Gravitation as a gauge theory*

Freydoon Mansouri
Department of Physics, Yale University, New Haven, Connecticut 06520

Lay Nam Chang
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174
(Received 18 February 1976)

We present theories of gravitation based, respectively, on the general linear group $GL(n,R)$ and its inhomogeneous extension $IGL(n,R)$ [$SO(n-1,1)$ and $ISO(n-1,1)$ for torsion-free manifolds]. Noting that the geometry of the conventional gauge theories can be described in terms of a fiber bundle, and that their action is a scalar in such a superspace, we construct principal fiber bundles based on the above gauge groups and propose to describe gravitation in terms of their corresponding scalar curvatures. To ensure that these manifolds do indeed have close ties with the space-time of general relativity, we make use of the notion of the parallel transport of vector fields in space-time to uniquely relate the connections in space-time to the gauge potentials in fiber bundles. The relations turn out to be similar to that suggested earlier by Yang. The actions we obtain are related to those of Einstein and Yang but are distinct from both and have an Einstein limit. The inclusion of internal symmetry leads to the analogs of Einstein-Yang-Mills equations. A number of variations and less attractive alternatives based on the above groups or their subgroups are also discussed.

I. INTRODUCTION

The enormous success of Einstein's theory of gravitation in providing a consistent description of the large-scale behavior of matter[1] necessitates very strong reasons for any attempt to modify it. It is therefore essential that we provide some motivation for a gauge theory of gravitation which in general is different from Einstein's.

We begin by noting that all the phenomena in which gravitation does not appear to play a significant role can be described by theories based on local gauge invariance. It is then natural to inquire whether gravitation can also be based on local gauge invariance. In fact, soon after Yang and Mills[2] extended the local gauge invariance of electrodynamics to a non-Abelian group, Utiyama[3] presented one such possibility in which the gauge group is the homogeneous Lorentz group. Later another theory with the Poincaré group as its gauge group was proposed by Kibble.[4] The question has been recently revived by Yang[5] in connection with his nonintegrable phase formulation of non-Abelian gauge theories. The main point of Yang's observation was that if, in his formulation, one indentifies the equations for the parallel displacement of gauge fields as equations for the gravitational field, then one is led to third-order differential equations for the components of the metric tensor.

The timeliness of Yang's observation has been further enhanced by recent results of Deser *et al*,[6] concerning the nonrenormalizability of Einstein's theory. If these results meet the test of time, as is presently thought, they lend support to the possibility that Einstein's theory might not be a gauge theory proper since massless gauge theories (at least in flat space-time) are known to be renormalizable. We could then entertain the possibility, among others, that some modification of Einstein's theory could be derivable from a bona fide gauge theory, thus making local gauge invariance the single unifying basis of theories describing all known interactions and offering hope that such a theory of gravitation would be renormalizable.

The manner in which we propose to implement this unified approach has much in common with that of Yang's, although it differs from it in details. We take as our starting point the observation that non-Abelian gauge theories can be viewed[7-9] as a geometrical theory in a superspace known in mathematics as a fiber bundle. In particular,[8] the gauge field part of the action of the gauge theories is proportional to the scalar curvature of the bundle manifold. To view gravitation as a gauge theory, we then demand that it must also be described in terms of the geometry of a suitable fiber bundle. Then the inclusion of internal gauge groups would result in a larger fiber bundle, the scalar curvature of which would describe both gravitation, matter gauge fields, and their couplings. We have also made progress in incorporating supergauge invariance in this geometry and have reported some of its consequences in Refs. 7 and 8.

To construct a fiber bundle for gravitation which is naturally related to the space-time of general relativity, let us recall[8] how we were led to the geometry of a fiber bundle for local gauge theories of internal symmetries. There we argued that by enlarging the four directions at each point in space-time to an extended tangent space of specific kind with horizontal and vertical sectors one arrives at a superspace which is a fiber bundle. Thus if one wants to describe local internal symmetries by a geometry, one is essentially forced to consider fiber bundles. For gravitation the situation is quite different, however. This is because for gravitation a geometrical theory does already exist. For example, Einstein's theory is invariant under all coordinate transformations of the space-time manifolds. These transformations already include such popular local transformation groups as the Lorentz, Poincaré, and general linear groups. So one cannot use the desire for inclusion of such local symmetries as an argument in favor of describing gravitation by a fiber bundle. (The situation would be different if one could start from a flat space-time and describe gravitation by a fiber bundle based on it. See Sec. VI for more details.) In fact, if such were the case, the problem would be quite trivial and academic. For example, one can get Einstein's theory from a trivial bundle with curved space-time as base manifold and the identity element of a real 4×4 matrix group as fiber. *The nontrivial problem is to justify what the structure of a fiber bundle has to do with the space-time of general relativity.* When there is no clear and convincing relation between the structure of these two spaces, any identification of the properties of such a fiber bundle with gravitation would be *ad hoc* and unsatisfactory.

To avoid an *ad hoc* choice of a fiber bundle for describing gravitation by a gauge theory, we make use of the notion of the parallel transport of vector fields in space-time to relate the connections in space-time to the gauge potentials (connections) in a fiber bundle. That this is possible follows from the identical transformation properties of gauge potentials and connection coefficients. This procedure ensures that any properties extraneous to the space-time manifold which one wants to consider are excluded from the fiber-bundle geometry by construction; it also leads naturally to the identification of the relevant gauge groups as $GL(n,R)$ or its inhomogeneous extension $IGL(n,R)$. These ideas are discussed in Sec. II.

In Sec. III, we give the details of a theory of gravitation with gauge group $GL(n,R)$ using the methods developed in Ref. 8. In particular we show that the action of a theory based on the scalar curvature of the fiber bundle differs from that of Einstein by a quadratic term which is proportional to that suggested by Yang. Thus, the theory is endowed with a parameter which we shall refer to as the "bundle manifold parameter," b_M. In the limit $b_M \to 0$, one recovers Einstein's theory. In Sec. IV we extend the theory of Sec. III to include local internal symmetry, and in Sec. V we consider $IGL(n,R)$ or its $ISO(n-1,1)$ subgroup as the gauge group for gravitation. We show that the latter theory in general involves two bundle parameters b_M and b_T and has certain advantages, in particular for the incorporation of local supersymmetries. Moreover, under a very reasonable requirement this theory reduces to that based on $GL(n,R)$ with the distinct advantage that now the metric tensor can also be expressed in terms of the gauge potentials of translations. Thus one arrives at a gauge theory of gravitation in which all the dynamical variables are gauge fields, the gauge potentials of translations being the independent degrees of freedom.

II. PARALLEL TRANSPORT AND THE GAUGE POTENTIALS IN SUPERSPACE

In this section we want to show the manner in which the notion of the parallel transport of vector fields in space-time, M, of general relativity uniquely relates its connection to the gauge potentials of a superspace which is a principlal fiber bundle $L(M)$. The discussion involved is of a rather technical nature. Since it is based on sound mathematical theorems,[10] the reader who is willing to take our word for it can proceed directly to Eq. (2.14), which is the main result of this section.

Before constructing the bundle of interest to us, it will be helpful to recall the properties of the horizontal tangent spaces to the bundles associated with local internal symmetries. We showed in Ref. 8 that local gauge transformations correspond to local rotations of bases in the horizontal tangent space. Thus, given a local frame at a point x, one can obtain the other frames at x by local gauge rotations. Conversely, the knowledge of all the frames in the horizontal tangent space is equivalent to the knowledge of the corresponding gauge group. With this in mind we now proceed to construct the bundle $L(M)$. Let M be an n-dimensional manifold. Let x be a point of M. One can construct a tangent space $T_x(M)$ at x by constructing the directional derivatives at x to all the curves in M. A linear frame u at a point $x\in M$ is a basis $\{\vec{e}_\mu\}$, $\mu=0,\ldots,n-1$ of this tangent space. We want to construct a superspace $L(M)$ in such a way that it will consist of all the linear frames u at all the points $x\in M$. This we do[8] by taking the basis in the horizontal tangent space of $L(M)$ to be isomorphic to $\{\vec{e}_\mu\}$ and then enlarge this by supplying an algebra of a group as the basis for the vertical part of the tangent space. Since we want $L(M)$ to include all the linear frames at all points of M, the group G must be chosen so as to ensure this. Given a frame u at x, we can ensure that all the other frames at x are included if G includes the transformations which rotate u to all the other frames at the same x. Then by letting x range over all the points in M, all the linear frames would be included in $L(M)$. The smallest group with this property is $GL(n,R)$. We thus take $GL(n,R)$ as the local gauge group (fiber) of a principal fiber bundle with base manifold M.

To introduce a basis in $L(M)$, let $\{x^\mu\}$, $\mu=0,\ldots,n-1$ be a local coordinate system in some neighborhood U of M. Then the set $\{\vec{e}_\mu\}=\{\vec{\partial}_\mu\}$ forms a basis in U. Every basis u at x can be obtained in the form

$$\vec{h}_\mu = h_\mu^\nu \vec{e}_\nu , \tag{2.1}$$

where h_μ^ν is a real nonsingular matrix. Since every basis at x can be determined in this way, the set $\{x^\mu\}$, together with the set $\{h_0^\nu, h_1^\nu,\ldots,h_{n-1}^\nu\}$, determines every frame in $L(M)$. Therefore, the set

$$\{y^i\}=\{x^\mu, h_0^\nu,\ldots,h_{n-1}^\nu\} \tag{2.2}$$

serves as a local coordinate system in $L(M)$. Here h_μ^ν stands for the νth component of the vector $\vec{h}_\mu$ with respect to the coordinate basis $\vec{e}_\mu$.

Following the approach explained in the Introduction, to describe gravitation as a gauge theory in the $L(M)$ superspace, we must establish a unique relation between its intrinsic characteristics and those of the manifold M which represents space-time of general relativity. This we do by establishing a unique relation between the connection forms in M and in $L(M)$ by exploiting the notion of the parallel transport of vectors along curves in M. The connection form in $L(M)$ then defines the gauge potentials in $L(M)$ as shown in Ref. 8. This is to be contrasted with conventional gauge theories in which there is no relation between the connection in M and the corresponding fiber bundle. Thus consider the connection Γ of the manifold M. Γ may be viewed as a rule for the parallel transport of vectors along any curve $C(\tau)$ in M. Let $x_1=C(\tau_1)$ be a point on this curve and $\{\vec{h}_\mu\}$ a basis at x. Then by (2.2) the set $\{\vec{x}_1, \vec{h}_\mu\}$ is a point u_1 in $L(M)$. By the parallel transport of the basis $\{\vec{h}_\mu\}$ along $C(\tau)$ from x_1 to $x_2=C(\tau_2)$ we obtain the unique set $\{x_2, \vec{h}_\mu\}$ which is a point u_2 in $L(M)$. The points $u_i=\{x_i, \vec{h}_\mu\}$ constitute a unique curve $\bar{C}(\tau)$ in $L(M)$ known in mathematical literature as the lift of $C(\tau)$. Clearly,

$$\bar{C}(\tau_1)=u_1, \quad \bar{C}(\tau_2)=u_2. \tag{2.3}$$

The curve $\bar{C}(\tau)$ can be specified in terms of the local coordinates established above. It is given by

$$\bar{C}(\tau) \to \{x^\mu(C(\tau)), h_\alpha^\beta(\tau)\}, \tag{2.4}$$

where at each τ, h_ν^μ is determined by the condition of parallel transport

$$\frac{dh_\alpha^\beta(\tau)}{d\tau} + h_\alpha^\gamma \Gamma_{\mu\gamma}^\beta \frac{dx^\mu(\tau)}{d\tau}=0. \tag{2.5}$$

Now we want to use this information to induce a basis in the horizontal sector of the tangent space to $L(M)$. To do this let $\{\vec{\partial}/\partial x^\mu, \vec{\partial}/\partial h_\alpha^\beta\}$ be a coordinate basis corresponding to the coordinates (2.2) introduced above. The horizontal part of the tangent space to $L(M)$ consists of the tangent vectors $\partial/\partial\tau$ to all the curves $\bar{C}$ through a given point u. Expanding $\partial/\partial\tau$ in terms of our coordinate bases we have, using (2.5),

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}\right)_{\bar{C}} &= \frac{dx^\mu}{d\tau}\vec{\partial}_\mu + \frac{dh_\alpha^\beta}{d\tau}\frac{\vec{\partial}}{\partial h_\alpha^\beta} \\ &= \frac{dx^\mu}{d\tau}\left(\vec{\partial}_\mu - h_\alpha^\gamma \Gamma_{\mu\gamma}^\beta \frac{\vec{\partial}}{\partial h_\alpha^\beta}\right),\end{aligned}$$

or defining

$$D_\mu = \vec{\partial}_\mu - \Gamma_{\mu\gamma}^\beta h_\alpha^\gamma \frac{\vec{\partial}}{\partial h_\alpha^\beta} \tag{2.6}$$

we have

$$\left(\frac{\partial}{\partial\tau}\right)_{\bar{C}} = \frac{dx^\mu}{d\tau}\vec{D}_\mu .$$

Thus $\vec{D}_\mu$ can serve as a basis for the horizontal tangent space of $L(M)$. The quantities $\vec{\partial}/\partial h_\alpha^\beta$ are coordinate bases associated with group parameters. As we have explained before,[8] it is more convenient to work with a noncoordinate basis in group space. So instead of $\vec{\partial}/\partial h_\alpha^\beta$ we shall work with

$$\vec{\omega}^\gamma_{\ \beta} = h_\alpha^\gamma \vec{\partial}/\partial h_\alpha^\beta, \tag{2.7}$$

which form a basis for the algebra of $GL(n,R)$:

$$[\vec{\omega}_{\alpha\beta}, \vec{\omega}_{\gamma\delta}] = f_{(\alpha\beta)(\gamma\delta)}^{(\epsilon\eta)} \omega_{\epsilon\eta} . \tag{2.8}$$

Here

$$\vec{\omega}_{\alpha\beta} = \eta_{\alpha\gamma}\vec{\omega}^\gamma_{\ \beta}.$$

Thus we arrive at a basis $\vec{E}_\mu$ induced by M in the horizontal section of the tangent space to $L(M)$:

$$\vec{E}_\mu = \vec{e}_\mu - \Gamma_{\mu\alpha}^\beta \eta^{\alpha\gamma}\vec{\omega}_{\beta\gamma}. \tag{2.9}$$

To see explicitly that this basis induces a connection in $L(M)$, we write down exactly as explained in Ref. 8 the gauge-covariant basis for the tangent space to the bundle manifold. These are the set

$$\{\vec{E}_\mu, \vec{\Omega}_{\alpha\beta}\}, \tag{2.10}$$

with

$$\begin{aligned}&[\vec{E}_\mu, \vec{E}_\nu] = +b_M F_{\mu\nu}^{\alpha\beta}\vec{\Omega}_{\alpha\beta}, \\ &[\vec{\Omega}_{\alpha\beta}, \vec{\Omega}_{\gamma\delta}] = f_{(\alpha\beta)(\gamma\delta)}^{(\eta\epsilon)}\vec{\Omega}_{\eta\epsilon}, \\ &[\vec{E}_\mu, \vec{\Omega}_{\alpha\beta}]=0.\end{aligned} \tag{2.11}$$

We shall refer to b_M as the "bundle manifold parameter." Whenever convenient we set it equal to unity. When $\vec{E}_\mu$ in this basis is expanded in the local direct product basis

$$\begin{aligned}
&[\vec{e}_\mu, \vec{e}_\nu] = 0,\\
&[\vec{\omega}_{\alpha\beta}, \vec{\omega}_{\gamma\delta}] = f^{(\eta\epsilon)}_{(\alpha\beta)(\gamma\delta)}\, \omega_{\eta\epsilon},\\
&[\vec{e}_\mu, \vec{\omega}_{\alpha\beta}] = 0,
\end{aligned} \tag{2.12}$$

one finds

$$\vec{E}_\mu = \vec{e}_\mu - N^{\alpha\beta}_\mu \vec{\omega}_{\alpha\beta}. \tag{2.13}$$

Comparing the two expressions for $\vec{E}_\mu$, we find

$$N^{\alpha\beta}_\mu = \eta^{\beta\gamma}\Gamma^\alpha_{\mu\gamma}. \tag{2.14}$$

Up to a multiplicative constant this establishes a unique correspondence between the connection in a curved space-time of general relativity and the gauge potentials of the bundle $L(M)$. Although known in mathematical literature for some time,[10] its relevance to physics first appeared in Yang's nonintegrable phase approach to gauge theories. Here it has been established by the parallel transport of vectors along curves in space-time.

Thus, in contrast to the case of local internal symmetry, where one can start from a flat space-time base and construct a curved superspace to set up a gauge theory, to have any control over the structure of the superspace, which describes gravitation, one must start from a curved space-time. The function of a fiber bundle would then be to relate such a manifold to gauge theories. In more practical terms this would suggest what kinds of actions and equations of motion for gravitation are compatible with the structure of gauge theories.

III. A GAUGE THEORY OF GRAVITATION BASED ON GL(n,R) OR SO$(n-1,1)$

We have described in the preceding section the manner in which a unique correspondence between the connection form in a principal fiber bundle can be related to the connection in a manifold describing gravitation. Also given a base manifold and a gauge group we have discussed in Ref. 8 various geometrical quantities associated with any fiber bundle and how to go about calculating them regardless of the bundle's physical implications. Since the superspace with which we wish to describe gravitation is one such bundle manifold, we will follow step by step the procedure described in Ref. 8 to calculate various geometrical quantities of this superspace without explicit use of the correspondence (2.14). In the end, we will make the necessary identification which distinguishes this superspace from any other. Again the reader who is willing to skip the geometrical details may proceed directly to the paragraph containing Eq. (3.13).

Thus we begin by describing a principal fiber bundle with a space-time (base) manifold M and a gauge group GL(n,R). We take the gauge-covariant basis

$$\{\vec{E}_i\} = \{\vec{E}_\mu, \Omega_{\alpha\beta}\}$$

to be that given by (2.10) and satisfying the commutation relations (2.11). Let the components of the metric tensor of the space-time manifold be $g_{\mu\nu}$ and that of GL(n,R) its corresponding Killing metric,

$$\bar{g}_{(\alpha\alpha')(\beta\beta')} = f^{(\delta\delta')}_{(\alpha\alpha')(\gamma\gamma')} f^{(\gamma\gamma')}_{(\delta\delta')(\beta\beta')}, \tag{3.1}$$

where the indices $\alpha, \alpha', \beta, \ldots$ each take on four values, so that $(\alpha\alpha')$ takes on 16 quantities. Whenever no confusion can arise, we will simplify the notation by writing $\hat{\alpha}$ for $(\alpha\alpha')$, etc., e.g.

$$\bar{g}_{(\alpha\alpha')(\beta\beta')} \to \bar{g}_{\hat{\alpha}\hat{\beta}}.$$

Let G_{ij} be the components of the metric tensor G of the superspace in the gauge-covariant basis. Then it has the form[8]

$$G_{ij} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \bar{g}_{\hat{\alpha}\hat{\beta}} \end{pmatrix} = \begin{pmatrix} \vec{E}_\mu \cdot \vec{E}_\nu & 0 \\ 0 & \vec{\Omega}_{\hat{\alpha}} \cdot \vec{\Omega}_{\hat{\beta}} \end{pmatrix}, \tag{3.2}$$

$$G \equiv \det(G_{ij}) = \det(g_{\mu\nu})\det(\bar{g}_{\hat{\alpha}\hat{\beta}}) \equiv g\bar{g}.$$

In this basis all the base vectors are of noncoordinate type but can be related to coordinate bases by (2.13) and (2.7).

We now proceed to obtain a number of structural relations using the Jacobi identities, etc. For example, the identity

$$\begin{aligned}0 = {}&[\vec{\Omega}_{\alpha\beta}, [\vec{E}_\mu, \vec{E}_\nu]] + [\vec{E}_\nu, [\vec{\Omega}_{\alpha\beta}, \vec{E}_\mu]]\\ &+ [\vec{E}_\mu, [\vec{E}_\nu, \vec{\Omega}_{\alpha\beta}]]\end{aligned}$$

leads to

$$\vec{\partial}_{\hat{\alpha}} F^{\hat{\beta}}_{\mu\nu} = -f^{\hat{\beta}}_{\hat{\alpha}\hat{\gamma}} F^{\hat{\gamma}}_{\mu\nu}. \tag{3.3}$$

Replacing $\vec{\Omega}_{\alpha\beta}$ with $\vec{E}_\gamma$ in the above identity one gets the Bianchi identity

$$F^{\alpha\beta}_{\mu\nu;\lambda} + F^{\alpha\beta}_{\lambda\mu;\nu} + F^{\alpha\beta}_{\nu\lambda;\mu} = 0. \tag{3.4}$$

The gauge potentials $N^{\alpha\beta}_\mu$ satisfy an equation similar to (3.3):

$$\vec{\partial}_{\hat{\alpha}} N^{\hat{\beta}}_\mu = -f^{\hat{\beta}}_{\hat{\alpha}\hat{\gamma}} N^{\hat{\gamma}}_\mu. \tag{3.5}$$

The gauge potentials and field tensors are related in the usual way,[8]

$$F^{\alpha\beta}_{\mu\nu} = N^{\alpha\beta}_{\mu,\nu} - N^{\alpha\beta}_{\nu,\mu} - \vec{\partial}_{(\gamma\gamma')} N^{\alpha\beta}_\mu N^{\gamma\gamma'}_\nu, \tag{3.6}$$

or using (3.5)

$$F^{\alpha\beta}_{\mu\nu} = N^{\alpha\beta}_{\mu,\nu} - N^{\alpha\beta}_{\nu,\mu} + f^{(\alpha\beta)}_{(\gamma\delta)(\epsilon\eta)} N^{\epsilon\eta}_{\mu} N^{\gamma\delta}_{\nu}. \tag{3.7}$$

Next we turn to the computation of connection coefficients Γ^i_{jk}. In general, if the manifold on which Γ^i_{jk} is imposed (a) admits a metric g with vanishing directional covariant derivatives

$$\nabla_{\vec{e}}\, g = 0, \tag{3.8}$$

and (b) is torsion-free, i.e.,

$$\nabla_{\vec{U}} \vec{V} - \nabla_{\vec{V}} \vec{U} = [\vec{U}, \vec{V}], \tag{3.9}$$

then

$$\Gamma^k_{ij} = \tfrac{1}{2} G^{kl} (G_{li,j} + G_{lj,i} - G_{ij,l} + C_{lij} + C_{lji} - C_{ijl}), \tag{3.10}$$

where

$$G_{ij} = \vec{E}_i \cdot \vec{E}_j, \qquad [\vec{E}_i, \vec{E}_j] = C^k_{ij} \vec{E}_k. \tag{3.11}$$

For the description of gravitation the space-time manifold M is normally taken to be torsion-free so that both (3.8) and (3.9) are satisfied. Since the connection in the fiber bundle of interest in the present case is induced from that in M using (2.14), it is also torsion-free. Conversely, we would be interested in fiber bundles with connections such that they determine via (2.14) torsion-free space-time manifolds. This will reduce the gauge group from $GL(n,R)$ to $SO(n-1,1)$. Therefore, Eq. (3.9) for the connection coefficients applies, and we get in our gauge-covariant basis

$$\begin{aligned}
&\Gamma^{(\alpha\alpha')}_{(\beta\beta')(\gamma\gamma')} \equiv \Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = -\tfrac{1}{2} f^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}, \\
&\Gamma^{\mu}_{\hat{\alpha}\hat{\beta}} = \Gamma^{\hat{\alpha}}_{\mu\hat{\beta}} = \Gamma^{\hat{\alpha}}_{\hat{\beta}\mu} = 0, \\
&\Gamma^{\lambda}_{\mu\hat{\alpha}} = \Gamma^{\lambda}_{\hat{\alpha}\mu} = \tfrac{1}{2} g^{\lambda\nu} \bar{g}_{\hat{\alpha}\hat{\beta}} F^{\hat{\beta}}_{\nu\mu}, \\
&\Gamma^{\hat{\alpha}}_{\mu\nu} = -\tfrac{1}{2} F^{\hat{\alpha}}_{\mu\nu}, \\
&\Gamma^{\lambda}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}_{\rm st},
\end{aligned} \tag{3.12}$$

where

$\left\{{\lambda \atop \mu\nu}\right\}_{\rm st} \equiv$ connection coefficients of the space-time manifold.

In a coordinate basis they reduce to the usual Christoffel symbols.

Finally, we compute the components of the Ricci tensor and the scalar curvature of the bundle manifold:

$$R_{(\alpha\alpha')(\beta\beta')} = \tfrac{1}{4} g^{\nu\rho} g^{\mu\lambda} \bar{g}_{(\alpha\alpha')(\gamma\gamma')} \bar{g}_{(\beta\beta')(\delta\delta')} F^{\gamma\gamma'}_{\rho\mu} F^{\delta\delta'}_{\lambda\nu} + R^G_{(\alpha\alpha')(\beta\beta')}, \tag{3.13}$$

$$R_{\mu\nu} = R^{\rm st}_{\mu\nu} - \tfrac{1}{2} g^{\lambda\rho} \bar{g}_{(\alpha\alpha')(\beta\beta')} F^{\alpha\alpha'}_{\lambda\nu} F^{\beta\beta'}_{\rho\mu}, \tag{3.14}$$

$R^G_{\hat{\alpha}\hat{\beta}} \equiv$ Ricci tensor of the group manifold,

$R^{\rm st}_{\mu\nu} \equiv$ Ricci tensor of the space-time manifold,

$$R = R^{\rm st} + R^G - \tfrac{1}{4} g^{\mu\nu} g^{\rho\lambda} \bar{g}_{(\alpha\alpha')(\beta\beta')} F^{\alpha\alpha'}_{\lambda\nu} F^{\beta\beta'}_{\rho\mu}, \tag{3.15}$$

$R^{\rm st}$ = curvature scalar of space-time,

R^G = curvature scalar of group manifold.

This expression for the scalar curvature of the bundle manifold is formally the same as that given in Ref. 8 for local gauge theories of internal symmetry. The essential difference is that now because of relation (2.14) the gauge potentials in the second term and the connection coefficients in $R^{\rm st}$ are related.

From the point of view we have adopted, the action function of gauge theories must depend on the geometrical quantities of the superspace, such as R, and must be a scalar in that space. This means that in the expression (3.15) for R it is not the individual terms but the whole expression which is geometrically significant (except, of course, for the constant terms R^G).

Thus we take for the theory of gravitation the action

$$\begin{aligned}
S &= \int d^n x\, dV_g \sqrt{-G}\, R \\
&= \int d^n x\, dV_g \sqrt{-g\bar{g}} \\
&\quad \times (R^{\rm st} + R_G - \tfrac{1}{4} g^{\mu\nu} g^{\rho\lambda} \bar{g}_{(\alpha\alpha')(\beta\beta')} F^{\alpha\alpha'}_{\lambda\nu} F^{\beta\beta'}_{\rho\mu}),
\end{aligned} \tag{3.16}$$

where R_G may or may not be subtracted out depending on whether one wishes to include a cosmological term or not.

As far as the classical theory is concerned, one can integrate over the group parameters in (3.16) and divide out by the group volume to obtain a classical action, S_C, which depends only on space-time coordinates:

$$S_C = \int d^n x \sqrt{-g} \times (R^{\rm st} - \tfrac{1}{4} b_M^2 g^{\mu\nu} g^{\rho\lambda} \bar{g}_{(\alpha\alpha')(\beta\beta')} F^{\alpha\alpha'}_{\lambda\nu} F^{\beta\beta'}_{\rho\mu}). \tag{3.17}$$

Now the gauge potentials in the second term are replaced by Christoffel symbols according to (2.14). Because of the presence of the second term in (3.17), the variation of this action leads to differential equations for $g_{\mu\nu}$ which are the same as Einstein's only in the limit $b_M \to 0$. To see its content more clearly, we note that the first term in the action (3.17) is just the Einstein action and the second term is proportional to the Yang action[5]:

$$S_C = S_E + b_M^2 S_Y. \tag{3.18}$$

Thus from the point of view of gauge theories as reflected upon the fiber-bundle geometry, it is the sum of the two terms which is the natural one.

We also note that the bundle parameter b_M is not determined by geometry. Therefore, insofar as the large-scale behavior of matter is concerned, b_M can be chosen small enough so that the weak-field consequences of Einstein's theory are not significantly altered. The more important question is whether this theory is renormalizable, if at all, for arbitrary values of b_M. Our optimism for the renormalizability of this theory rests on its close geometrical relation to the non-Abelian gauge theories. We note, however, that in the case of gravitation one is still dealing with a curved space-time manifold.

As for the solutions of our differential equations, one can see from (3.18) that any solution which simultaneously extremizes the actions of Einstein and of Yang also extremizes our action. A large class of related solutions has already been found in connection with Yang's equation.[11] We hope to return to a more careful analysis of the possible solutions and their physical interpretation elsewhere.

IV. THE INCLUSION OF LOCAL INTERNAL SYMMETRY

Once the structure of the action for the gravitational field is determined, the inclusion of local internal symmetry proceeds in exactly the same way as that carried out for the Einstein-Yang-Mills theory in Ref. 8. Here we extend the gauge group from GL(n,R) to GL(n,R)×G, where G is the local internal-symmetry group to be considered. The gauge-covariant basis (2.10) is now extended to the set

$$\{\vec{E}_\mu, \vec{\Omega}_{\alpha\beta}, \vec{E}_A\}, \tag{4.1}$$

where in addition to (2.11) one has

$$[\vec{E}_A, \vec{E}_B] = f^C_{AB}\vec{E}_C, \tag{4.2}$$

$$[\vec{E}_A, \vec{E}_\mu] = [\vec{E}_A, \vec{\Omega}_{\alpha\beta}] = 0. \tag{4.3}$$

The analog of (2.13) is

$$\vec{E}_\mu = \vec{e}_\mu - N_\mu^{\alpha\beta}\vec{\omega}_{\alpha\beta} - N_\mu^A\vec{e}_A, \tag{4.4}$$

with the result that

$$[\vec{E}_\mu, \vec{E}_\nu] = +b_M F_{\mu\nu}^{\alpha\beta}\vec{\Omega}_{\alpha\beta} + g F_{\mu\nu}^A\vec{E}_A. \tag{4.5}$$

From here on it is only a matter of endurance to compute various geometrical quantities of the superspace with space-time manifold and gauge group GL(n,R)×G. Since the procedure is practically identical to those of the preceding section and of Ref. 8, it is sufficient to quote the expression for the scalar curvature of this bundle:

$$R^I = R^{\text{st}} + R^G - \tfrac{1}{4}b_M^2 g^{\mu\nu}g^{\rho\lambda}\bar{g}_{(\alpha\alpha')(\beta\beta')}F_{\lambda\nu}^{\alpha\alpha'}F_{\rho\mu}^{\beta\beta'} - \tfrac{1}{4}g^2 F_{\mu\nu}^A F_A^{\mu\nu}. \tag{4.6}$$

Comparing this with (3.15), we see that the difference is the last quadratic term which one expects for non-Abelian gauge theories. We emphasize again that it is not the individual terms but the entire expression for R^I, which from our point of view is geometrically significant. This means that in the scalar action

$$S^I = \int d^n x\, dV_g\sqrt{-G}\,R \tag{4.7}$$

all the couplings and the relative coefficients of various terms are completely fixed. Again one can integrate over the group parameters to obtain the classical action

$$S_C^I = \int d^n x\sqrt{-g}\,\left(R^{\text{st}} + R^G - \tfrac{1}{4}b_M^2 g^{\mu\nu}g^{\rho\lambda}\bar{g}_{(\alpha\alpha')(\beta\beta')}F_{\lambda\nu}^{\alpha\alpha'}F_{\rho\mu}^{\beta\beta'} - \tfrac{1}{4}g^2 F_{\mu\nu}^A F_A^{\mu\nu}\right). \tag{4.8}$$

This action differs from that of Einstein-Yang-Mills by the quadratic gravitational piece. It would be interesting to see how the solutions of equations following from S_C^I are related to those of Einstein-Yang-Mills equations.

The inclusion of other matter fields is also straightforward. One makes use of the so-called associated bundle.[10] This will be discussed elsewhere.

V. GAUGE THEORY BASED ON IGL(n,R) OR ISO(n-1,1)

The theory based on homogeneous GL(n,R) is interesting in that it relates the connection in the space-time manifold to the gauge potentials of a gauge theory. However, the other independent dynamical variable, i.e., the metric, is a concept directly carried over from the space-time manifold, and its relation to a gauge potential is not clear. To allow for such a relationship, we consider in this section a gauge theory of gravity based on the inhomogeneous GL(n,R) group IGL(n,R) [ISO($n-1,1$) for torsion-free manifolds]. There are other reasons for including translations in such a theory. For example, in supersymmetric theories[12] the translation operators enter in an essential way in the formalism, so that to describe local supersymmetries, one must of necessity consider a gauge group such as IGL(n,R) or some subgroup of it including translations.

Thus we consider a superspace $A(M)$ with IGL(n,R) as its gauge group and M its space-time manifold. Once again to relate the geometrical characteristics of $A(M)$ to those of the space-time manifold M, we consider parallel displacement of vector fields in M. The procedure is very similar to that described in Sec. II for GL(n,R). So to avoid repetition we omit most of the details. We begin by extending the gauge-covariant basis (2.10) for GL(n,R) to include translations:

$$\{\vec{E}_\mu, \vec{\Omega}_{\alpha\beta}, \vec{E}_\alpha\}, \quad \mu, \alpha = 0, \ldots, n-1. \tag{5.1}$$

The corresponding direct product basis is

$$\{\vec{h}, \vec{\omega}_{\alpha\beta}, \vec{h}_\alpha\}. \tag{5.2}$$

Then the bases for the horizontal sector of the tangent space are related according to

$$\vec{E}_\mu = \vec{h}_\mu - N_\mu^\alpha \vec{h}_\alpha - N_\mu^{\alpha\beta} \vec{\omega}_{\alpha\beta}. \tag{5.3}$$

Comparing this expression with the basis induced from the space-time manifold by parallel transport of vector fields, one finds, up to multiplicative constants,

$$N_\mu^{\alpha\beta} = \eta^{\alpha\gamma} \Gamma_{\mu\gamma}^\beta, \tag{5.4}$$

$$N_\mu^\alpha = K_\mu^\alpha. \tag{5.5}$$

So in this case the requirement of parallel transport leads to two conditions. The first one is the same as (2.14) obtained from $GL(n,R)$. The second one relates N_μ^α to a (1, 1) tensor K_μ^α of the space-time manifold. It is this latter condition that can be used to express the metric tensor $g_{\mu\nu}$ in terms of the gauge potentials N_μ^α.

It is to be emphasized that conditions (5.4) and (5.5) are consequences of a single requirement and must therefore be considered simultaneously. For example, if one considers translations alone by setting $N_\mu^{\alpha\beta} = 0$ everywhere, then (5.4) indicates that in the base manifold $\Gamma_{\mu\gamma}^\beta = 0$ everywhere, in which case (5.5) would hold only in flat space-time. Interestingly enough this was the case considered by Kibble.[4] It is for these reasons that the $GL(n,R)$ part of the gauge group is essential to relate the fiber bundle to the curved space-time of general relativity.

To see how the metric tensor of space-time can be related to the gauge potentials N_μ^α, we note that the quantities K_μ^α may be taken to be the tetrad coefficients which relate the coordinate basis $\{\vec{e}_\mu\}$ of the sapce-time manifold to a noncoordinate one $\{\vec{d}_\mu\}$:

$$\begin{aligned} \vec{e}_\mu &= K_\mu^\alpha \vec{d}_\alpha \equiv N_\mu^\alpha \vec{d}_\alpha, \\ \vec{d}_\alpha &= K_\alpha^\mu \vec{e}_\mu \equiv N_\alpha^\mu \vec{d}_\mu, \\ \vec{d}_\alpha \cdot \vec{d}_\beta &= \eta_{\alpha\beta}, \end{aligned} \tag{5.6}$$

where

$$N_\mu^\alpha N_\alpha^\nu = \delta_\mu^\nu, \quad N_\mu^\alpha N_\beta^\mu = \delta_\alpha^\beta. \tag{5.7}$$

Then one can write

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu = N_\mu^\alpha N_\nu^\beta \eta_{\alpha\beta}, \tag{5.8}$$

where $\eta_{\alpha\beta}$ is the flat metric $(1, -1, -1, -1)$. Once this identification is made, it is of course not necessary to work in the basis $\{\vec{d}_\mu\}$. One can set up any convenient basis in the bundle manifold and compute various geometrical quantities such as its scalar curvature R'. Then in the end $g_{\mu\nu}$ is systematically eliminated in favor of N_μ^α by Eq. (5.8), or vice versa.

The computation of the geometrical quantities in the bundle $A(M)$ proceeds as in the case of $GL(n,R)$. We omit most of the details. The reader interested in the final expression can go directly to Eqs. (5.13) and (5.14). We take the metric for the translation subgroup to be Minkowskian $\eta_{\alpha\beta}$, so that the components of the metric tensor G in the gauge-covariant basis are

$$G_{ij} = \begin{bmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & \bar{g}_{\hat{\alpha}\hat{\beta}} & 0 \\ 0 & 0 & \eta_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \vec{E}_\mu \cdot \vec{E}_\nu & 0 & 0 \\ 0 & \vec{\Omega}_{\hat{\alpha}} \cdot \vec{\Omega}_{\hat{\beta}} & 0 \\ 0 & 0 & \vec{E}_\alpha \cdot \vec{E}_\beta \end{bmatrix} \tag{5.9}$$

The commutation coefficients of the basis (5.1) are

$$\begin{aligned} [\vec{E}_\mu, \vec{E}_\nu] &= b_M F_{\mu\nu}^{\alpha\beta} \vec{\Omega}_{\alpha\beta} + b_T H_{\mu\nu}^\alpha \vec{E}_\alpha, \\ [\vec{\Omega}_{\hat{\alpha}}, \vec{\Omega}_{\hat{\beta}}] &= f_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \vec{\Omega}_{\hat{\gamma}}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} [\vec{E}_\alpha, \vec{\Omega}_{\beta\gamma}] &= f_{\alpha(\beta\gamma)}^\delta \vec{E}_\delta, \\ [\vec{E}_\alpha, \vec{E}_\beta] &= 0 = [\vec{E}_\mu, \vec{E}_\alpha] = [\vec{E}_\mu, \vec{\Omega}_{\alpha\beta}]. \end{aligned} \tag{5.11}$$

In addition to the structural equations which were given in Sec. III, we note the following which arise because of the addition of translations:

$$\begin{aligned} \vec{E}_{(\alpha\alpha')} F_{\mu\nu}^{\beta\beta'} &= -f_{(\alpha\alpha')(\gamma\gamma')}^{(\beta\beta')} F_{\mu\nu}^{\gamma\gamma'}, \\ \vec{E}_{(\alpha\alpha')} H_{\mu\nu}^\beta &= -f_{(\alpha\alpha')\gamma}^\beta H_{\mu\nu}^\gamma, \\ \vec{E}_\alpha H_{\mu\nu}^\beta &= -f_{\cdot(\gamma\gamma')}^\beta F_{\mu\nu}^{\gamma\gamma'}, \\ \vec{E}_\alpha F_{\mu\nu}^{\beta\gamma} &= \vec{E}_\alpha N_\mu^{\beta\gamma} = 0, \\ \vec{E}_\alpha N_\mu^\gamma &\equiv N_{\mu,\alpha}^\gamma = -f_{\alpha(\beta\beta')}^\gamma N_\mu^{\beta\beta'}, \\ \vec{E}_{\alpha\alpha'} N_\mu^{\beta\beta'} &= -f_{(\alpha\alpha')(\gamma\gamma')}^{(\beta\beta')} N_\mu^{(\gamma\gamma')}, \\ \vec{E}_{\alpha\alpha'} N_\mu^\beta &= -f_{\alpha\alpha'\gamma}^\beta N_\mu^\gamma. \end{aligned} \tag{5.12}$$

The covariant field tensors $H_{\mu\nu}^\alpha$ and $F_{\mu\nu}^{\alpha\beta}$ are given in terms of gauge potentials as follows:

$$H_{\mu\nu}^\alpha = N_{\mu,\nu}^\alpha - N_{\nu,\mu}^\alpha - \tfrac{1}{2} f_{\beta(\gamma\gamma')}^\alpha (N_\mu^\beta N_\nu^{\gamma\gamma'} - N_\nu^\beta N_\mu^{\gamma\gamma'}), \tag{5.13}$$

$$F_{\mu\nu}^{\alpha\beta} = N_{\mu,\nu}^{\alpha\beta} - N_{\nu,\mu}^{\alpha\beta} + f_{(\gamma\delta)(\eta\epsilon)}^{(\alpha\beta)} N_\mu^{\gamma\delta} N_\nu^{\eta\epsilon}. \tag{5.14}$$

From these the scalar curvature R' can be calculated. The result is, for torsion-free manifolds for which the gauge group is $ISO(n-1, 1)$,

$$R' = R^{st} + R^G - \tfrac{1}{4} b_T^2 \eta_{\alpha\beta} g^{\mu\rho} g^{\nu\lambda} H_{\mu\nu}^\alpha H_{\rho\lambda}^\beta - \tfrac{1}{4} b_M^2 g^{\mu\rho} g^{\nu\lambda} \bar{g}_{(\alpha\alpha')(\beta\beta')} F_{\mu\nu}^{\alpha\alpha'} F_{\rho\lambda}^{\beta\beta'}. \tag{5.15}$$

Then without any further requirements one can take for the action

$$S' = \int d^n x \, dV_g \sqrt{-G} R' \tag{5.16}$$

and proceed to obtain the analog of (3.17) for this case. We note, however, an interesting alternative that can occur as a consequence of the condition (3.8) for the space-time manifold:

$$\nabla_\mu g_{\rho\lambda} = \eta_{\alpha\beta} \nabla_\mu (N^\alpha_\rho N^\beta_\lambda) = 0. \tag{5.17}$$

Using (5.4) we can write (5.13) as

$$H^\alpha_{\mu\nu} = \nabla_\nu N^\alpha_\mu - \nabla_\mu N^\alpha_\nu. \tag{5.18}$$

Now to satisfy (5.17) it is sufficient (but not necessary) to require

$$\nabla_\mu N^\alpha_\rho = 0. \tag{5.19}$$

Therefore, with this condition we have

$$H^\alpha_{\mu\nu} = 0, \tag{5.20}$$

so that (5.15) and (3.15) become identical:

$$R' \to R.$$

Thus with the condition (5.19) one recovers the action, S_C, given by (3.17) for the SO$(n-1,1)$ bundle, with the important bonus that now one can express the $g_{\mu\nu}$ in S_C in terms of the gauge potentials of translation. Thus it appears that one is led uniquely to a gauge theory of gravitation based on the action (3.17). The function of the inhomogeneous transformations is to turn (3.17) into a bona fide gauge theory. It is well to keep in mind that all this is the consequence of the conditions (5.4) and (5.5) arising from the parallel transport requirement. Without them the connections in space-time and the fiber bundle would have been unrelated, (5.13) could not have been written in the form (5.18), and (5.17) would not have been of any help. We also note that by expressing $N^{\alpha\beta}_\mu$ in terms of N^α_μ one can write the action (3.17) entirely in terms of the gauge potentials of translations. The inclusion of internal symmetry in this scheme is clearly the same as that in Sec. IV.

VI. DISCUSSION OF THE ALTERNATIVES AND CONCLUSIONS

The main result of this paper is the action (3.17) for a gauge theory of gravitation. By making GL(n,R) inhomogeneous and imposing the requirement (5.19), $\nabla_\mu N^\alpha_\nu = 0$, even the metric tensor can be replaced by gauge potentials of the translation group, thus making (3.17) a genuine gauge theory. Throughout this work we have gone to a considerable length to clarify why one should or should not interpret a gauge theory based on a given fiber bundle as having anything to do with gravitation. We have shown that the requirement of parallel transport provides a key link between the space-time of general relativity and a fiber bundle which represents a gauge theory. We have also shown how local internal symmetry is incorporated in this scheme.

For the convenience of the reader who might wonder how our conclusions would be affected if our assumptions as to the choice of group, etc., were relaxed or replaced, we briefly discuss a number of alternatives.

(i) Lorentz and Poincaré groups with parallel transport. As we have seen in preceding sections, in these cases everything we have said about GL(n,R) and IGL(n,R) will go through, except that the reduction of GL(n,R) to SO(3, 1) would force $N^{\alpha\beta}_\mu$ to be antisymmetric in α and β. Since parallel transport relates $N^{\alpha\beta}_\mu$ to the connection coefficients in the space-time manifold, this would restrict the class of space-time theories one can consider. For example, for space-time manifolds which are pseudo-Riemannian and torsion-free, one can obtain (3.17) with SO(3, 1) or Poincaré groups as gauge groups. These conclusions are true, of course, only if one takes, as we have, the geometrically significant quantities to be those associated with the bundle.

(ii) Lorentz and Poincaré groups without parallel transport. Once the requirement of parallel transport is relaxed one may take the space-time manifold to be either flat or curved. Suppose we consider the Poincaré group on a *flat* base manifold, as was done by Kibble.[4] In this case the scalar curvature of the bundle would be similar to the last three terms of (5.15), where the contributions of potentials are both quadratic. If one imposes (5.19), $\nabla_\mu N^\alpha_\nu = 0$, as was done by Kibble, then the translation part drops out. He, however, chose to work with a differenct object constructed out of $F^{\alpha\beta}_{\mu\nu}$ and N^μ_α in order to get second-order equations. As was pointed out by Kibble,[4] one would have to identify such a theory with some space-time manifold at a later stage. As a result, although this is a gauge theory, one has no control over the nature of the space-time manifold to which it may correspond. In particular, it appears that a theory of this kind could not correspond to a torsion-free space-time manifold.

On the other hand, if one takes the base manifold to be curved, it fixes the space-time in question. Then Lorentz or Poincaré groups will play the role of internal-symmetry groups and have *a priori* nothing to do with space-time. Such a theory as a gauge theory of space-time alone is clearly not tenable since in addition to the connection in space-time one has gauge potentials of the gauge groups involved. This example serves to illustrate the power and beauty of the parallel transport requirement. Without it one would be hard-pressed to provide a reason why a gauge group should be linked to space-time.

(iii) Einstein's theory as a limit. As was pointed out in the Introduction there is no problem in getting Einstein's theory from a fiber bundle, because it is already a geometrical theory based on a pseudo-Riemannian manifold. Any such manifold may be viewed as a trivial fiber bundle in which the gauge group is the identity subgroup of GL(n,R). Then clearly the only nontrivial contribution to the scalar curvature of the bundle manifold would come from the space-time manifold: $R = R_{st}$. One can improve the appearance of such a theory by writing it in the Cartan's formulation of Einstein's theory, i.e., by going to a suitable noncoordinate basis, defining an antisymmetric object, and relating R to the square of it. Although this is no more or no less than Einstein's theory, its relation to a gauge group is not clear.

A more sophisticated-looking version of this approach may be obtained by noting that tetrad coefficients look like gauge potentials of translations, so that one may be tempted to take translations as a gauge group. As can be seen from (5.4) and (5.5) this can be done consistently either if translations are the inhomogeneous part of GL(n,R) or if the space-time is flat. Otherwise, the connection in the fiber bundle would not be related to a connection in the space-time manifold.

From our point of view, the most natural way of obtaining Einstein's theory is to note that the two terms in (3.17) and (3.18) are separately gauge invariant. Therefore, one can take the limit $b_M \to 0$ *after* obtaining the final action.

Although the examples we have given are by no means exhaustive, we hope to have convinced the reader that the requirement of the parallel transport of vector fields seems to be the only condition which provides a direct link between the space-time of general relativity and a fiber bundle which implements a gauge theory of gravitation. Therefore, our results appear to be fairly unique. In fact, the basic assumption underlying such a theory is insensitive to whether or not gravitons are elementary objects. If it turns out that they are composite, as speculated by several authors,[8,13] this theory will still be viable in such a context.

ACKNOWLEDGMENTS

We would like to thank our theoretical colleagues for stimulating discussions. In particular, we are indebted to Feza Gürsey for his continued interest in this work over the last two years and many helpful suggestions, to Kenneth Macrae for many enlightening discussions on the mathematical properties of fiber bundles and the choice of gauge groups, and to Douglas Eardley for a critical reading of the manuscript and several helpful remarks. We would also like to express our appreciation to the Aspen Center for Physics for the hospitality extended to us in the summer of 1975 and for providing a stimulating atmosphere which helped substantially in the completion of this work.

---

*Research supported in part by the U. S. Energy Research and Development Administration.

[1]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), part IX.

[2]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[3]R. Utiyama, Phys. Rev. 101, 1597 (1956).

[4]T. W. B. Kibble, J. Math. Phys. 2, 212 (1961).

[5]C. N. Yang, Phys. Rev. Lett. 33, 445 (1974).

[6]S. Deser, in *Gauge Theories and Modern Field Theory*, edited by R. Arnowitt and P. Nath (M.I.T. Press, Cambridge, Mass., 1976).

[7]L. N. Chang, K. Macrae, and F. Mansouri, Phys. Lett. 57B, 59 (1975).

[8]L. N. Chang, K. Macrae, and F. Mansouri, Phys. Rev. D 13, 235 (1976). References to earlier attempts are given in this paper.

[9]T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975). Earlier works on the nonintegrable phase formalism are quoted in this paper.

[10]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963, Vol. 1 and 1969, Vol. 2); R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds* (Academic, New York, 1964).

[11]W. T. Ni, Phys. Rev. Lett. 35, 319 (1975); A. H. Thompson, *ibid.* 35, 320 (1975). Further references can be traced from these works.

[12]J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974); A. Salam and J. Strathdee, *ibid.* B70, 477 (1974); D. Volkov and P. Akulov, Phys. Lett. 46B, 109 (1974).

[13]F. Gürsey (private communication); A. B. Borisov and V. I. Ogieveskii, Theor. Math. Phys. 21, 329 (1974); S. MacDowell (private communication); S. Weinberg, invited talk, Proceedings of Eastern Theoretical Physics Conference, Philadelphia, Pennsylvania, 1975 (unpublished); B. Zumino (private communication); P. G. O. Freund and Y. M. Cho, Phys. Rev. D 12, 1711 (1975).