Stimulated-emission effects in particle creation near black holes*

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It has recently been shown that if a black hole is formed by gravitational collapse, spontaneous particle creation will occur and a thermal spectrum of all species of particles will be emitted to infinity if the quantum matter was initially in the vacuum state. In this paper we investigate the stimulated-emission effects which occur if particles are present initially. We show in general that for a Hermitian scalar field in an external potential or in curved, asymptotically flat spacetime, stimulated-emission effects can occur precisely in those modes for which there is spontaneous particle creation from the vacuum. For the case of a Schwarzschild black hole, this result appears paradoxical, since spontaneous emission occurs at late times but there is no classical analog of stimulated emission at late times. The resolution of this paradox is that in order to induce emission of particles which emerge at late times one must send in particles at early times, so that they reach the black hole very near the instant of its formation. However, enormous energy is required of these incoming particles in order to stimulate emission of particles which emerge at late times. Thus, for a Schwarzschild black hole, even if particles are initially present (with limited energy) they will induce emission only at early times; at late times one will see only the spontaneously emitted blackbody thermal radiation. For the case of a Kerr black hole stimulated emission can be induced by particles sent in at late times with the appropriate frequencies and angular dependence. If the number of incoming particles is large, this quantum stimulated emission just gives the classical superradiant scattering.

I. INTRODUCTION

It is generally accepted as true that spontaneous emission of particles by a quantum system occurs if and only if stimulated emission of these particles can be induced if one perturbs the system by having these particles present initially. Thus, for example, if a spontaneous transition between two levels in an atom with emission of a photon occurs, then it must be possible to induce transitions between these levels by perturbing the atom with a photon, and conversely. An argument showing that this must be the case in atomic physics was first given by Einstein in 1916. He assumed that the probability of inducing an atomic transition between two levels was proportional to the intensity of the perturbing electromagnetic radiation. Furthermore, he assumed that if an atom is placed in a thermal blackbody radiation bath, equilibrium will be reached when the population of the atomic levels is given by the Boltzmann factor $\exp(-E/kT)$. Using the Planck formula for the blackbody spectrum, it is easy to show that these assumptions imply that the transition probability for spontaneous emission must be proportional to that for stimulated emission.

In black-hole physics, it has been known for several years that if one scatters classical scalar, electromagnetic, or gravitational waves off a rotating Kerr black hole, then if $0<\omega<m\Omega_H$ (where ω is the frequency of the wave, $e^{im\varphi}$ is its azimuthal dependence, and Ω_H is the angular velocity of the black hole) the scattered wave will have *greater* amplitude than the incident wave. This phenomenon, known as superradiant[1] scattering, is analogous to stimulated emission in atomic physics if one views a rotating black hole as an excited state of a nonrotating black hole. If the connection between spontaneous and stimulated emission applies to this context, the existence of classical superradiant scattering suggests that quantum processes will result in the spontaneous emission of these particles. On the basis of this reasoning Zel'dovich[2] and Starobinski[3] first proposed that rotating black holes spontaneously emit particles. Formulas for the spontaneous-emission rate were then derived by Unruh[4] using quantum field theory.

Recently, Hawking[5] showed that if a body undergoes gravitational collapse and forms a black hole, spontaneous particle creation occurs *even if the black hole is nonrotating*. In fact, in the nonrotating, Schwarzschild case, it has been shown[5,6] that one gets a steady rate of spontaneous emission at late times which is identical in all aspects to blackbody thermal emission at temperature $kT=\hbar\kappa/2\pi$, where κ is the surface gravity of the black hole. For a Kerr black hole, the spontaneous emission is not thermal[5,6]; in the limit $\kappa\to 0$ one gets the emission at superradiant frequencies originally derived by Unruh.[4]

The nature of the spontaneous particle emission from Schwarzschild and Kerr black holes has been analyzed in detail.[5,6] In the present paper, we analyze the effects of stimulated emission. Specifically, the aim of this paper is to answer the following questions: (1) In the context of quantum field theory in curved spacetime, is it generally true that stimulated emission occurs whenever spontaneous emission occurs and vice versa? (2) What is the nature of the quantum stimulated-emission effects for black holes?

As shown in Sec. II the answer to the first question is "yes." For a Hermitian scalar field in curved, asymptotically flat spacetime, we derive the expression for the state vector describing the final state assuming one or more particles are originally present. For a given outgoing mode, we show that it is always possible to choose an n-particle initial state such that the expected number of outgoing particles in the given mode is greater than or equal to n plus the expected number of particles spontaneously created from the vacuum, with strict inequality possible precisely when spontaneous emission occurs. The analysis of Sec. II follows closely and extends the analysis given in Ref. 6 for spontaneous particle creation from the vacuum.

Stimulated-emission effects for black holes are analyzed in Sec. III. The result of Sec. II that spontaneous emission implies that stimulated emission can be induced may seem, at first glance, paradoxical for the case of a Schwarzschild black hole, since spontaneous emission occurs at late times, but there is no classical analog of stimulated emission for wave packets sent in at late times. The resolution of this paradox is that stimulated emission at late times for a Schwarzschild black hole can indeed be induced, but only by sending in particles at early times, so that they reach the black hole very near the instant of its formation. Since black holes are formed by the collapse of matter, one might expect to have many incoming particles near the instant of black-hole formation. Therefore, it might be thought that the stimulated-emission effects caused by these particles will affect the predictions of the emission one expects to see at late times. However, we show that the energy required of an incoming particle to induce emission of particles which emerge at retarded time t varies as $\exp(t\kappa)$. Hence, enormous incoming energies are required to induce emission at late times. If the energy of the incoming particles is bounded, one will see only the spontaneous blackbody thermal radiation at late times.

Finally, we analyze stimulated emission for a Kerr black hole. Here it is found that stimulated emission occurs for particles sent in at late times with superradiant frequencies. We derive the probability distribution for finding k outgoing particles in a superradiant mode assuming n particles were sent in. The expected number of outgoing particles is just the sum of the spontaneous emission plus the superradiance factor times the number of incoming particles. For a large number of incoming particles the quantum stimulated emission agrees with classical superradiant scattering.

II. STIMULATED EMISSION IN QUANTUM FIELD THEORY IN CURVED, ASYMPTOTICALLY FLAT SPACETIME

In this section we derive expressions for the state vector and expected number of particles in the final state for a Hermitian scalar field in curved, asymptotically flat spacetime (or in an external potential) assuming that particles are present initially. The results given here extend the analysis of Ref. 6, where the case of no particles initially present (i.e., spontaneous particle creation from the vacuum) was analyzed in detail. We first briefly review the notation and relevant results of Ref. 6.

We wish to consider a quantum theory of the Hermitian scalar field interacting with gravity in the approximation in which the gravitational field is treated classically. We represent the scalar field ϕ as an operator-valued distribution acting on a Hilbert space of states. ϕ satisfies

$$[\nabla_\mu \nabla^\mu + m^2 + V(x)]\phi = 0, \tag{2.1}$$

where ∇_μ denotes the covariant derivative, and, for generality, we have allowed for the possibility of an external potential $V(x)$ acting on the system. Furthermore, if the spacetime is asymptotically flat, we assume that in the distant past and future, the states of the actual, interacting Hilbert space can be identified with states of the free-field Fock space such that the action of the interacting-field operator approaches that of the free-field operator.

Our notation is as follows: We denote vectors in the symmetric Fock space $\mathfrak{F}(\mathcal{H})$ associated with the one-particle Hilbert space $\mathcal{H}$ by capital Greek letters. We denote n-particle states by lower case Greek letters with n upper Latin indices. Thus, a vector $\Psi \in \mathfrak{F}(\mathcal{H})$ can be written as

$$\Psi = (c, \psi^a, \psi^{ab}, \psi^{abc}, \dots), \tag{2.2}$$

where c is the vacuum amplitude, ψ^a is the one-particle amplitude, etc. For one-particle states, the upper Latin index will often be dropped, e.g. $\psi \in \mathcal{H}$. We will explicity deal with the dual Hilbert space $\overline{\mathcal{H}}$ so that all our maps will be linear rather than antilinear. The vector in $\overline{\mathcal{H}}$ naturally associated with $\psi^a \in \mathcal{H}$ will be denoted $\overline{\psi}_a$, or simply $\overline{\psi}$. The annihilation operator for a state $\overline{\sigma} \in \overline{\mathcal{H}}$ is defined by

$$a(\overline{\sigma})\Psi = (\psi^a \overline{\sigma}_a, \sqrt{2}\, \psi^{ab}\overline{\sigma}_b, \sqrt{3}\, \psi^{abc}\overline{\sigma}_c, \dots), \tag{2.3}$$

and the creation operator for $\sigma \in \mathcal{H}$ is given by

$$a^\dagger(\sigma)\Psi = (0, c\sigma^a, \sqrt{2}\,\sigma^{(a}\psi^{b)}, \sqrt{3}\,\sigma^{(a}\psi^{bc)}, \ldots). \quad (2.4)$$

The relevant information for the outcomes of all possible scattering experiments is contained in the S matrix, $S : \mathfrak{F}_{\text{in}}(\mathcal{H}) \to \mathfrak{F}_{\text{out}}(\mathcal{H})$, which associates to every incoming free particle state the outgoing free particle state which it approaches in the future. It was shown in Ref. 6 that the above-stated minimal assumptions about the nature of quantum field theory in curved spacetime suffice to uniquely determine S and that the theory one obtains is well behaved mathematically. To explicitly construct the quantum S matrix given the spacetime curvature and external potential, it is convenient to introduce the operators $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{H} \to \overline{\mathcal{H}}$, $C : \mathcal{H} \to \mathcal{H}$, and $D : \mathcal{H} \to \overline{\mathcal{H}}$, which are determined by the curvature and potential and which may be thought of as classical S-matrix operators for the scattering of positive-frequency waves. The operator A is defined as follows: A vector $\sigma \in \mathcal{H}$ is naturally associated with a positive-frequency solution of the free Klein-Gordon equation. One gives this solution as data in the asymptotic future in the curved spacetime, and propagates it backward into the past. In the distant past, the wave will again look like a solution of the free Klein-Gordon equation. $A\sigma$ is defined as the state associated with the positive-frequency part of this solution in the past. The definition of B is similar except one takes the negative-frequency part in the past. The definitions of C and D are the time reverse of those of A and B. As shown in Ref. 6 these operators satisfy the following relations:

$$A^\dagger A - B^\dagger B = I, \quad (2.5a)$$

$$C^\dagger C - D^\dagger D = I, \quad (2.5b)$$

$$B^\dagger \overline{A} = A^\dagger \overline{B}, \quad (2.6a)$$

$$D^\dagger \overline{C} = C^\dagger \overline{D}, \quad (2.6b)$$

$$A = C^\dagger, \quad (2.7a)$$

$$B = -\overline{D}^\dagger. \quad (2.7b)$$

Let $a, a^\dagger$ denote the annihilation and creation operators on $\mathfrak{F}_{\text{in}}(\mathcal{H})$, let $b, b^\dagger$ denote the annihilation and creation operators on $\mathfrak{F}_{\text{out}}(\mathcal{H})$. As shown in Ref. 6, the annihilation and creation operators for the "in" and "out" states are related by

$$Sa^\dagger(\sigma)S^{-1} = b^\dagger(C\sigma) - b(D\sigma). \quad (2.8)$$

Similarly, we have

$$S^{-1}b^\dagger(\sigma)S = a^\dagger(A\sigma) - a(B\sigma). \quad (2.9)$$

Finally, the formula for $S\Psi_0$, where Ψ_0 is the vacuum "in" state, can be derived from the adjoint of Eq. (2.8). The result is as follows: Let $E = \overline{D}\overline{C}^{-1}$. One can view $E : \overline{\mathcal{H}} \to \mathcal{H}$ as an element of $\mathcal{H} \otimes \mathcal{H}$. By Eq. (2.6b) E is symmetric and, provided $\operatorname{tr}(E^\dagger E) < \infty$, one can associate E with a two-particle state ϵ^{ab}. Then we have[6]

$$S\Psi_0 = c\left(1, 0, (\tfrac{1}{2})^{1/2}\epsilon^{ab}, 0, \left(\frac{3\times 1}{4\times 2}\right)^{1/2} \epsilon^{(ab}\epsilon^{cd)}, 0, \ldots\right), \quad (2.10)$$

where c is a normalization constant. This completes the review of the results of Ref. 6.

It is now not difficult to derive the expression for the state vector describing the final state of the system corresponding to starting initially with a single particle in the state σ. We simply apply the operator on the left-hand side of Eq. (2.8) to the state on the left-hand side of Eq. (2.10) and equate it with the result obtained by applying the right-hand side of Eq. (2.8) to the right-hand side of Eq. (2.10). We obtain

$$S(a^\dagger(\sigma)\Psi_0) = [b^\dagger(C\sigma) - b(D\sigma)] \times c(1, 0, (\tfrac{1}{2})^{1/2}\epsilon^{ab}, 0, \ldots). \quad (2.11)$$

Using Eqs. (2.5b), (2.6b), (2.7a), and the definition of ϵ^{ab} in terms of $E = \overline{D}\,\overline{C}^{-1}$, Eq. (2.11) can be simplified as

$$S(a^\dagger(\sigma)\Psi_0) = c\left(0, \gamma^a, 0, (\tfrac{3}{2})^{1/2}\gamma^{(a}\epsilon^{bc)}, 0, \left(\frac{5\times 3}{4\times 2}\right)^{1/2}\gamma^{(a}\epsilon^{bc}\epsilon^{de)}, 0, \ldots\right), \quad (2.12)$$

where

$$\gamma = A^{-1}\sigma. \quad (2.13)$$

Notice that the amplitude for finding an even number of particles in the final state is zero, as must be the case if all created particles come in pairs. The final state corresponding to any incoming state can be found by repeated application of the operators of Eq. (2.8) to the states of Eq. (2.10). In particular, if the initial state consists of n incoming particles in states, $\sigma_1, \sigma_2, \ldots, \sigma_n$, the final state is

$$S\left(\prod_{i=1}^{n} a^\dagger(\sigma_i)\Psi_0\right) = \left\{\prod_{i=1}^{n}[b^\dagger(C\sigma_i) - b(D\sigma_i)]\right\} c(1, 0, (\tfrac{1}{2})^{1/2}\epsilon^{ab}, 0, \ldots). \quad (2.14)$$

The most relevant quantity insofar as stimulated emission is concerned is the expected number of particles, $\langle N\rangle$, in the final state. This quantity, as well as everything else about the final state, can, of course, be calculated directly from the above expressions for the final state vector. However, it is simpler in practice to derive an expression for $\langle N\rangle$ directly as follows: The number operator for particles in the final state τ is $b^\dagger(\tau)b(\overline{\tau})$. Hence, if one starts with a single particle in the state σ, the expected number of particles in the state τ at late times is given by

$$\begin{aligned}\langle N(\tau)\rangle &= (Sa^\dagger(\sigma)\Psi_0, b^\dagger(\tau)b(\overline{\tau})Sa^\dagger(\sigma)\Psi_0)\\ &= (a^\dagger(\sigma)\Psi_0, S^{-1}b^\dagger(\tau)SS^{-1}b(\overline{\tau})Sa^\dagger(\sigma)\Psi_0)\\ &= (\Psi_0, a(\overline{\sigma})[a^\dagger(A\tau)-a(B\tau)][a(\overline{A}\overline{\tau})-a^\dagger(\overline{B}\overline{\tau})]a^\dagger(\sigma)\Psi_0),\end{aligned} \tag{2.15}$$

where Eq. (2.9) has been used in the last step. An elementary calculation now yields

$$\langle N(\tau)\rangle = |(A\tau,\sigma)|^2 + |(\overline{B}\overline{\tau},\sigma)|^2 + ||B\tau||^2 \tag{2.16}$$

$$= |(\tau, C\sigma)|^2 + |(\tau,\overline{D}\overline{\sigma})|^2 + ||B\tau||^2, \tag{2.17}$$

where Eqs. (2.7a) and (2.7b) were used in obtaining the second expression. The expected total number of particles in the final state is obtained by summing Eq. (2.17) over an orthonormal basis of states τ_i,

$$\begin{aligned}\langle N\rangle &= \sum_i \langle N(\tau_i)\rangle\\ &= \sum_i [|(\tau_i, C\sigma)|^2 + |(\tau_i,\overline{D}\overline{\sigma})|^2 + ||B\tau_i||^2]\\ &= ||C\sigma||^2 + ||D\sigma||^2 + \mathrm{tr}(B^\dagger B).\end{aligned} \tag{2.18}$$

These results are easily generalized to the case where initially we have n incoming particles in states $\sigma_1, \ldots, \sigma_n$ (where the σ_j need not be distinct). We obtain

$$\langle N(\tau)\rangle = \sum_{j=1}^{n} [|(A\tau,\sigma_j)|^2 + |(\overline{B}\overline{\tau},\sigma_j)|^2] + ||B\tau||^2 \tag{2.19}$$

and

$$\langle N\rangle = \sum_{j=1}^{n} (||C\sigma_j||^2 + ||D\sigma_j||^2) + \mathrm{tr}(B^\dagger B). \tag{2.20}$$

In particular, for n incoming particles all in the state σ, we have

$$\langle N(\tau)\rangle = n[|(A\tau,\sigma)|^2 + |(\overline{B}\overline{\tau},\sigma)|^2] + ||B\tau||^2 \tag{2.21}$$

and

$$\langle N\rangle = n(||C\sigma||^2 + ||D\sigma||^2) + \mathrm{tr}(B^\dagger B). \tag{2.22}$$

For $n=0$, i.e., no incoming particles, Eqs. (2.21) and (2.22) reduce to the previously obtained results for spontaneous emission[6]

$$\langle N(\tau)\rangle = ||B\tau||^2, \tag{2.23}$$

$$\langle N\rangle = \mathrm{tr}(B^\dagger B). \tag{2.24}$$

Let us now examine the relationship between spontaneous emission and the emission resulting when one has a single particle σ in the initial state. Suppose there is no spontaneous emission of particles in the state τ. By Eq. (2.23) this is equivalent to the condition $B\tau=0$. It follows from Eq. (2.5a) that $||A\tau||=1$. Hence, by Eq. (2.16), no matter what incoming one-particle state σ we start with, we always have $\langle N(\tau)\rangle \leq 1$. Equality is achieved when $\sigma = A\tau = C^{-1}\tau$, i.e., when the initial state corresponds to a wave packet which classically would be scattered into the state τ. Since $\epsilon^{ab}\overline{\tau}_b = 0$, it can be seen from the expression for the state vector, Eq. (2.12), that if the incoming state is $\sigma = A\tau$, the final state corresponds to a simple product state of a single particle in the state τ with the vector $S\Psi_0$ describing spontaneous particle creation. Thus we conclude that there are no stimulated-emission effects in modes for which there is no spontaneous particle creation from the vacuum.

On the other hand, suppose particles in the state τ are spontaneously produced. Then $B\tau \neq 0$ and by Eq. (2.5a) we have $||A\tau|| > 1$. If we choose $\sigma = A\tau/||A\tau||$ as our initial incoming one-particle state we get

$$\begin{aligned}\langle N(\tau)\rangle &\geq ||A\tau||^2 + ||B\tau||^2\\ &> 1 + ||B\tau||^2.\end{aligned} \tag{2.25}$$

Thus, the number of τ particles in the final state is greater than the number of spontaneously produced particles ($||B\tau||^2$) plus the initial number of particles (1). We interpret this as stimulated emission induced by the initial particle $\sigma = A\tau/||A\tau||$. [If $A\tau$ and $\overline{B}\overline{\tau}$ are not orthogonal $\langle N(\tau)\rangle$ can be made even greater, i.e., more emission can be induced, by choosing σ to be an optimal linear combination of $A\tau$ and $\overline{B}\overline{\tau}$.] In fact, stimulated emission occurs for every one-particle initial state σ with $D\sigma \neq 0$. Namely, by Eq. (2.18) the expected total number of outgoing particles is

$$\begin{aligned}\langle N\rangle &= \|C\sigma\|^2+\|D\sigma\|^2+\mathrm{tr}(B^\dagger B)\\ &=1+\mathrm{tr}(B^\dagger B)+2\|D\sigma\|^2,\end{aligned} \tag{2.26}$$

where Eq. (2.5b) was used. Thus, $\langle N\rangle$ is always greater than or equal to the expected spontaneous emission ($\mathrm{tr}B^\dagger B$) plus the initial number of particles (1) and is strictly greater whenever $D\sigma\neq 0$. (When $D\sigma=0$, the final state vector is simply the product of the particle $C\sigma$ with the spontaneous emission $S\Psi_0$.)

These results generalize in a straightforward manner to the case where initially one has n particles in the states $\sigma_i, \ldots, \sigma_n$. The expected total number of particles in the final state is

$$\langle N\rangle = n+\mathrm{tr}(B^\dagger B)+2\sum_{i=1}^{n}\|D\sigma_i\|^2, \tag{2.27}$$

which again is at least as great as the initial number of particles n plus the expected spontaneous emission. Thus, in this case stimulated emission always occurs unless each of the incoming particles satisfies $D\sigma_i=0$. It is not difficult to show that this conclusion is also valid for an arbitrary n-particle initial state (i.e., one that cannot necessarily be written as a simple product of one-particle states).

Thus, we have demonstrated the existence of stimulated-emission effects in quantum field theory in curved spacetime. Furthermore, in the sense described above, stimulated emission is possible in precisely those situations where spontaneous emission occurs. We should emphasize, however, that the above results apply only to the case where the initial state has a definite number of particles. If the incoming state is not an eigenstate of the total number operator, it is no longer true that the outgoing $\langle N\rangle$ must be greater than or equal to $\mathrm{Tr}(B^\dagger B)$ plus the expected number of incoming particles. Indeed, if the initial state is chosen to be $S^{-1}\Psi_0$, there will be no particles in the final state.

In the next section we apply these results to the case of particle emission by black holes.

III. STIMULATED EMISSION BY BLACK HOLES

In the preceding section we analyzed stimulated emission for a quantized scalar field in curved spacetime and established its general relationship with spontaneous emission. In this section, we apply these results to the case of particle production by black holes formed by gravitational collapse. We begin by analyzing the case of spherical gravitational collapse to a Schwarzschild black hole. We shall discuss the massless scalar field in order to avoid some technical difficulties in defining asymptotic states. If there are no incoming particles, then at late times there will be a steady rate of emission of particles to infinity with a thermal spectrum at temperature $kT=\hbar\kappa/2\pi$, where κ is the surface gravity of the black hole. What stimulated-emission effects are possible if particles are present initially?

As is well known, an observer at infinity never sees a black hole form; the collapsing body appears only to asymptotically approach its Schwarzschild radius. However, in the following manner one can define a retarded time t_0 such that for all practical purposes an observer at infinity would say that a black hole has formed at time $\sim t_0$. Let v_0 be the advanced time corresponding to the formation of the black hole, i.e., radial null geodesics emanating from $\mathscr{I}^-$ at time v_0 reach the origin of spherical symmetry at the instant when the event horizon forms. Radial null geodesics from $\mathscr{I}^-$ prior to v_0 will pass through the origin and reach $\mathscr{I}^+$. We define t_0 as the retarded time at which radial null geodesics which leave $\mathscr{I}^-$ at time $\kappa^{-1}\approx 2\times 10^{-5}(M/M_0)$ sec prior to v_0 reach $\mathscr{I}^+$. Roughly speaking at time t_0, the collapsing body will appear to be within a light travel time of forming a black hole. The intensity light received from the surface to the collapsing body will be greatly diminished (as well as red-shifted) by time t_0 and will continue to decay exponentially in time.[7] Thus, in practice, an observer at infinity would say a black hole is present at $t\gtrsim t_0$.

We wish to analyze the nature of the incoming single-particle states which induce emission of particles which reach $\mathscr{I}^+$ at time $t\gtrsim t_0$. Let τ be such a late time one-particle state at $\mathscr{I}^+$. (For example, we may take τ to be the state associated with the wave packet P_{jn} of Refs. 5 and 6 with n sufficiently large.) From Eq. (2.16) we see that in order to produce more particles in the state τ than would be spontaneously emitted, we need the incoming particle state σ to be a linear combination of $A\tau$ and $\overline{B}\overline{\tau}$.

Recall the definitions of $A\tau$ and $B\tau$: If we propagate the wave packet corresponding to the state τ at $\mathscr{I}^+$ backward into the past, $A\tau$ and $B\tau$ correspond, respectively, to its positive- and negative-frequency parts at $\mathscr{I}^-$. There are two contributions[5] to $A\tau$: Part of the wave packet $(A\tau)_1$ is scattered by the static Schwarzschild geometry and reaches $\mathscr{I}^-$ at late advanced times ($>v_0$). The other part $(A\tau)_2$ is scattered through the collapsing body and reaches $\mathscr{I}^-$ just prior to v_0. Only the part of the wave packet which is scattered through the collapsing body contributes to $B\tau$ [i.e., $(B\tau)_1=0$] since scattering by the static Schwarzschild geometry does not change frequencies.

Suppose we choose the incoming particle σ to correspond to the first contribution to $A\tau$, i.e., $\sigma = (A\tau)_1/\|(A\tau)_1\|$. Then it is not difficult to see from Eq. (2.16) that the expected number of τ particles at $\mathscr{I}^+$ is simply the spontaneous emission $\|B\tau\|^2$ plus the fraction of the wave packet σ which does not get absorbed by the black hole. In fact for an appropriate choice of definition of positive frequency at the future event horizon—namely, one which agrees with that obtained from the Killing parameter on the horizon at late times—we have $D\sigma = 0$, and the final state vector is simply the product state of the scattered wave $C\sigma$ with the spontaneous emission $S\Psi_0$. (With a different definition of positive frequency on the horizon, $D\sigma$ has a nonzero horizon part and one might say that there is stimulated emission of particles, all of which go into the black hole. However, the physical interpretation of "particles which go into the black hole" is, at present, unclear. Only particles which reach infinity have an unambiguous physical interpretation and, as shown in Ref. 6, the density matrix describing these particles is independent of this ambiguity in the definition of positive frequency on the horizon.) Thus, there is no stimulated-emission effect for incoming particles sent in at late times; one gets only the classical scattering of these particles superimposed on the spontaneous emission.

In order to get stimulated emission we must choose the incoming particle σ to be a linear combination of the early time components $(A\tau)_2$ and $(\overline{B}\overline{\tau})_2$. We can roughly estimate the properties of σ as follows: A past-directed radial null geodesic which leaves $\mathscr{I}^+$ at retarded time $t \gtrsim t_0$ will reach $\mathscr{I}^-$ at advanced time v given by

$$v_0 - v = \kappa^{-1}\exp[-\kappa(t-t_0)]. \tag{3.1}$$

Hence, we expect that if τ represents a particle which reaches $\mathscr{I}^+$ at time $\sim t \gtrsim t_0$, then $A\tau$ and $\overline{B}\overline{\tau}$ will have a time spread Δv around v_0 at $\mathscr{I}^-$ given by

$$\Delta v \sim \kappa^{-1}\exp[-\kappa(t-t_0)]. \tag{3.2}$$

By the uncertainty relation, we have $\Delta\omega\Delta v \sim 1$ so $A\tau$ and $\overline{B}\overline{\tau}$ must be composed of frequencies at least as large as

$$\omega \sim \kappa\exp[\kappa(t-t_0)]. \tag{3.3}$$

Thus, in order to stimulate emission of particles which reach $\mathscr{I}^+$ at time t, it appears that we must send in particles within a time spread around v_0 given by Eq. (3.2) and with frequencies given by Eq. (3.3). Hence, for a solar-mass black hole, in order to stimulate emission of particles which reach $\mathscr{I}^+$ only one second after time t_0, one must send in a particle within $10^{-20\,000}$ sec of v_0 with energy $\hbar\omega \sim 10^{20\,000}$ erg. For smaller black holes for which quantum emission is important (say $M \sim 10^{15}$ g), κ is much larger and the energy and time requirements for the incoming particle to stimulate emission are enormously stronger. Clearly, if any reasonable contraints are placed on the energy and time resolution of the incoming particle, there will be no stimulated emission seen at $\mathscr{I}^+$ very shortly after the black hole is formed. At late times one will see only the spontaneously emitted thermal radiation.

The above conclusions were based on the rough estimates of Δv and ω, Eqs. (3.2) and (3.3). However, for the case where τ corresponds to a P_{jn} wave packet at $\mathscr{I}^+$, it is not difficult to show from the results of Ref. 6 that Eqs. (3.2) and (3.3) are indeed valid [although the frequency spectrum is quite flat so that the energy of an incoming particle does not have to be quite as high as indicated by (3.3) to produce some stimulated emission]. Explicitly, Eq. (3.2) follows from the fact that the total wave packet has this time resolution [see Eq. (4.14) of Ref. 6], while the positive- or negative-frequency part of the wave packet can be expressed as a sum of the total wave packet and its time reflection about v_0. The explicit expressions for $A\tau$ and $B\tau$ are given, respectively, in Eqs. (A.7) and (A.9) of Ref. 6.

Thus, we conclude that stimulated emission is of no importance for particle creation near a Schwarzschild black hole formed by spherical collapse. Particles sent in so that they reach the black hole very near to the instant of its formation can, in principle, stimulate emission but the energy required of these particles to stimulate emission of particles which come out at even moderately late times is absolutely enormous. If particles are sent in at late times, the outgoing state is merely the classical scattering of these particles superimposed upon the spontaneous emission.

If the gravitational collapse is not spherical but still results in a Schwarzschild black hole, then $(A\tau)_2$ and $(\overline{B}\overline{\tau})_2$ will differ and thus the precise state of an incoming particle required to stimulate emission in a given outgoing mode τ will be different. However, the nature of the required state (i.e., its narrow time resolution about v_0 and its enormous energy) will be the same as in the exactly spherical case, and the conclusions about stimulated emission remain unchanged. For the same reasons, these conclusions also apply to stimulated emission in nonsuperradiant modes when the gravitational collapse produces a Kerr black hole. However, for the superradiant modes $0<\omega<m\,\Omega_H$—where ω is the frequency, m is the azimuthal quantum number of the wave, and Ω_H is the angular velocity of the black hole—the situa-

tion is different.

Suppose that we are interested in inducing emission into a superradiant mode τ at late times. $(A\tau)_2$ and $(\overline{B}\overline{\tau})_2$ will again be states of enormously high energy and narrow time resolution about v_0, so we cannot, in practice, hope to stimulate emission by having incoming particles in these states. However, now we have $\|(A\tau)_1\|^2 = |r|^2 > 1$, where r is the reflection amplitude for the mode τ. Furthermore, $D(A\tau)_1 \neq 0$. Hence, if we send in a particle in the state $\sigma = (A\tau)_1/\|(A\tau)_1\|$ we shall get stimulated emission. This is, of course, the quantum analog of superradiant scattering.

It is interesting to examine some of the properties of the state vector Eq. (2.14) describing the outgoing emission in the classically superradiant mode τ assuming one sends in n particles in the state $\sigma = (A\tau)_1/\|(A\tau)_1\|$. In the limit $\kappa \to 0$ corresponding to no thermal emission, it is a straightforward matter to compute the probability P_k of observing k outgoing particles in the state τ, using Eq. (2.14) and the expression for ϵ^{ab} given in Eq. (5.6) of Ref. 6. One finds that $P_k = 0$ for $k < n$ and for $k \geq n$

$$P_k \propto \frac{k!}{(k-n)!} \frac{|t|^{2k}}{|r|^{2k}}, \tag{3.4}$$

where t is the transmission amplitude of the mode (and thus $|r|^2 - |t|^2 = 1$). This "negative binomial" distribution was previously obtained by Page (unpublished) from heuristic considerations. For $n = 0$ it reduces to the previously derived probability distribution[6] for spontaneous superradiant emission.

The mean number of outgoing particles for the distribution (3.4) is given by

$$\langle k \rangle = n|r|^2 + |t|^2. \tag{3.5}$$

[This equation can also be obtained directly from Eq. (2.21).] For large n we have $\langle k \rangle \approx n|r|^2$, in agreement with classical superradiance, since $|r|^2$ is just the amplification factor. The standard deviation of the distribution (3.4) is

$$[\langle (k - \langle k \rangle)^2 \rangle]^{1/2} = (n+1)^{1/2} |t|\,|r|. \tag{3.6}$$

Thus, for large n the distribution is sharply peaked about the mean number of particles Eq. (3.5) and the quantum stimulated emission goes over to classical superradiant scattering.

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[1]The term "superradiance" unfortunately has previously been used in atomic physics to denote a process which is *not* analogous to superradiance in black-hole physics. I thank I. Abella for pointing out to me this possible source of confusion.

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