Zeros of the multiparticle generating function in hadronic scattering models*

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Feynman and Wilson have proposed an analogy between the physics of multiparticle production and the description of a gas of molecules in statistical mechanics. The zeros of the grand partition function are known to play an essential role in the thermodynamic description of a gas. Motivated by the Feynman-Wilson gas analogy, Khuri has recently shown that the zeros of the multiparticle generating function play an equally important role in hadronic scattering. In particular, Khuri has shown that hadronic theories satisfying unitarity and the Froissart bound and having a nonshrinking nearest zero of the multiparticle generating function will have an improved Froissart bound on the total hadronic cross section. The improvement of the bound resulting from a nonshrinking nearest zero is essentially from $[\ln(s/s_0)]^2$ to $\ln(s/s_0)$. This work applies a number of mathematical techniques of statistical mechanics to study the behavior of the nearest zero of the multiparticle generating function in a general class of hadronic production models. Sufficient conditions are found which control the shrinkage of the nearest zero in production models with general, multibody interactions. The reduction of the conditions to the simpler case of two-body interactions and the physical interpretation of the sufficient conditions are discussed. Multiperipheral models are shown to be a small subclass of the two-body interaction models.

I. INTRODUCTION

Feynman[1] and Wilson[2] have conjectured that the physical description of multiparticle production may be analogous to the physics of gas molecules in statistical mechanics. The description of multiparticle production begins by defining the multiparticle generating function,

$$\Omega(z, s) = 1 + \sum_{n=1}^{N(s)} \sigma_n(s) z^n \tag{1}$$

with a form analogous to the grand partition function in statistical mechanics. The n-particle hadronic production cross sections, $\sigma_n(s)$, are assumed to be analogous to the n-particle partition functions of a gas, while some increasing function of the center-of-mass energy squared, s, is analogous to the volume of the gas system. The expansion parameter z is in general unspecified, though in certain hadronic models it may have a clear physical interpretation. For example, in multiperipheral models z is the square of the strong-coupling constant.

As the collision energy s increases, the number of produced particles may increase in principle as fast as $\sqrt{s}$, and is observed experimentally[3] to increase roughly like $\ln s$. For s sufficiently large, one does not seek a description of multiparticle production which entails the position and momentum of each outgoing particle, but rather a collective description of the multiparticle system. From the multiparticle generating function one can define collective parameters in the same way one obtains thermodynamic parameters from the grand partition function. The hope of the Feynman-Wilson gas analogy is that these collective parameters will be meaningful in describing multiparticle production.

Several authors[4] have studied the mathematical relation between field theory models of hadronic scattering and the statistical-mechanical description of a gas of molecules. Chang, Yan, Yao, and Campbell have used the cluster decomposition techniques of statistical mechanics to study inclusive and exclusive multiparticle spectra in a number of simple hadronic models. Lee has put the Feynman-Wilson gas analogy on a firmer footing by explicitly demonstrating the correspondence between the ϕ^3 multiperipheral production model[5] and a one-dimensional ring of gas molecules.

Zeros of the grand partition function are known to play an important role in statistical mechanics. Yang and Lee[6] pointed out that zeros accumulating on the positive real fugacity axis in the infinite-volume limit provide a mathematical mechanism for phase transitions. Bogoliubov and Khatset,[7] Groeneveld,[8] Ruelle,[9] and Penrose[10] have studied the behavior of zeros in the complex fugacity plane to prove the existence of a zero-free region around the origin and thus, the existence of the gas phase, in the limit of infinite volume.

Recently, Khuri[11] pointed out that zeros of the multiparticle generating function play an equally important role in hadronic scattering. In particular, Khuri showed for theories satisfying unitarity and the Froissart bound that if the nearest zero to the origin in the complex z plane does not collapse to the origin in the limit of infinite energy, then one has the bound

$$\sigma_{\rm tot}(s) \underset{s\to\infty}{\leqslant} C \ln(s/s_0)\ln\ln(s/s_0).$$

This is essentially an improvement of the Froissart bound by a factor of $\ln(s/s_0)$. It is important to emphasize that Khuri's result is independent of the validity of the gas analogy. One can start with Eq. (1) defining the multiparticle generating function and study the relation between its zeros and the Froissart bound without any reference to statistical mechanics.

This paper studies the conditions insuring a nonshrinking nearest zero of the multiparticle generating function in the limit of infinite energy. While formally the existence of the gas analogy is unnecessary, the mathematical techniques developed to study zeros of the grand partition function are valuable in studying the zeros of the multiparticle generating function. Using some methods of statistical mechanics, a set of sufficient conditions is found insuring a nonshrinking nearest zero for production models having general, multibody interactions. Unfortunately, as in statistical mechanics, when working with multibody interactions the conditions giving a zero-free region are not physically simple. Such conditions constitute only a first step toward understanding the properties of general theories with nonshrinking zeros, and define a class of models which may be useful in the study of the gas analogy.

It is essential that the sufficient conditions for nonshrinking zeros do not trivially restrict the growth of the elastic cross section with energy. A simple example shows that this does not necessarily happen. Consider the ansatz

$$\bar{\sigma}_n(s) = \frac{[2\sigma_{\rm el}(s)]^{n/2}}{n!}, \quad n \geqslant 1. \tag{2}$$

From Eq. (1) it is clear that the multiparticle generating function has a nonshrinking zero-free region for large s, since

$$\Omega(z, s) = e^{z[2\sigma_{\rm el}(s)]^{1/2}}. \tag{3}$$

Thus, Khuri's theorem guarantees that

$$\bar{\sigma}_{\rm tot}(s) \leqslant C \ln(s/s_0)\ln\ln(s/s_0). \tag{4}$$

This bound results from the n dependence of the n-particle production cross sections, *without any initial restriction on the s dependence of $\sigma_{\rm el}$ or the n-particle cross sections*. It is this restriction on the n dependence of the production amplitudes which will be essential in obtaining a nonshrinking nearest zero in more realistic production models as well. In the simple ansatz above, Eq. (4) can be found without studying the zeros of the generating function, using the relation $\Omega(1, s) \sim \bar{\sigma}_{\rm tot}(s)$ and the Froissart bound. However, in more complicated production theories the study of zeros provides a general and powerful technique limiting the growth of the total cross section.

Section II of this paper reviews the connection between the zeros of the generating function and the Froissart bound shown by Khuri. It also contains a simple theorem giving sufficient conditions to relate nearest zeros of models or approximate theories to the nearest zero of the true multiparticle generating function. Section III describes conditions sufficient to control the nearest zero of the multiparticle generating function for multibody interaction models and proves that those conditions restrict the shrinkage of zeros. Section III B discusses alternative conditions on the potentials leading to a nonshrinking nearest zero, and Sec. III C describes the conditions sufficient to guarantee a nonshrinking nearest zero in the simpler case of strictly two-body interactions.

II. ZEROS OF THE MULTIPARTICLE GENERATING FUNCTION

This chapter contains two theorems on zeros of the multiparticle generating function. The first theorem reviews the work of Khuri showing the connection between the shrinkage of the nearest zero and the bound on the total cross section. The second theorem points out the essential features which tie together the shrinkage of the nearest zero in some approximate theory and the nearest zero of the true multiparticle generating function.

A. Nearest zeros and the Froissart bound

For simplicity, consider a theory in which only one kind of particle is present, having mass m. Then $\sigma_2(s) = \sigma_{\rm el}(s)$. $\sigma_1(s)$ appearing in Eq. (1) is a free parameter corresponding to the phase-space volume, and has been included to allow a more general class of models in which $\sigma_{\rm el}(s)$ is nontrivial. The upper limit on $N(s)$, the number of produced particles, may grow like $\sqrt{s}/m$, so there may be at most $\sqrt{s}/m$ zeros of $\Omega(z, s)$ for fixed s. Since all the coefficients $\sigma_n(s)$ are positive no zeros will lie on the positive real axis for finite s; and since $\Omega(0, s) = 1$, $\Omega(z, s)$ will have no zeros at the origin for finite values of s. Only in the limit of infinite values of s may zeros collapse to the z-plane origin.

For fixed s the zeros of $\Omega(z, s)$ in the complex

z plane may be ordered according to their distance from the origin:

$$|z_0(s)| \leq |z_1(s)| \leq \cdots \leq |z_n(s)|. \tag{5}$$

The circle around the origin in which $\Omega(z, s)$ is free of zeros has a radius $R(s) = |z_0(s)|$. $R(s)$ is shown by the following generalization of Khuri's work to determine the bound on the total cross section.

Theorem I[12]: Theories or models satisfying unitarity and the Froissart bound, and having a multiparticle generating function free of zeros inside a circle of radius $R(s)$, satisfy the following bounds for s sufficiently large:

$$\sigma_n(s) \leq \frac{C_n}{[R(s)]^n}[\ln\ln(s/s_0)]^n, \tag{6}$$

$$\sigma_{\text{tot}}(s) \leq \frac{C}{R(s)}\ln(s/s_0)\ln\ln(s/s_0). \tag{7}$$

Proof of theorem I makes use of the derivative form of Caratheodory's inequality[13]:

Consider the function $f(z)$ analytic for $|z| < R(s)$ with $f(0) = 0$. Define $A(r)$ to be the maximum of $\text{Re}f(z)$ on $|z| = r$, and assume $A(R(s)) > 0$. Then,

$$\max_{|z|=r}\left|\frac{d^n}{dz^n}f(z)\right| \leq \frac{2^{n+2}n!\,R(s)}{[R(s)-r]^{n+1}}A(R(s)). \tag{8}$$

For the case at hand, let $f(z) = \ln\Omega(z, s)$. From the definition of $\Omega(z, s)$, $f(z)$ satisfies the conditions of Caratheodory's inequality inside $|z| = R(s)$. Thus

$$\max_{|z|=r}\left|\frac{d^n}{dz^n}\ln\Omega(z, s)\right| \leq \frac{2^{n+2}n!\,R(s)}{[R(s)-r]^{n+1}}\ln[\max_{\varphi}|\Omega(R(s)e^{i\varphi}, s)|]. \tag{9}$$

The inequality takes a simpler form for specific values of n.

For $n = 1$,

$$\max_{|z|=r}\left|\frac{\sum_{n=1}^{N(s)} n\sigma_n(s)z^{n-1}}{\Omega(z, s)}\right| \leq \frac{2^3R(s)}{[R(s)-r]^2}\ln[\max_{\varphi}|\Omega(R(s)e^{i\varphi}, s)|]. \tag{10}$$

Consider the case $R(s) < 1$, applying the inequality on the circle $|z| = r = 1/s^p$, $p > 0$, a circle shrinking to zero in the limit of large s. In that limit the inequality becomes

$$|\sigma_1(s) + O(1/s^p)| \leq \frac{2^3}{R(s)}\ln[1 + \sigma_1(s) + \sigma_{\text{tot}}(s)]. \tag{11}$$

Choosing the phase-space parameter $\sigma_1(s)$ so that it does not increase as fast as $\sigma_{\text{tot}}(s)$ for large s, and using the Froissart bound

$$\sigma_{\text{tot}}(s) \underset{s\to\infty}{\leq} C[\ln(s/s_0)]^2 \tag{12}$$

on the right-hand side leads to the following bound for $n = 1$:

$$\sigma_1(s) \underset{s\to\infty}{\leq} \frac{C_1}{R(s)}\ln\ln(s/s_0). \tag{13}$$

There is no physical content to the bound, since it relates the unphysical free phase-space parameter to the radius of the zero-free region. More interesting results are obtained for higher values of n.

For $n = 2$, double differentiation on the left-hand side and application of the inequality on the circle $|z| = r = 1/s^p$, $p > 0$, give the inequality

$$|2\sigma_2(s) - \sigma_1^2(s) + O(1/s^p)| \leq \frac{2^4}{[R(s)]^2}\ln\ln(s/s_0), \tag{14}$$

where the Froissart bound has again been used on the right-hand side. Use of Eq. (13) giving an upper bound on $\sigma_1(s)$ results in the bound

$$\sigma_2(s) \underset{s\to\infty}{\leq} \frac{C_2}{[R(s)]^2}[\ln\ln(s/s_0)]^2. \tag{15}$$

For a theory with only one particle type $\sigma_2(s) = \sigma_{\text{el}}(s)$; for theories with more than one kind of hadron, $\sigma_{\text{el}}(s) < \sigma_2(s)$, so in either case, Eq. (15) gives a bound on the elastic cross section in terms of $R(s)$.

For $n > 2$, application of Eq. (9) for higher values of n on a circle of shrinking radius $r = 1/s^p$ and use of the Froissart bound results in upper bounds on all n-body production cross sections in terms of $R(s)$:

$$\sigma_n(s) \underset{s\to\infty}{\leq} \frac{C_n}{[R(s)]^n}[\ln\ln(s/s_0)]^n. \tag{6}$$

The bound on the total cross section in terms of $R(s)$ follows easily from the bound obtained in Eq. (15) on $\sigma_{\text{el}}(s)$. Using only unitarity, Martin[14] has obtained the following inequality:

$$\sigma_{\text{tot}}(s) \underset{s\to\infty}{\leq} C\ln(s/s_0)[\sigma_{\text{el}}(s)]^{1/2}. \tag{16}$$

Inequalities (15) and (16) together yield

$$\sigma_{\text{tot}}(s) \underset{s\to\infty}{\leq} \frac{C\ln(s/s_0)\ln\ln(s/s_0)}{R(s)}. \tag{7}$$

These results conclude the proof of theorem I, and emphasize the importance of nearest zero of the multiparticle generating function. Inequalities (6) and (7) follow strictly from the assumptions of unitarity and the Froissart bound. From Eq. (7) one can see that a nonshrinking zero-free region in the large-s limit leads to the bound on the total cross section

$$\sigma_{tot}(s) \underset{s\to\infty}{\leq} C \ln(s/s_0) \ln\ln(s/s_0). \quad (17)$$

This is essentially an improvement of $\ln(s/s_0)$ over the Froissart bound. Even in the case of $R(s)$ shrinking slowly such that

$$R(s) \underset{s\to\infty}{\geq} C[\ln(s/s_0)]^{\epsilon-1}, \quad (18)$$

with $\epsilon > 0$ but arbitrarily small, Eq. (7) yields an improvement of the Froissart bound.

B. Connection between zeros of approximate theories and zeros of the true generating function

In the next chapter, a number of models will be shown to have nonshrinking nearest zeros of their multiparticle generating functions under conditions on the interactions. According to theorem I, such a property can lead to an improved Froissart bound for the total cross section in those models. However, one is interested in improving the Froissart bound on the true total cross section rather than the cross section in some model of hadronic scattering.

This section gives a simple theorem explaining how closely an approximate theory having nonshrinking zeros must come to the true hadronic production theory in order to restrict the shrinkage of zeros for the ture multiparticle generating function, such that the true total cross section has an improved Froissart bound.

Consider Eq. (1) to define the true multiparticle generating function with $\sigma_n(s)$ the true n-particle production cross sections. Let

$$\bar{\Omega}(z, s) = 1 + \sum_{n=1}^{N(s)} z^n \bar{\sigma}_n(s) \quad (19)$$

define the multiparticle generating function for some approximate theory, $\bar{\sigma}_n(s)$ being the approximate n-particle production cross sections.

Theorem II: If $\bar{\Omega}(z, s)$ has a nonshrinking zero-free region with constant radius $\bar{R}$ in the limit of large s, and if

$$\text{(i)}\ |\sigma_{el}(s) - \bar{\sigma}_{el}(s)| \underset{s\to\infty}{\leq} C[\ln(s/s_0)]^{2(1-\epsilon)},$$

$$\text{(ii)}\ |\bar{\sigma}_n(s)| \underset{s\to\infty}{\leq} C'[\ln(s/s_0)]^{n(1-\epsilon)}, \quad n \geq 3,$$

ϵ positive but arbitrarily small, then $\Omega(z, s)$ has a zero-free region with radius

$$R(s) \underset{s\to\infty}{\geq} C[\ln(s/s_0)]^{\epsilon-1}, \quad (18)$$

giving an improved Froissart bound for the true total cross section.

The term $|\bar{\sigma}_n(s)|$, rather than $|\bar{\sigma}_n(s) - \sigma_n(s)|$, appears on the left-hand side of condition (ii) by use of the Froissart bound to restrict the growth of the physical n-particle production cross sections, $\sigma_n(s) \leq C[\ln(s/s_0)]^2$.

The following proof of theorem II uses Rouche's theorem[15]:

If two functions, $\Omega(z, s)$ and $\bar{\Omega}(z, s)$ are analytic inside and on a closed contour C in the z plane for fixed s, and $|\Omega(z, s) - \bar{\Omega}(z, s)| < |\bar{\Omega}(z, s)|$ on C, then $\Omega(z, s)$ and $\bar{\Omega}(z, s)$ have the same number of zeros inside C.

The generating functions defined by Eqs. (1) and (19) are polynomials in z for fixed s, and are thus analytic inside and on any closed circle about the z-plane origin. The contour C is chosen to be the circle $|z| = R(s)$, $R(s) < \bar{R} < 1$. $\bar{R}$ is the constant radius within which $\bar{\Omega}(z, s)$ has no zeros. If it can be established that

$$|\Omega(z, s) - \bar{\Omega}(z, s)| < |\bar{\Omega}(z, s)| \quad \text{on}\ |z| = R(s), \quad (20)$$

then Rouche's theorem will guarantee that $\Omega(z, s)$ is free of zeros inside a circle of radius $R(s)$.

Conditions sufficient to guarantee inequality (20) are established by finding an upper bound for the left-hand side of the inequality and a lower bound for the right-hand side. The free phase-space parameter, $\sigma_1(s)$, may be chosen to be identical for the approximate and true theories. Using the definitions of the generating functions, the upper bound on the circle $|z| = R(s)$ is

$$\begin{aligned} |\Omega(z, s) - \bar{\Omega}(z, s)| &= \left| \sum_{n=2}^{N(s)} z^n [\sigma_n(s) - \bar{\sigma}_n(s)] \right| \\ &\leq R^2(s) |\sigma_{el}(s) - \bar{\sigma}_{el}(s)| \\ &\quad + \sum_{n=3}^{N(s)} R^n(s) |\sigma_n(s) - \bar{\sigma}_n(s)| . \end{aligned} \quad (21)$$

A lower bound for $|\bar{\Omega}(z, s)|$ on the circle $R(s)$ is found by applying the maximum-modulus principle to the inverse of $\bar{\Omega}(z, s)$. $\bar{\Omega}(z, s)$ has no zeros inside or on the circle $|z| = \bar{R}$, so

$$|\bar{\Omega}(\bar{R}e^{i\varphi}, s)| \geq \delta, \quad \delta > 0. \quad (22)$$

Thus, $[\bar{\Omega}(z, s)]^{-1}$ is analytic inside and on $|z| = \bar{R}$, having the upper bound $1/\delta$. Applying the maximum-modulus principle to this quantity and inverting yields

$$\bar{\Omega}(R(s)e^{i\varphi}, s) > \delta, \quad \text{for}\ R(s) < \bar{R}. \quad (23)$$

Thus, if it can be established that

$$[R(s)]^2 |\sigma_{el}(s) - \bar{\sigma}_{el}(s)| + \sum_{n=3}^{N(s)} R^n(s) |\sigma_n(s) - \bar{\sigma}_n(s)| < \delta, \quad (24)$$

then the conditions of Rouche's theorem will be satisfied, giving $\Omega(z, s)$ a zero-free region inside $|z| = R(s)$. $R(s)$ determines the bound on the true total cross section, and as pointed out in the previous section, the Froissart bound is improved as long as

$$R(s) \underset{s\to\infty}{\gtrsim} C[\ln(s/s_0)]^{\epsilon-1}, \quad \epsilon > 0 \text{ but arbitrarily small.} \tag{18}$$

Assuming Eq. (18), inequality (24) is minimally satisfied under conditions (i) and (ii) of theorem II. Thus, Rouche's theorem is satisfied within a circle of radius $R(s) \geq C[\ln(s/s_0)]^{\epsilon-1}$. Since $\overline{\Omega}(z, s)$ has no zeros inside this region, $\Omega(z, s)$ will also be free of zeros in the region. This yields an improved Froissart bound on the true total cross section.

Use has been made of the Froissart bound to restrict the growth of the true n-particle cross sections

$$\sigma_n(s) \leq \sigma_{\text{tot}}(s) \leq C[\ln(s/s_0)]^2, \quad n \geq 3, \tag{25}$$

so that inclusion of $\sigma_n(s)$ in the left-hand side of condition (ii) is unnecessary. In fact, if $\overline{\sigma}_n(s)$ is known to satisfy the Froissart bound, then condition (ii) of theorem II is completely unnecessary.

The proof of theorem II assumes that $\overline{\Omega}(z, s)$ has a nonshrinking region free of zeros as $s \to \infty$. The same theorem can be proved under the weaker condition that $\overline{\Omega}(z, s)$ has a slowly shrinking zero-free region, such that

$$\overline{R}(s) \geq C[\ln(s/s_0)]^{\epsilon-1}, \quad \overline{R}(s) > R(s). \tag{26}$$

The nearest zero of the true generating function and the nearest zero of the approximate model can be tied together by conditions (i) and (ii) even when the nearest zero in the model shrinks slowly to the origin.

The proof with $\overline{R}(s)$ shrinking proceeds in the same way as above, the only change being that the lower bound on $|\overline{\Omega}(z, s)|$ is found on a circle of shrinking radius. Conditions (i) and (ii) are again sufficient to insure an improved Froissart bound on the true total cross section.

Theorem II recovers a well-known result of unitarity[14]: If the Froissart bound is saturated by a $[\ln(s/s_0)]^2$ energy dependence, then the elastic cross section must grow like $[\ln(s/s_0)]^2$. This follows strictly from unitarity using Eq. (16). The result can also be seen from the conditions of theorem II. If $\sigma_{el}(s) \leq C[\ln(s/s_0)]^{2(1-\epsilon)}$, then a trivial model with constant n-particle cross sections can be found satisfying conditions (i) and (ii), insuring an improved bound on the true total cross section. Thus, the only way the total cross section can saturate the Froissart bound is with the elastic cross section also saturating the bound.

There is no stipulation in theorem II that the approximate theories must satisfy unitarity or the Froissart bound. It is only necessary that the approximate theories have elastic cross sections which roughly correspond to the true elastic cross section asymptotically and have well-behaved n-particle production cross sections. This opens up the investigation of nearest zeros to a much wider class of hadronic theories which may lead to an improved Froissart bound on the true total cross section.

III. CONDITIONS FOR A NONSHRINKING NEAREST ZERO

In this chapter, conditions are found on hadronic scattering models guaranteeing a nonshrinking nearest zero of the multiparticle generating function in the limit of large s. A general class of multibody interaction models is discussed. The sufficient conditions for multibody interaction models are complicated, so to better understand the sufficient conditions reduction to the case of two-body interactions is studied.

A. Multibody interaction models

The study of zeros in systems having multibody interactions is new to both statistical mechanics and particle physics.[16] In a gas of molecules the assumption of two-body interactions is physically justifiable. As the gas becomes sufficiently dilute the effect of a third particle on the interaction between any pair of particles should become small. There is no such compelling physical justification for strictly two-body interactions in hadronic production. The study of zeros in systems having multibody interactions is essential in dealing with hadronic production models. The goal of this work is to set up the mathematical tools to study zeros in production models with general n-body interactions.

First, one must define what is meant by an interaction in multiparticle production. The definition chosen here is motivated by the requirement that the interparticle potentials have properties similar to the intermolecular potentials of a gas system. A reasonable procedure for defining the potentials corresponding to a ϕ^3 multiperipheral model has been given by Lee.[4] A generalization of his procedure will be used here to define the interactions of general hadronic production models.

The phase-space integral for the n-particle production cross section is defined to be

$$\sigma_n(s)=\frac{1}{2s}\int\frac{d^3q_1}{(2\pi)^3 2\omega_1}\cdots\frac{d^3q_n}{(2\pi)^3 2\omega_n}(2\pi)^4\delta^4\left(p_a+p_b-\sum_{i=1}^{n}q_i\right)|T_n(q_1,\ldots,q_n)|^2, \tag{27}$$

where T_n is the production amplitude for two hadrons to scatter into n hadrons. The potentials are defined by the production amplitudes integrated over transverse-momentum variables:

$$\chi(x_1)\cdots\chi(x_n)e^{-U_n(x_1,\ldots,x_n)}=\frac{1}{2s}\int\frac{d^2q_{1\perp}}{(2\pi)^3}\cdots\frac{d^2q_{n\perp}}{(2\pi)^3}\delta^3\left(p_a+p_b-\sum_{i=1}^{n}q_i\right)|T_n(q_1,\ldots,q_n)|^2. \tag{28}$$

The variables $x_1,\ldots,x_n$ are continuous, increasing functions of the longitudinal momenta, $q_{1_L},\ldots,q_{n_L}$. For example, $x_1,\ldots,x_n$ may be the rapidities of the produced particles. The factors $\chi(x_1),\ldots,\chi(x_n)$ are characteristic single-particle functions which have been included to allow for a general phase-space shape. The n-particle production cross sections are now given by a one-dimensional (longitudinal) phase-space integral:

$$\sigma_n(s)=\int dx_1\cdots dx_n\,\delta(g(x_1,\ldots,x_n),s)\chi(x_1)\cdots\chi(x_n)e^{-U_n(x_1,\ldots,x_n)}. \tag{29}$$

By analogy to statistical mechanics, the n-body potential, U_n, is defined to include general, multibody interactions:

$$U_n(x_1,\ldots,x_n)=\sum_{k=2}^{n}\sum_{1\le i_1<\cdots<i_k\le n}\varphi_k(x_{i_1},\ldots,x_{i_k}). \tag{30}$$

The k-body interactions may be found by induction.[17] The two-body potential between particles with longitudinal momenta x_i and x_j is found from U_n, $n\gg 2$, by letting all longitudinal variables other than x_i and x_j go to infinity. The remaining quantity is $\phi_2(x_i,x_j)$. The three-body potential among particles with momenta x_i, x_j, and x_k is found from U_n, $n\gg 3$, by letting all other variables go to infinity. The remaining quantity is

$$\varphi_3(x_i,x_j,x_k)+\varphi_2(x_i,x_j)+\varphi_2(x_i,x_k)+\varphi_2(x_j,x_k).$$

The strictly three-body potential is found by subtracting the two-body potentials which were found explicitly above. Continuing this method, one can find by induction all k-body potentials. In principle, any production amplitude may be reduced to multiparticle potentials in this way. In practice, we will be restricted to a general class of amplitudes which lead to potentials satisfying reasonable boundedness and falloff properties.

The one-dimensional δ function remaining in Eq. (29) carries the s dependence of the n-particle production cross sections. The presence of the δ function poses a minor obstruction to the general proof of a nonshrinking nearest zero. At this stage we must restrict the class of hadronic models to those models in which the energy dependence of the δ function can be transferred directly to the phase-space integrals. For example, in a hadronic production model in which the produced particles are ordered in energy or rapidity, it is convenient to choose $x_1,\ldots,x_n$ to be the rapidities of the outgoing particles. Then, the n-particle production cross sections are given by[18]

$$\sigma_n(s)=\int_0^\infty dx_n\int_0^{x_n}dx_{n-1}\cdots\int_0^{x_2}dx_1\chi(x_1)\cdots\chi(x_n)\delta(x_1-a_1)\delta(x_n-Y+a_n)e^{-U_n(x_1,\ldots,x_n)}, \tag{31}$$

where $Y\sim\ln s$. The s dependence of the n-particle cross sections can be transferred directly to the limits of integration

$$\begin{aligned}\sigma_n(s)&=\int_0^Y dx_n\int_0^{x_n}dx_{n-1}\cdots\int_0^{x_2}dx_1\chi(x_1)\cdots\chi(x_n)e^{-U_n(x_1,\ldots,x_n)}\\&=\frac{1}{n!}\int_0^Y dx_n\cdots\int_0^Y dx_1\chi(x_1)\cdots\chi(x_n)e^{-U_n(x_1,\ldots,x_n)},\end{aligned} \tag{32}$$

using the symmetry of the potential U_n under the interchange of particles and the fact that a_n is small compared to Y. Such a simplification relaxes strict energy-momentum conservation, but will not alter the s dependence of the production cross sections for potentials with reasonable asymptotic properties.

The dynamics of the production models is contained in the interparticle potentials, U_n.

Equations (1) and (32) define the multiparticle generating function for hadronic scattering. The potential U_n is composed of multibody interactions, as shown in Eq. (30). The following theorem gives a set of conditions on the potentials sufficient to guarantee a zero-free region in the limit of infinite energy.

Theorem III: For multibody potentials satisfying the boundedness condition

$$\text{(i)}\quad \int dx_{k+1}\cdots\int dx_{k+m}\left|\sum_{n=0}^{m}(-1)^{m+n}\binom{m}{n}\exp[-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots,x_{k+n})]\right|\leq C^k D^m, \tag{33}$$

where C and D are independent of s, and (ii) $|\chi(x)|\leq 1$ for all values of x, the multiparticle generating function has a nonshrinking zero-free region around the origin in the complex z plane in the limit of large s.

Proof of theorem III proceeds by defining distribution functions and deriving Kirkwood-Salsburg equations[19] in the presence of multibody interactions. The Kirkwood-Salsburg equations are used to verify that the conditions of theorem III are sufficient to give a zero-free region, following the methods used by Penrose to study zeros of two-body interaction models in statistical mechanics.[10]

The multiparticle generating function and the k-particle distribution functions are defined by

$$\Omega(z,s)=1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_0^Y dx_1\cdots\int_0^Y dx_n\,\chi(x_1)\cdots\chi(x_n)e^{-U_n(x_1,\ldots,x_n)}, \tag{34}$$

$$\rho_k(x_1,\ldots,x_k|z)=\frac{1}{\Omega(z,s)}\sum_{n=0}^{\infty}\frac{z^{n+k}}{n!}\int_0^Y dx_{k+1}\cdots\int_0^Y dx_{k+n}\,\chi(x_1)\cdots\chi(x_{k+n})e^{-U_{k+n}(x_1,\ldots,x_{k+n})}, \tag{35}$$

with U_n composed of multibody interactions according to Eq. (30). The Kirkwood-Salsburg equations are derived using the relation

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}=\begin{cases}1, & m=0\\ 0, & m\geq 1,\ \text{integer},\end{cases} \tag{36}$$

to write down the expression

$$\begin{aligned}e^{-U_k(x_1,\ldots,x_k)}&=\sum_{m=0}^{\infty}\int dx_{k+1}\cdots\int dx_{k+m}\,\chi(x_1)\cdots\chi(x_{k+m})e^{-U_{k+m}(x_1,\ldots,x_{k+m})}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_{k+1}\cdots\int dx_{k+n}(-1)^n\\&\qquad\times\sum_{m=0}^{\infty}\frac{1}{m!}\int dx_{k+n+1}\cdots\int dx_{k+n+m}\,\chi(x_1)\cdots\chi(x_{k+n+m})e^{-U_{k+n+m}(x_1,\ldots,x_{k+n+m})}.\end{aligned} \tag{37}$$

Using the identity

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{n!\,m!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-z)^n}{n!}\frac{z^m}{m!} \tag{38}$$

and Eq. (35) defining the distribution functions gives the identity

$$e^{-U_k(x_1,\ldots,x_k)}=z^{-k}\Omega(z,s)\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_{k+1}\cdots\int dx_{k+n}(-1)^n\rho_{k+n}(x_1,\ldots,x_{k+n}|z). \tag{39}$$

Equation (35) defining the distribution functions can be rewritten as

$$\begin{aligned}\rho_k(x_1,\ldots,x_k|z)=\frac{1}{\Omega(z,s)}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_{k+1}\cdots\int dx_{k+n}\,\chi(x_1)\cdots\chi(x_{k+n})\\ \times e^{-U_{k+n-1}(x_2,\ldots,x_{k+n})}\left(e^{-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots,x_{k+n})}\right).\end{aligned} \tag{40}$$

Using definition (39) in place of $e^{-U_{k+n-1}(x_2,\ldots,x_{k+n})}$ yields recursion relations among the distribution functions,

$$\rho_k(x_1,\ldots,x_k\,|\,z)$$

$$= z\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!}\int dx_{k+1}\cdots\int dx_{k+n}\,\frac{1}{m!}\int dx_{k+n+1}\cdots\int dx_{k+n+m}(-1)^m\chi(x_1)\cdots\chi(x_{k+n})\rho_{k+n+m-1}(x_2,\ldots,x_{k+n+m})$$
$$\times\left(e^{-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots,x_{k+n})}\right)$$

$$= z\sum_{m=0}^{\infty}\frac{1}{m!}\int dx_{k+1}\cdots\int dx_{k+m}\rho_{k+m-1}(x_2,\ldots,x_{k+m}|\,z)$$
$$\times\left[\sum_{n=0}^{m}(-1)^{n+m}\binom{m}{n}\chi(x_1)\cdots\chi(x_{k+n})e^{-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots,x_{k+n})}\right]$$
$$k=1,2,\ldots,$$

$$\rho_0 = 1, \quad k=0. \tag{41}$$

These are the Kirkwood-Salsburg equations for multibody interactions.

We are now prepared to prove theorem III, following the method of Penrose.[10] First, expand the logarithm of the generating function and each of the k-particle distribution functions in power series in z:

$$\frac{1}{V(s)}\ln\Omega(z,s)=\sum_{l=0}^{\infty}z^l b_l(s), \tag{42}$$

$$\rho_k(x_1,\ldots,x_k\,|\,z)=\sum_{l=0}^{\infty}n_{k,l}(x_1,\ldots,x_k)z^{k+l}. \tag{43}$$

$V(s)$ is the phase-space volume,

$$V(s)=\int_0^Y dx\,\chi(x). \tag{44}$$

The definitions of the generating function and distribution functions yield the relation

$$\int dx_1\,\rho_1(x_1|\,z)=z\frac{d}{dz}\ln\Omega(z,s), \tag{45}$$

and use of the power series expansions for ρ_1 and $\ln\Omega$ gives

$$\int dx_1\,n_{1,l-1}(x_1)=V(s)\,l\,b_l(s). \tag{46}$$

The radius of convergence of the power series in Eq. (42) is

$$R(s)=\liminf_{l\to\infty}|\,b_l(s)|^{-1/l}. \tag{47}$$

The multiparticle generating function is free of zeros inside a circle about the origin with radius $R(s)$. A lower bound on $R(s)$ for s large is established by finding upper bounds on the coefficients $b_l(s)$, or on the coefficient functions $n_{1,l-1}(x_1)$.

From Eqs. (34), (35), and (43), the values of the lowest coefficient functions are found:

$$n_{1,0}(x_1)=\chi(x_1),\qquad n_{0,l}=\delta_{0,l}. \tag{48}$$

The higher coefficient functions are found from these lower ones by writing the multibody Kirkwood-Salsburg equations in terms of the coefficient functions using Eq. (43):

$$n_{k,l}(x_1,\ldots,x_k)=\sum_{m=0}^{l}\frac{1}{m!}\int dx_{k+1}\cdots\int dx_{k+m}n_{k+m-1,\,l-m}(x_2,\ldots,x_{k+m})$$
$$\times\left[\sum_{n=0}^{m}(-1)^{m+n}\binom{m}{n}\chi(x_1)\cdots\chi(x_{k+n})e^{-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots x_{k+n})}\right]. \tag{49}$$

Conditions are sought under which there are upper bounds on the coefficient functions of the form

$$|n_{k,d-k}(x_1,\ldots,x_k)|\le B_{k,d-k},\quad k=1,2,\ldots . \tag{50}$$

Such bounds hold for $k=0$ and for $k=1$, $d=1$ from Eq. (48) and condition (ii) of theorem III with

$$\begin{aligned} &B_{1,0}=1,\\ &B_{0,l}=\delta_{0,l}. \end{aligned} \tag{51}$$

This establishes the bounds for all nonvanishing coefficient functions with $d=1$. By induction, the bounds of Eq. (50) can be established by assuming for fixed $k=1,2,\ldots$ that bounds exist for $d=M-1$, and proving their existence for $d=M$. If the bounds of Eq. (50) hold for fixed k and $d=M-1$, then

$$|n_{m+k-1,M-k-m}(x_1,\ldots,x_{m+k-1})|\le B_{m+k-1,M-k-m}. \tag{52}$$

Letting $l=M-k$ in the Kirkwood-Salsburg equations for the coefficient functions and using Eq. (52) gives the bound

$$|n_{k,M-k}(x_1,\ldots,x_n)|\le\sum_{m=0}^{m-k}\frac{1}{m!}\int dx_{k+1}\cdots\int dx_{k+m}B_{k+m-1,M-k-m}\times\left|\sum_{n=0}^{m}(-1)^{m+n}\binom{m}{n}e^{-U_{k+n}(x_1,\ldots,x_{k+n})+U_{k+n-1}(x_2,\ldots,x_{k+n})}\right|. \tag{53}$$

Condition (i) of theorem III leads to the upper bound

$$|n_{k,M-k}(x_1,\ldots,x_k)|\le\sum_{m=0}^{M-k}\frac{C^kD^m}{m!}B_{k+m-1,M-k-m}. \tag{54}$$

In deriving the multibody Kirkwood-Salsburg equations the variable x_1 has been singled out for special treatment. By singling out $x_2,\ldots,x_k$ in the same manner, k inequalities such as Eq. (54) can be derived. $|n_{k,M-k}|$ is less than the geometric mean of the k different right-hand sides, resulting in the bound

$$|n_{k,M-k}(x_1,\ldots,x_k)|\le C\sum_{m=0}^{M-k}\frac{D^m}{m!}B_{k+m-1,M-k-m}. \tag{55}$$

Thus, the bounds of Eq. (52) hold for $d=M$ if the upper bounds satisfy the inequality

$$C\sum_{m=0}^{M-k}\frac{D^m}{m!}B_{k+m-1,M-k-m}\le B_{k,M-k}. \tag{56}$$

The bounds of Eq. (52) follow by induction if the bounds $B_{k,l}$ satisfy Eq. (56) for all other positive values of k and l. These conditions can be satisfied, and the best solutions to the conditions are the upper bounds

$$B_{k,l}=\frac{k(k+l)^{l-1}}{l!}D^l\times\begin{cases}C^l, & k=0,1\\ C^{k+l-1}, & k\ge2.\end{cases} \tag{57}$$

The lower bound on the radius of convergence is established using the particular set of upper bounds

$$|n_{1,l-1}(x_1)|\le B_{1,l-1}=\frac{l^{l-1}}{l!}(CD)^{l-1}. \tag{58}$$

Using Eqs. (46) and (47) yields the lower bound on the zero-free region;

$$R(s)=\liminf_{l\to\infty}|b_l(s)|^{-1/l}\ge(eCD)^{-1}. \tag{59}$$

Note that as long as C and D appearing in condition (i) of theorem III are independent of s, the radius of the zero-free region has a lower bound independent of s:

$$R=\lim_{s\to\infty}R(s)\ge(eCD)^{-1}, \tag{60}$$

and theorem III is proved.

The proof of the theorem also shows how a slowly shrinking nearest zero may occur. If instead of being constant in s, C, or D appearing in condition (i) of theorem III is an increasing function of s, then $R(s)$ will be bounded below by the decreasing function

$$R(s)\ge\frac{1}{eC(s)D(s)}. \tag{61}$$

B. Alternative conditions on the potentials

The crux of the proof of a zero-free region in theorem III is condition (i) on the multibody potentials. Unless the multibody potentials have a particularly simple form, the content of condition (i) is not transparent. In this section, physically clearer conditions sufficient to guarantee a nonshrinking nearest zero are discussed.

Condition (i) contains implicitly the conditions of stability and regularity[20] of the potentials for multibody interactions. Multibody stability and regularity conditions can be individually separated out of condition (i) by factoring all potentials which are not functions of the integration variables $x_{k+1}, \ldots, x_{k+n}$:

$$\left[\chi(x_1)\cdots\chi(x_k)e^{-U_k(x_1,\ldots,x_k)+U_{k-1}(x_2,\ldots,x_k)}\right]\int dx_{k+1}\cdots\int dx_{k+m}$$
$$\times\left|\sum_{n=0}^{m}(-1)^{m+n}\binom{m}{n}\chi(x_{k+1})\cdots\chi(x_{k+n})\exp\left[-\sum_{S\subset\{x_2,\ldots,x_{k+n}\}}\varphi(x_1,S)\right]\right|<\infty, \quad (62)$$

where, in the sum over S, S must contain one or more elements of $\{x_{k+1},\ldots,x_{k+n}\}$. The expression in the first set of brackets occurs in all terms of the series and involves interactions solely among the first k particles. The remaining potentials inside the integrals include all possible choices of variables $x_2,\ldots,x_{k+n}$ containing at least one element of the set $x_{k+1},\ldots,x_{k+n}$.

The finiteness of the expression in brackets in Eq. (62) for all values of the variables $x_1,\ldots,x_k$ is the multibody extension of the stability condition. Using the condition that the characteristic functions, $\chi(x)$, have bounded moduli, a sufficient condition for multibody stability is

$$U_k(x_1,\ldots,x_k)-U_{k-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)\geqslant -kB \quad (63)$$

for all values of $x_1,\ldots,x_k$, $1\leqslant i\leqslant k$, with B independent of s.

The finiteness of the integral in Eq. (62) leads to the multibody regularity condition. Greenberg[16] has derived a set of conditions sufficient to guarantee the convergence of the integral, one of which is interpreted to be the multibody regularity condition:

(i) the n-particle potentials, ϕ_n, are continuous and bounded below for all values of their arguments,

(ii) $\phi_2(x)\to\infty$ as $x\to 0$,

(iii) multibody regularity:

$$\lim_{V(s)\to\infty}\sup_{\{x_1,\ldots,x_n\}}\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_{k+1}\cdots\int dx_{k+n}\left|\exp\left[-\sum_{S\subset\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k\}}\varphi(x_i,S,x_{k+1},\ldots,x_{k+n})\right]-1\right|<\infty, \quad (64)$$

assuming as before that the characteristic functions have bounded moduli. Note that Greenberg has succeeded in rewriting the regularity condition in a form which is similar to the two-body regularity condition. In particular, every integration variable appears in the potential, and the validity of the regularity condition depends on the falloff of the n-particle potentials at infinity.

To illustrate the content of Eq. (64), consider the case where the set of variables $x_1,\ldots,x_k$ consists only of x_1. Further, assume that the example has only two-, three-, and four-body interactions. Then, Eq. (64) requires that

$$\sup_{x_1}\left(\int dx_2|e^{-\varphi_2(x_1,x_2)}-1|+\frac{1}{2!}\int dx_2dx_3|e^{-\varphi_3(x_1,x_2,x_3)}-1|+\frac{1}{3!}\int dx_2dx_3dx_4|e^{-\varphi_4(x_1,x_2,x_3,x_4)}-1|\right)<\infty. \quad (65)$$

This simplified example makes it clear that multibody regularity requires the potentials to go to zero sufficiently fast as each variable becomes large. This kind of fast decrease of multibody potentials is precisely what Lee found for potentials corresponding to the ϕ^3 model, and is not an unreasonable condition for the potentials of hadronic production models.

It is not difficult to construct multibody interaction models which satisfy the regularity condition. A simple example is a model with multibody interactions which decrease exponentially as any of its variables becomes large:

$$\varphi_k(x_1, \ldots, x_k) = f_k e^{-x_1 - x_2 - \cdots - x_k} \tag{66}$$

One can also establish that such models do not have a trivially improved Froissart bound, or more precisely, that $\sigma_2(s)$ has a sufficiently general behavior in such models. $\sigma_2(s)$ is given by

$$\sigma_2(s) = \frac{1}{2!} \int dx_1 dx_2\, \chi(x_1)\chi(x_2) e^{-U_2(x_1, x_2)}. \tag{67}$$

If the variables x_1 and x_2 are the rapidities of the elastically scattered particles, the characteristic functions are constants, and the two-body potential satisfies the regularity condition, then the model has the apparent behavior

$$\sigma_2(s) \lesssim Y^2 = [C \ln(s/s_0)]^2. \tag{68}$$

Such an apparent bound is general enough to allow saturation of the Froissart bound. It is only when one studies the detailed behavior of the nearest zero of the multiparticle generating function that one finds the apparent bound on the cross section can be improved.

To summarize the supplementary conditions giving a nonshrinking nearest zero in a system with multibody interactions, we state the following theorem.

Theorem IV: The multiparticle generating function has a nonshrinking zero-free region around the z-plane origin given the following conditions:

(i) $|\chi(x)| < 1$ for all values of x;

(ii) the n-particle potentials, ϕ_n, are bounded below and continuous for all values of their arguments;

(iii) $\phi_2(x) \to \infty$ as $x \to 0$;

(iv) multibody stability:

$$U_k(x_1, \ldots, x_k) - U_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \geq -kB$$

for all values of $x_1, \ldots, x_k$, $1 \leq i \leq k$;

(v) multibody regularity:

$$\lim_{V(s)\to\infty} \sup_{\{x_1, \ldots, x_k\}} \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_{k+1} \cdots \int dx_{k+n} \left| \exp\left[-\sum_{S \subset \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}} \varphi(x_i, S, x_{k+1}, \ldots, x_{k+n}) \right] - 1 \right| < \infty.$$

It is possible that a weaker set of sufficient conditions may be found insuring a nonshrinking zero-free region. Theorems III and IV constitute a first attempt to give a set of reasonable conditions constraining the nearest zero of the multiparticle generating function for models having multibody interactions.

C. Simplified interaction models[21]

To clarify the physical content of the sufficient conditions for a nonshrinking nearest zero, it is helpful to consider the above conditions in the case of strictly two-body interactions. In that case, Eq. (30) is replaced by the definition

$$U_n(x_1, \ldots, x_n) = \sum_{i<j}^{n} \varphi_2(x_i, x_j). \tag{69}$$

In production models with the n-particle production amplitudes independent of transverse momenta, the two-body assumption corresponds to nonplanar production diagrams with propagators between all outgoing particles of the form

$$K(x_i, x_j) = e^{-\varphi_2(x_i, x_j)}. \tag{70}$$

Hadronic models having this general two-body factorization of the longitudinal production amplitudes have recently been studied by Arnold and Thomas.[22]

In the case of two-body interactions, condition (i) of theorem III reduces to the condition

$$\left[\prod_{i=2}^{k} K(x_1, x_i)\right] \int dx_{k+1} \cdots \int dx_{k+m} \times \prod_{j=k+1}^{k+m} |K(x_1, x_j) - 1| < C^k D^m. \tag{71}$$

This condition immediately reduces to the well-known stability and regularity conditions for two-body interactions, giving the following theorem.

Theorem V: The multiparticle generating function

$$\Omega(z, s) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int dx_1 \cdots \int dx_n \prod_{i<j}^{n} K(x_i, x_j) \times \chi(x_1) \cdots \chi(x_n) \tag{72}$$

has a nonshrinking zero-free region in the limit of infinite s under the following conditions:

(i) $|\chi(x)| \leq 1$,

(ii) two-body stability:

$$|K(x)| = C(s) < \infty, \quad \text{for all values of } x,$$

(iii) two-body regularity:

$$\int |K(x) - 1| \, dx = D(s) < \infty.$$

Conditions (ii) and (iii) guarantee the validity of condition (i) of theorem III[21] in the case of two-body interactions. As in theorem III, if conditions (ii) and (iii) are weakened to the condition

$$C(s)D(s) \underset{s\to\infty}{\leqslant} C\left(\ln \frac{s}{s_0}\right)^{1-\epsilon}, \tag{73}$$

$\epsilon \to \infty$ but arbitrarily small, then the zero-free region has the slowly shrinking lower bound

$$R(s) \geqslant C\left(\ln \frac{s}{s_0}\right)^{\epsilon - 1}. \tag{74}$$

Theorem V can also be proved directly using the two-body Kirkwood-Salsburg equations and either the Banach space proof of Ruelle[9] or the induction proof of Penrose.[10]

An even more intuitive production model results from the further restriction to nearest-neighbor two-body interactions. The nearest-neighbor assumption requires that the longitudinal production amplitudes factorize in the form

$$\left| \frac{1}{s} T_n(x_1, \ldots, x_n) \right|^2 = g^{2n} K(x_1 - x_2) K(x_2 - x_3) \cdots K(x_{n-1} - x_n). \tag{75}$$

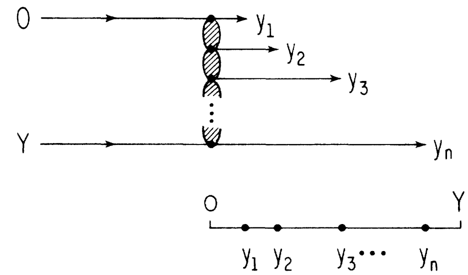

FIG. 1. The top figure shows a chainlike multiperipheral production mechanism with the exchanged quantities unspecified. The produced particles have ordered rapidities $y_1, \ldots, y_n$, shown schematically by the length of the outgoing lines. The lower figure is the analogous one-dimensional gas model, the positions of the gas molecules on the line being analogous to the rapidities of the produced particles.

Such an assumption corresponds to the simple multiperipheral production model shown in Fig. 1, with the propagators of the exchanged objects, $K(x)$, unspecified. This corresponds precisely to the generalized Chew-Pignotti model studied by DeTar.[18]

The conditions for a nonshrinking nearest zero in the nearest-neignbor model are even simpler than those for a general two-body model. In the nearest neighbor case, the Kirkwood-Salsburg equations can be replaced by simple recursion relations among the distribution functions:

$$\rho_k(x_1, \ldots, x_k | z) = z\chi(x_1) K(x_1 - x_2) \rho_{k-1}(x_2, \ldots, x_k | z), \quad k \geqslant 2. \tag{76}$$

Applying the induction method of Penrose leads to a final theorem.[21]

Theorem VI: The multiparticle generating function

$$\Omega(z, s) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_0^Y dx_1 \cdots \int_0^Y dx_n \chi(x_1) \cdots \chi(x_n) K(x_1 - x_2) \cdots K(x_{n-1} - x_n)$$

has a nonshrinking zero-free region in the limit of infinite s given the conditions

(i) $|\chi(x)| \leq 1$,

(ii) $|K(x)| = C < \infty$, for all values of x.

The models discussed above do not exhaust the list of models in which the multiparticle generating function can be shown to have a nonshrinking zero-free region. Any of the proofs given above can be extended to include cluster production models.[23] All that is needed is to consider a production mechanism in which $\sigma_2(s)$ has the same form as above. That is, clusters are not produced at the elastic vertices, but may be produced at any internal vertex. In such cluster models the production amplitudes should factorize into kernels which depend on the rapidity differences between produced clusters. With conditions on the kernels identical to those on models already studied, the cluster models can be shown to have multiparticle generating functions with nonshrinking zero-free regions.

IV. CONCLUSIONS

Conditions have been investigated leading to a nonshrinking zero-free region for the multiparticle generating function around the z-plane origin in the limit of infinite energy. Such conditions have been shown to hold in a number of nontrivial hadronic production models. Equally important, the mathematical techniques of statistical mechanics have been shown to provide powerful tools for the study of properties of hadronic scattering. While this work has concentrated on the induction method of Penrose,[10] the Banach space techniques of Ruelle[9] can be used with equal effectiveness.

The importance of a nonshrinking nearest zero has been pointed out by Khuri[11]: In theories satisfying unitarity and the Froissart bound, this property leads to an improved Froissart bound, as well as bounds on all n-particle production cross sections. It is also pointed out that a theory which roughly approximates the true elastic scattering and has a nonshrinking nearest zero can lead to an improved bound on the true total cross section.

The sufficient conditions for a nonshrinking nearest zero in multibody interaction theories are not physically simple. It is only in the reduction to two-body interactions that the physics is apparent. However, such multibody conditions constitute a first step toward understanding general theories with nonshrinking zeros, and define a class of models which may be useful in studying the gas analogy. The ultimate goal of this approach would be to derive the sufficient conditions for a nonshrinking nearest zero directly from the axioms of field theory. This is clearly a difficult problem, which must await further study.

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[22]R. C. Arnold and G. H. Thomas, ANL-HEP Report No. 75-21, 1975 (unpublished).

[23]See, for example, D. Horn and F. Zachariasen, *Hadron Physics at Very High Energies* (Benjamin, Reading, Mass., 1973), p. 56; R. D. Peccei, Nucl. Phys. B81, 301 (1974).