tfrac{3}{2}]+14\xi+12\xi'-3\}\,, \tag{44}$$

$$T_R(2)=m^2(64\pi^2a^2)^{-1}\{(m^2a^2+12\xi-2)[2\,\mathrm{Re}\,\psi(\tfrac{3}{2}+i\overline{m}a)-\ln(m^2a^2)-1] + m^2a^2-\tfrac{2}{3}-2(m^2a^2)^{-1}[(6\xi-1)^2-\tfrac{1}{30}]\}\,. \tag{45}$$

Setting ω equal to 2 in (43) gives the "field equation" corresponding to the renormalized Lagrangian (32). The third term on the left now disappears (α, β, and γ are finite) and the final result is the same as that which we would have got by dropping terms of order a^{-4} at the outset, as did Candelas and Raine.

For certain purposes it is convenient to develop the arguments for arbitrary ω. However, in order to avoid problems of competing limiting processes we prefer now to work with the expressions at the physical ω point, such as (44) and (45).

Particular interest attaches itself to the conformally invariant ("improved") Eq. (20), with $\xi=(\omega-1)/(4\omega-2)$. This means we set $\xi=\frac{1}{6}$, $\xi'=\frac{1}{18}$, and $\overline{m}^2a^2=m^2a^2-\frac{1}{4}$ in (44) and (45). We would also be interested in the case $m=0$, when, independently of the value of ξ, $T(2)$ vanishes.

For $\xi=\frac{1}{6}$ and $m=0$, the massless conformally invariant case $T_R(2)$ takes the value

$$T_R(2)=(960\pi^2a^4)^{-1}\,. \tag{46}$$

Note that this value is the result of subtracting the T_2/a^4 term in (42). It should not be confused with the nonzero vacuum average of $T^{\mu\nu}$ obtained by Ford[18] by a Casimir-type calculation for closed Robertson-Walker metrics.

Ford's nonzero value in de Sitter space is a consequence of his using a subtraction procedure which violates de Sitter invariance. Our calculation is de Sitter-invariant throughout and we naturally obtain a vanishing vacuum average for the massless case.[19]

In this connection we note that when $\xi=\frac{1}{6}$ the pole term in (44), the unrenormalized T, is independent of the geometry and we can then apply Ford's version of Casimir's argument. This fact agrees with a remark of Ford's.[18] However, the method gives a zero answer for $m=0$, as expected. If m does not vanish we find the value

$$T_C(2)=m^4(64\pi^2)^{-1}[2\,\mathrm{Re}\psi(\tfrac{3}{2}+i\overline{m}a)-\ln(m^2a^2)]$$

for this "Casimir renormalized" T.

It is amusing and probably not significant to notice that if we set $\lambda=0$ in Eq. (43), for $\omega=2$, and then use (46) on the right-hand side, we obtain $a\approx 10^{-34}$ cm. Thus a massless, conformally invariant scalar field can support self-consistently through its vacuum fluctuations a de Sitter universe of typically quantum geometric dimensions.

VI. DISCUSSION

In view of recent papers[4,11,13,18,20,21] on the subject of vacuum energy in curved spaces it seems unnecessary to give a review of the background material.

We have considered the coupled Einstein-Klein-Gordon system for a given (de Sitter) background gravitational field and have renormalized the field equations in the traditional fashion.[15] The problem, if there is one, seems to be the interpretation of the result. Can we take finite terms from the left-hand side of Einstein's equations onto the right and still call this the vacuum stress tensor of the scalar field? It is our opinion that we can.

The result is a $\langle T^{\mu\nu}\rangle_{\rm ren}$, which in the massless, conformally invariant case is not traceless. We do not view this as a real difficulty. The only conformal rescalings allowed, if we are to remain in de Sitter space, are constant rescalings.

At the more technical level, instead of the dimensional regularization method it would have been possible to use the zeta-function approach outlined in Sec. II. This would have involved a discussion of zeta functions on spheres, an interesting subject in its own right and probably worth pursuing from a formal angle. However, the result for us would have been the same.

The advantage of the particular dimensional regularization used in this paper (and earlier in Ref. 3) is that all quantities are displayed as closed hypergeometric expressions. In I we sketched an alternative scheme which also leads to similar results. Briefly, Eq. (25) is written as a sum over classical paths by using a Mehler-Dirichlet integral for P_n and then a θ-function transformation. In the resulting equation for G the integrations can be performed for $\omega<1$ and we find an expression for $G_\infty(x,x,\omega)$ which differs from the Candelas and Raine form (28) but which produces the same renormalization theory. For this reason we have not employed it here, although it possesses certain advantages and allows a comparison with the proper-time method with a minimum of new equations.

An interesting feature of our renormalization procedure is that the quadratic terms in the bare gravitational Lagrangian give contributions to the field equations that do not vanish in the four-dimensional limit because the unrenormalized coupling constants have poles at this point.

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