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Geometrical packing and the solidification of neutron-star matter*

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We examine the role, if any, of geometrical packing or "caging" of the neutron repulsive cores in the solidification of neutron matter at neutron-star densities. A search for a stable solid phase retaining only the repulsive part of the Reid ${}^1S_0$ potential and Bose statistics shows that this model system cannot sustain a solid core until densities much beyond 2 fm^{-3}. This indicates that the n-n repulsive core is too soft to induce solidification at neutron-star densities. Using the analogy of helium it is argued that including the intermediate-range attractive well of the n-n potential would not lower the solidification density significantly. In support of this, a second search was conducted with a potential which was a density-dependent average of the ${}^1S_0$ and ${}^1D_2$ Reid potentials (including the attractive part). Again no stable solid solution was found.

I. INTRODUCTION

Recently we presented a new variational procedure for dealing with quantum crystals[1] which successfully predicts both the solidification density and correlation energy of ^{4}He.[2] In this paper we apply the same method to examine whether neutron matter solidifies at the high densities found in the interior of neutron stars ($0.5 \leqslant \rho \leqslant 2.0$ fm^{-3}). Apart from the existence of such a solid core having its own fundamental interest, it has been proposed as a possible cause of the speed-up in the period of the Vela pulsar,[3] and the on-off cycle of the x-ray source Her X – 1.[4]

At the densities characteristic of neutron-star interiors the average spacing between the neutrons is comparable to the radius of the n-n potential's repulsive core. An analogous situation is found in liquid ^{3}He and ^{4}He where the average interparticle spacing at equilibrium density is comparable to the repulsive core radius of the He-He potential. This similarity was exploited by Anderson and Palmer,[5] who assumed that the n-n potential has a form similar to the potential between two atoms, and then used the law of corresponding states to estimate that the solidification density of neutron matter was about 0.24 fm^{-3}. Clark and Chao[6] independently obtained a similar value for the solidification density also using this law but with a different potential which fitted the n-n potential better. More recently Miller, Nosanow, and Parish[7] have examined in considerable detail the application of the law of corresponding states in the general phenomena of melting.

However, having pointed out this similarity, we must hasten to add that the He-He potential is much the more singular potential at short distances. For small r the He-He potential goes as r^{-n}, $8 \lesssim n \lesssim 16$, whereas the n-n potential goes roughly as r^{-2} to r^{-3}. Accordingly the He-He potential should induce far stronger short-range correlations.

The primary mechanism in the solidification of ^{3}He and ^{4}He is known to be "geometrical caging" caused by the repulsive cores. The attractive part of the potential (and exchange effects in the case of ^{3}He) have little effect in determining the solidification density. Hansen and Pollock,[8] for example, found in their Monte Carlo calculation for ^{4}He that they could completely remove the attractive well of the Lennard-Jones potential without altering their calculated solidification density. Since the relative volumes occupied by the repulsive cores for helium and neutron-star matter are so close, it is of interest to see whether the softer n-n potential core is still sufficiently repulsive to cause solidification by the same caging mechanism.

With this limited aim in mind, we only consider the most repulsive components of the short-range n-n potential. This permits us to work with central, state-independent potentials. The repulsive part of the ${}^1S_0$ Reid[9] potential is

$$v_{\mathrm{I}}(r) = 46.78\, \hbar c \frac{e^{-7ar}}{r}, \quad a = 0.7\ \mathrm{fm}^{-1}. \qquad (1)$$

This state-independent potential is used in the so-called "homework model"[10-17] which was devised at the 1973 Urbana workshop on dense matter primarily as a test of different many-body techniques. A second reason for using $v_{\mathrm{I}}(r)$ for the interaction between all two-nucleon states is that it permits us to determine whether geometrical caging *by itself* can cause solidification at neutron-star densities. For the same reason we use Bose rather than Fermi statistics for "homework model" neutrons (recalling the relative unimportance of exchange effects in ^{3}He solidification).

To check the effect of including at least part of the attractive region of the n-n potential, we also carried out the calculation with a second potential which is a density-dependent average over the $^1S_0$ and $^1D_2$ Reid potentials[18]

$$v_{\rm II}(r) = \hbar c\left[0.075\frac{e^{-ar}}{r} - \frac{0.089\alpha}{1+\alpha}e^{-2ar} - \frac{(11.91+8.03\alpha)}{1+\alpha}\frac{e^{-4ar}}{r} + 46.78\frac{e^{-7ar}}{r}\right], \tag{2}$$

where $a = 0.7\ {\rm fm}^{-1}$, $\alpha^3 = (b\rho)^2$, $b = 0.94\ {\rm fm}^3$.

Using either potential we came to identical conclusions about the absence of solidification within the relevant density range (up to $2.0\ {\rm fm}^{-3}$). This suggests that a calculation which realistically treats the noncentral and state-dependent attractive part of the n-n potential would likewise reach a similar negative conclusion. Pion condensation as a possible alternate solidification mechanism[19,20] is outside the realm of this discussion and we do not consider it.

In Sec. II we describe how we applied our variational method to search for stable solid solutions in neutron-star matter, and in Sec. III we present our results for the potentials $v_{\rm I}(r)$ and $v_{\rm II}(r)$. Finally, in the Appendix we discuss briefly Feenberg's arguments[21] that a translationally invariant wave function is quite capable of completely describing the internal structure of a solid, and Clark and Sandler's[17] subsequent search for a "floating solid" in the solutions of Shen *et al.*[15] for liquid neutron matter.

II. VARIATIONAL CALCULATION

In this section we apply the variational procedure outlined in Ref. 1 to the problem of solid neutron-star matter. The Hamiltonian for the system is

$$H = -\sum_{i=1}^{N}\nabla_i^2 + \sum_{i<j=1}^{N} v(r_{ij}), \tag{3}$$

where the state-independent, central potential is either $v_{\rm I}(r)$ or $v_{\rm II}(r)$.

Since Shen *et al.*[15] have shown that the three-particle factors in the translationally invariant wave function ψ_L contribute only negligibly to the energy, we may justifiably neglect these factors in the solid wave function ψ_S, since their contribution to ψ_S should be even smaller. We may accordingly restrict our class of variational wave functions for the solid to

$$\psi_S(\vec{r}_1,\ldots,\vec{r}_N) = \prod_{i=1}^{N}\varphi_S(\vec{r}_i)\prod_{j<k=1}^{N} e^{u_S(r_{jk})/2}. \tag{4}$$

Note that ψ_S will have the same crystal symmetry as the single-particle wave function $\varphi_S(\vec{r})$.

We define

$$\psi_L(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N) = \prod_{i<j=1}^{N} e^{u_L(r_{ij})/2}, \tag{5}$$

$$E_L = \langle\psi_L|H|\psi_L\rangle/\langle\psi_L|\psi_L\rangle, \tag{6}$$

and

$$E_S = \langle\psi_S|H|\psi_S\rangle/\langle\psi_S|\psi_S\rangle, \tag{7}$$

and denote the minimum values of the expectation values by $\tilde{E}_L$ and $\tilde{E}_S$, respectively. The restriction

$$\tilde{E}_S \geq \tilde{E}_L \tag{8}$$

follows from the discussion in the Appendix since the three-particle and higher correlations in the wave functions ψ_L and ψ_S are negligible. If the ground state is crystalline, there must exist some nonconstant $\varphi_S(\vec{r})$ with the symmetry of the crystal for which the equality in Eq. (8) is satisfied. If no such $\varphi_S(\vec{r})$ exists, the liquid is the stable phase.

We may write the exact expressions for the energies E_L and E_S as

$$\begin{aligned} E_L &= \tfrac{1}{2}\int d\vec{r}_1 d\vec{r}_2 \tilde{v}_L(r_{12}) P_L^{(2)}(\vec{r}_1,\vec{r}_2), \\ E_S &= -\tfrac{1}{2}\int d\vec{r}_1 P_S^{(1)}(\vec{r}_1)\nabla_1^2 \ln\varphi_S(\vec{r}_1) \\ &\quad + \tfrac{1}{2}\int d\vec{r}_1 d\vec{r}_2 \tilde{v}_S(r_{12}) P_S^{(2)}(\vec{r}_1,\vec{r}_2), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \tilde{v}_L(r) &= v(r) - \tfrac{1}{2}\nabla^2 u_L(r), \\ \tilde{v}_S(r) &= v(r) - \tfrac{1}{2}\nabla^2 u_S(r). \end{aligned} \tag{10}$$

$P_L^{(n)}(\vec{r}_1,\ldots,\vec{r}_n)$ and $P_S^{(n)}(\vec{r}_1,\ldots,\vec{r}_n)$ are the n-particle distribution functions for the liquid and solid, respectively.

From the symmetry of ψ_L, $P_L^{(1)}(\vec{r}_1)$ is a constant equal to the average density ρ, and $P_L^{(2)}(\vec{r}_1,\vec{r}_2)$ is a function only of r_{12}. We may obtain an expression for $P_L^{(2)}(r_{12})$ by differentiating it with respect to $\vec{r}_1$:

$$\vec{\nabla}_1 P_L^{(2)}(r_{12}) = \frac{\vec{\nabla}_1 N(N-1)\int d\vec{r}_3\cdots d\vec{r}_N \exp\left[\sum_{i<j} u_L(r_{ij})\right]}{\int d\vec{r}_1\cdots d\vec{r}_N \exp\left[\sum_{i<j} u_L(r_{ij})\right]} \tag{11}$$

$$= P_L^{(2)}(r_{12})\vec{\nabla}_1 u_L(r_{12}) + \int d\vec{r}_3 P_L^{(3)}(r_{12},r_{13},r_{23})\vec{\nabla}_1 u_L(r_{13}). \tag{12}$$

Defining the pair correlation function $g_L(r)$,

$$P_L^{(2)}(r) = \rho^2 g_L(r), \tag{13}$$

and using Kirkwood's superposition approximation[22]

$$P_L^{(3)}(r_{12},r_{13},r_{23}) \approx \rho^2 g_L(r_{12}) g_L(r_{13}) g_L(r_{23}), \tag{14}$$

Eq. (12) reduces to the Bogoliubov-Born-Green-

Kirkwood-Yvon equation[23]

$$\vec{\nabla}_1 \ln g_L(r_{12}) = \vec{\nabla}_1 u_L(r_{12}) + \rho \int d\vec{r}_3\, g_L(r_{13}) g_L(r_{23}) \vec{\nabla}_1 u_L(r_{13}). \qquad (15)$$

The energy E_L may then be expressed in terms of the resultant $g_L(r)$:

$$E_L = N \tfrac{1}{2} \rho \int d\vec{r}\, \tilde{v}(r) g_L(r). \qquad (16)$$

Differentiating $P_S^{(1)}(\vec{r}_1)$ and $P_S^{(2)}(\vec{r}_1, \vec{r}_2)$ with respect to $\vec{r}_1$ we obtain[24]

$$\vec{\nabla}_1 P_S^{(1)}(\vec{r}_1) = P_S^{(1)}(\vec{r}_1) \vec{\nabla}_1 \ln \varphi_S{}^2(\vec{r}_1) + \int d\vec{r}_2\, P_S^{(2)}(\vec{r}_1, \vec{r}_2) \vec{\nabla}_1 u_S(r_{12}) \qquad (17)$$

and

$$\vec{\nabla}_1 P_S^{(2)}(\vec{r}_1, \vec{r}_2) = P_S^{(2)}(\vec{r}_1, \vec{r}_2) [\vec{\nabla}_1 \ln \varphi_S{}^2(\vec{r}_1) + \vec{\nabla}_1 u_S(r_{12})] + \int d\vec{r}_3\, P_S^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \vec{\nabla}_1 u_S(r_{13}). \qquad (18)$$

Kirkwood's superposition approximation cannot be used directly with a function such as $P_S^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ which has long-range order. However, since the long-range order in $P_S^{(3)}$ arises solely from the periodic nature of the solid wave function ψ_S, we may factor it out before applying a generalized Kirkwood superposition approximation to the remainder,[2]

$$P_S^{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \approx P_S^{(1)}(\vec{r}_1) P_S^{(1)}(\vec{r}_2) P_S^{(1)}(\vec{r}_3) \times g_S(\vec{r}_1, \vec{r}_2) g_S(\vec{r}_1, \vec{r}_3) g_S(\vec{r}_2, \vec{r}_3), \qquad (19)$$

where the pair correlation function for the solid $g_S(\vec{r}_1, \vec{r}_2)$ is defined as

$$P_S^{(2)}(\vec{r}_1, \vec{r}_2) = P_S^{(1)}(\vec{r}_1) P_S^{(1)}(\vec{r}_2) g_S(\vec{r}_1, \vec{r}_2). \qquad (20)$$

Equations (17) and (18) then reduce to coupled integro-differential equations for the functions $P_S^{(1)}(\vec{r})$ and $g_S(\vec{r}_1, \vec{r}_2)$,

$$\vec{\nabla}_1 \ln P_S^{(1)}(\vec{r}_1) = \vec{\nabla}_1 \ln \varphi_S{}^2(\vec{r}_1) + \int d\vec{r}_2\, P_S^{(1)}(\vec{r}_2) g_S(\vec{r}_1, \vec{r}_2) \vec{\nabla}_1 u_S(r_{12}) \qquad (21)$$

and

$$\vec{\nabla}_1 \ln g_S(\vec{r}_1, \vec{r}_2) = \vec{\nabla}_1 u_S(r_{12}) + \int d\vec{r}_3\, P_S^{(1)}(\vec{r}_3) g_S(\vec{r}_1, \vec{r}_3) \times [g_S(\vec{r}_2, \vec{r}_3) - 1] \vec{\nabla}_1 u_S(r_{13}), \qquad (22)$$

where we have used Eq. (21) to eliminate the term $\vec{\nabla}_1 \ln \varphi_S{}^2(\vec{r}_1)$ from Eq. (22).

In principle these two coupled equations can be solved by initially guessing a form for $P_S^{(1)}(\vec{r})$, solving Eq. (22) for $g_S(\vec{r}_1, \vec{r}_2)$, then solving Eq. (21) for $P_S^{(1)}(\vec{r})$, and so on, until the two functions simultaneously satisfy both equations. This involves prohibitively long computation times. In practice, we simply solved each equation once. The initial form we chose for $P_S^{(1)}(\vec{r})$ was a constant. In this case Eq. (22) reduces to the BBGKY equation (15) for the liquid and $g_S(\vec{r}_1, \vec{r}_2)$ becomes a function of r_{12} only. Any other approximation for $P_S^{(1)}(\vec{r})$ leads to a $g_S(\vec{r}_1, \vec{r}_2)$ which is explicitly a function of $\vec{r}_1$ and $\vec{r}_2$ separately, so that should we for convenience wish to approximate g_S by a function of r_{12} then

$$g_S(\vec{r}_1, \vec{r}_2) \approx g_L(r_{12}) \qquad (23)$$

would be the only approximation consistent with Eq. (22). Just how good an approximation this is is currently under investigation. It should be pointed out that in previous quantum crystal calculations $g_S(\vec{r}_1, \vec{r}_2)$ corresponds either to $g_L(r_{12})$ or, worse yet, $e^{u_L(r_{12})}$.

Having solved Eq. (15) for $g_S(r)$ [equal to $g_L(r)$] we must solve Eq. (21) for $P_S^{(1)}(\vec{r})$. Using the property that both $P_S^{(1)}(\vec{r})$ and the ground state $\varphi_S(\vec{r})$ are non-negative and also that $P_S^{(1)}(\vec{r})$ has the same crystal symmetry as $\varphi_S(\vec{r})$, we may make Fourier expansions as follows:

$$\ln \varphi_S(\vec{r}) = \sum_{\vec{G}} t_{\vec{G}}\, e^{i\vec{G}\cdot\vec{r}}, \qquad (24)$$

$$P_S^{(1)}(\vec{r}) = \sum_{\vec{G}} s_{\vec{G}}\, e^{i\vec{G}\cdot\vec{r}}, \qquad (25)$$

and

$$\ln \lambda P_S^{(1)}(\vec{r}) = \sum_{\vec{G}} q_{\vec{G}}\, e^{i\vec{G}\cdot\vec{r}}, \qquad (26)$$

where the summations are over the reciprocal vectors $\vec{G}$ of the lattice with the symmetry of $\varphi_S(\vec{r})$, the coefficients $t_{\vec{G}}$ are the variational parameters for $\varphi_S(\vec{r})$ within the selected lattice structure [they determine the shape of $\varphi_S(\vec{r})$ about each lattice site], and λ is a constant which is determined by the normalization

$$\int d\vec{r}\, P_S^{(1)}(\vec{r}) = N. \qquad (27)$$

The coefficients $s_{\vec{G}}$ or $q_{\vec{G}}$ specify $P_S^{(1)}(\vec{r})$ and it is these we must determine.

Using these expansions we may reduce Eq. (21) to a set of algebraic equations relating $q_{\vec{G}}$ and $s_{\vec{G}}$ to the variational parameters $t_{\vec{G}}$,

$$q_{\vec{G}} = \alpha_G s_{\vec{G}} + 2 t_{\vec{G}}, \qquad (28)$$

where

$$\alpha_G = \frac{1}{|\vec{G}|} \int d\vec{r}\, \hat{G} \cdot \hat{r}\, e^{i\vec{G}\cdot\vec{r}} u'(r) g_S(r). \tag{29}$$

A second relation between $s_{\vec{G}}$ and $q_{\vec{G}}$ may be obtained by combining Eqs. (25) and (26):

$$s_{\vec{G}} = \frac{1}{\lambda} \int d\vec{r}\, e^{-i\vec{G}\cdot\vec{r}} \exp\Big(\sum_{\vec{G}'} q_{\vec{G}'} e^{i\vec{G}'\cdot\vec{r}}\Big). \tag{30}$$

Equations (28) and (30) together determine the coefficients $s_{\vec{G}}$ or $q_{\vec{G}}$ in terms of $g_S(r)$ and the variational parameters $t_{\vec{G}}$.

Having determined $P_S^{(1)}(\vec{r})$ and $P_S^{(2)}(\vec{r}_1, \vec{r}_2)$ for a given form of the variational functions $\varphi_S(\vec{r})$ and $u_S(r)$, we may calculate the corresponding energy E_S using Eq. (9). Alternatively we may express E_S directly in terms of $t_{\vec{G}}$ and $s_{\vec{G}}$ as

$$E_S = N \sum_{\vec{G}} \tfrac{1}{2} s_{\vec{G}} t_{\vec{G}} |\vec{G}|^2 + s_{\vec{G}}^2 \gamma_G, \tag{31}$$

where

$$\gamma_G = \tfrac{1}{2}\rho \int d\vec{r}\, \tilde{v}(r) g_S(r) e^{i\vec{G}\cdot\vec{r}}. \tag{32}$$

III. RESULTS OF CALCULATION AND CONCLUSION

Our variational search divides rather naturally into two parts. The first concerns the choice of functional form for the two-body Jastrow wave function $u_S(r)$. For the potential $v_{\mathrm{I}}(r)$, Shen *et al.*[15] have calculated the optimized Jastrow form for the liquid $\tilde{u}_L(r)$, and while this serves as a convenient starting point, our experience with solid ^{4}He indicates that the best $u_S(r)$ for the solid is somewhat less localized than the best $u_L(r)$ for the liquid at the same density.[2] The second part of the search concerns the variational parameters $t_{\vec{G}}$ which determine the functional form of the single-particle wave function $\varphi_S(\vec{r})$. Again from our experience with solid ^{4}He, where we found that the minimizing values of $t_{\vec{G}}$ fell off rapidly with increasing $\vec{G}$ so that the dominating coefficient invariably was for the smallest nonzero $|\vec{G}|$, we retained only the two leading coefficients $t_{(1)}$ and $t_{(2)}$. These are the values of $t_{\vec{G}}$ for $|\vec{G}|$ equal to the nearest and second nearest neighbor distance in the reciprocal lattice. (We note that $t_{\vec{G}}$ is a function only of $|\vec{G}|$ if $|\vec{G}|$ is less than the fifth nearest neighbor distance.) Fixing the $\vec{r}$ origin on a lattice site, $t_{(1)}$ must be positive if $\varphi_S(\vec{r})$ is to be maximized on each site.

We took for $u_L(r)$ and $u_S(r)$ the parametrized forms

$$u_L(r) = -\left(\frac{b_L}{r}\right)^3 \tag{33}$$

and

$$u_S(r) = -\left(\frac{b_S}{r}\right)^3. \tag{34}$$

For the potential $v_{\mathrm{I}}(r)$ we determined the optimal liquid energy $\tilde{E}_L$ by taking the best energy obtainable from (33) and then subtracting the paired-phonon analysis (PPA) energy correction term.[15] For $v_{\mathrm{II}}(r)$ we simply equated $\tilde{E}_L$ to the minimum energy for the form(33).

In Figs. 1 and 2 we show representative results from our variational search using the potentials $v_{\mathrm{I}}(r)$ and $v_{\mathrm{II}}(r)$, respectively. Since the solid energies were lower for an fcc crystal structure than for a bcc, these results are all for fcc. We searched through values of b_S encompassing the minimizing value of b_L, which was approximately 0.7 fm for $v_{\mathrm{I}}(r)$ and 0.5 fm for $v_{\mathrm{II}}(r)$. Each curve is for a fixed b_S plotted as a function of $t_{(1)}$. The parameter $t_{(2)}$ was freely varied, although we found it invariably to be small compared to $t_{(1)}$.

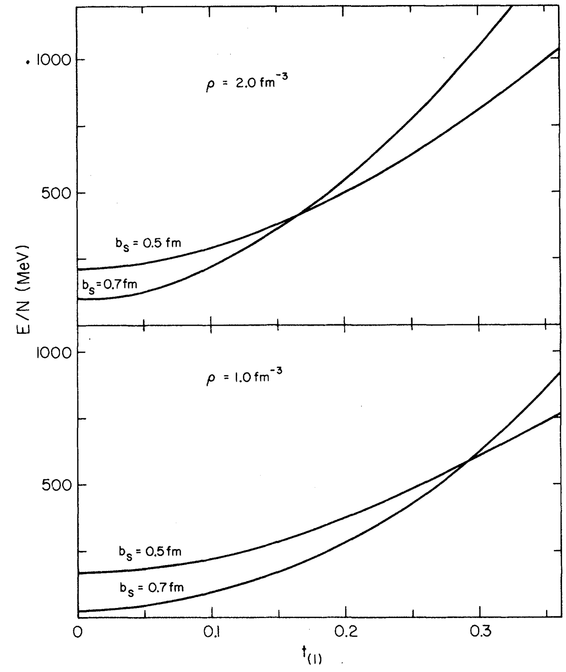

FIG. 1. Energy of the solid relative to the minimum liquid energy as a function of the variational parameter $t_{(1)}$. The potential used is $v_{\mathrm{I}}(r)$, the repulsive part of the $^1S_0$ Reid potential. Each curve is for a fixed value of the parameter b_S. The parameter $t_{(2)}$ is allowed to vary freely. The two densities are labeled by ρ. Note that each energy curve monotonically increases with $t_{(1)}$ (increasing localization). The solid configurations characterized by nonzero $t_{(1)}$ all have energies above the minimum liquid energy. This implies a stable liquid phase up to 2 fm^{-3}. The crossing of the curves has no particular significance in the present context.

The energy scale is relative to $\tilde{E}_L$. Other values of b_S above and below the best b_L gave qualitatively similar curves. For the potential $v_{\rm I}(r)$, we also tried using the optimized liquid form $\tilde{u}_L(r)$ for $u_S(r)$. The curves we obtained were again similar to those shown. Note that the energy curves all monotonically increase with increasing $t_{(1)}$. We found that the solid energy E_S only equaled $\tilde{E}_L$ when $u_S(r)$ was identical to the best liquid $u_L(r)$ and

$$t_{(1)} = t_{(2)} = 0 .$$

Our results thus indicate that for either potential all solid configurations have energies higher than the liquid. We conclude that the repulsive core of the Reid potential is too soft to induce solidification of neutron matter below 2 fm^{-3}. Our calculations with $v_{II}(r)$ suggest that, as with ^{4}He, the central attractive potential is not the key mechanism for inducing solidification.

Qualitatively the results look similar to our previously calculated results for ^{4}He well below the solidification density.[2] For the sake of comparison, Fig. 3 shows the results for ^{4}He about the solidification density. Above the solidification density we see the energy curve dipping quite dramatically down to $\tilde{E}_L$, indicating a stable solid solution. A numerical comparison of the two sets of results suggests that even 2.0 fm^{-3} is not close to the solidification density, and so it would appear likely that a more refined calculation using realistic state-dependent and noncentral potentials would still find no solidification up to 2 fm^{-3}.

There remain two mechanisms which could possibly induce a solid neutron-star core. They are first, an enhancement of the tensor component of the n-n interaction by a π^0 condensate[20] or second, a more repulsive neutron core due to size effects: If neutrons should be treated as extended composite objects,[25] when two neutron

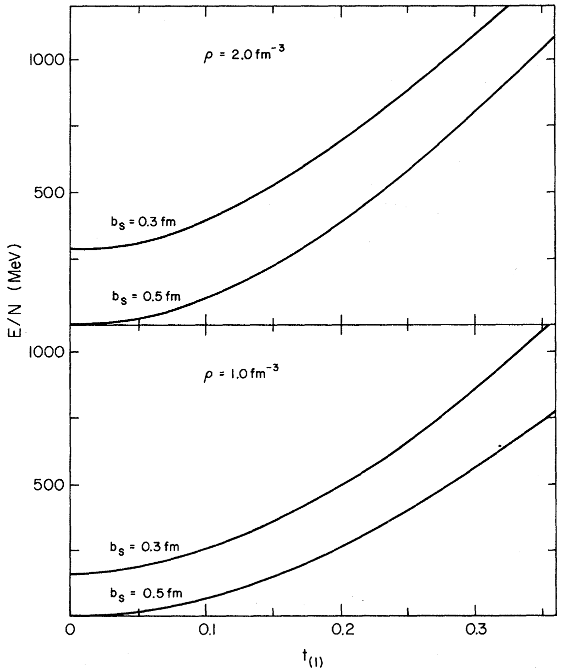

FIG. 2. Energy of the solid for the potential $v_{\rm II}(r)$, a density-dependent average over the complete $^1S_0$ and $^1D_2$ Reid potentials, plotted as a function of $t_{(1)}$. The labels are described in Fig. 1. The results again imply a stable liquid phase up to 2 fm^{-3} for this potential.

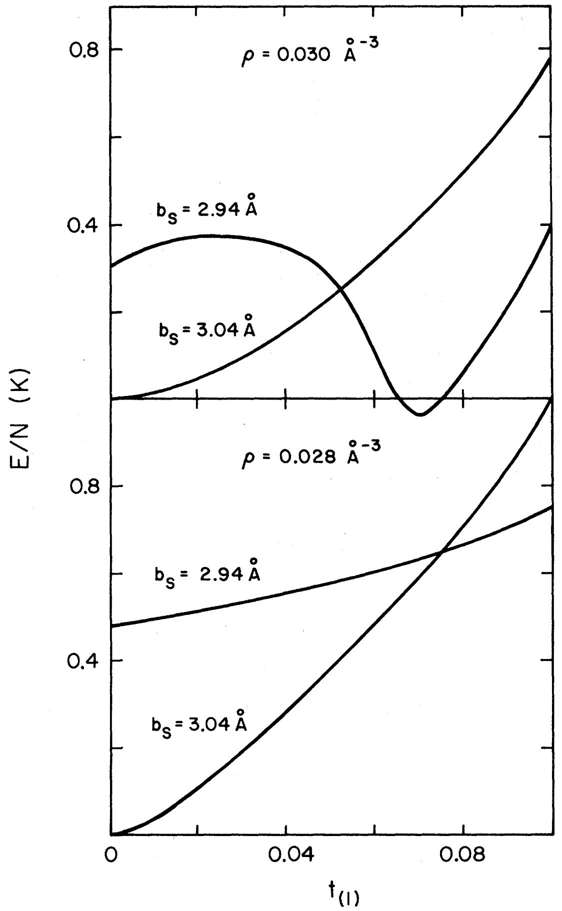

FIG. 3. Energy of solid ^{4}He calculated in Ref. 2 as a function of $t_{(1)}$ at two densities near the predicted solidification density. The labels are analogous to those in Fig. 1. The parametrized form used for $u_S(r)$ was $-(b_S/r)^5$. Note that above the solidification density (0.030 Å^{-3}), the solid energy for certain nonzero values of $t_{(1)}$ dips down to the liquid energy, implying a stable solid phase. In the present calculation no similar dip was observed. The calculated solidification density for ^{4}He was within 10% of the observed solidification density.

surfaces touch the repulsive effects of exchange at the surfaces could be very large. The solidification density would be sensitive to any appreciable hardening of the core.

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APPENDIX: FLOATING SOLID

For the "homework model" potential $v_{\mathrm{I}}(r)$, Shen *et al.*[15] have obtained accurate ground-state wave functions and energies for liquid neutron-star matter. Starting with a parametrized Jastrow function

$$\psi_L(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N)=\prod_{i<j=1}^{N} e^{u_L(r_{ij})/2}, \tag{A1}$$

they first minimized the energy expectation value with respect to the variational parameters in $u_L(r)$. Next, they subjected the resultant wave function to a paired-phonon analysis,[26] which was shown by Campbell and Feenberg[27] to lead to the *optimized* Jastrow function. This is the Jastrow function which minimizes the energy irrespective of the particular parametrized form initially selected for $u_L(r)$. Finally, corrections due to three-particle correlations as they appeared in the improved trial wave function

$$\psi_L(\vec{r}_1,\ldots,\vec{r}_N)=\prod_{i<j=1}^{N} e^{u_L(r_{ij})/2} \times \prod_{k<l<m=1}^{N} e^{w(r_{kl},r_{lm},r_{mk})/2} \tag{A2}$$

were perturbatively estimated. They were found to be insignificant ($\lesssim 1\%$) throughout the range of densities considered ($0.5 \leqslant \rho \leqslant 2.0\ \mathrm{fm}^{-3}$). This indicates that higher-order correlations can all be neglected, and the Jastrow approximation is valid for neutron liquids.

Feenberg[21] has pointed out that a translationally invariant wave function of the general form

$$\psi(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N)=\prod_{i<j=1}^{N} e^{u(r_{ij})/2} \times \prod_{k<l<m=1}^{N} e^{w(r_{kl},r_{lm},r_{mk})/2}\cdots \tag{A3}$$

completely describes a "floating solid." This is a solid which has been averaged over all positions of its lattice sites and over all orientations of its crystal axes relative to a fixed frame. Clark and Sandler[17] argued that the results of the Shen *et al.* liquid calculation consequently remain valid at densities for which the solid is the stable phase. In this sense the minimum energy of the "liquid" calculation must equal the ground-state energy of the system. If the energy is always calculated assuming a single phase, then the liquid-solid phase transition region should make its appearance by displaying an unphysical inflection in the energy-density curve. Clark and Sandler concluded from the fact that no such inflection is observed in the Shen *et al.* energy curve that no liquid-solid transition occurs below 2.0 fm^{-3}. Although in principle we believe this inference, a valid objection is that the densities Shen *et al.* considered were rather widely spaced, up to 0.6 fm^{-3} apart, and consequently an inflection could have escaped detection. Also, the inflection itself may be small, as is the case in ^{4}He.[2] The procedure presented in this paper directly determines whether the stable phase at any given density is solid or liquid and thus overcomes this objection.

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