Concepts of multiparticle time delay*

D. Bollé[†‡]

Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
and Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

T. A. Osborn

Department of Physics, Brooklyn College, Brooklyn, New York 11210
and Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

(Received 28 August 1975)

This paper studies two concepts of time delay in few-particle scattering. The first is a global time delay that refers to the total advancement or retardation of the entire wave-packet motion owing to the presence of interactions not contained in the asymptotic Hamiltonian. The second type, the angular time delay, is the early or late arrival of a particle in a counter subtending an angle θ with respect to the incident beam direction. In the two-body problem the magnitude of this time delay is known to be $(d/dE)f(E,\theta)$, where $f(E,\theta)$ is the scattering amplitude at energy E. We discuss the definition of these two kinds of time delay in the three-body problem. We provide a generalization of the relation between angular time delay and the scattering amplitude that is valid for elastic, rearrangement, and breakup scattering. The interdependence of these two kinds of multichannel time delay is established. Possible physical applications of the resulting theory are discussed.

I. INTRODUCTION

This paper comprises a study of two, physically distinct concepts of time delay that occur in multiparticle scattering theory. The first of these is related to the total advancement or retardation of the wave-packet motion due to the presence of interactions *not* contained in the asymptotic Hamiltonians. We shall characterize this kind of time delay as "global." The second form of time delay is one appropriate for a scattering observed by counters in a differential cross-section measurement. We shall call this latter concept the "angular" time delay. In the main body of this article we develop both of these concepts for the three-body problem within the theoretical context of Faddeev's time-dependent multichannel scattering theory.[1] However, it is our belief that the three-body problem provides a paradigm within which we may explicitly state our analysis. The simple and general nature of our results suggest a much wider range of validity.

In the multiparticle time-delay phenomena the definitions, the theory, and the associated derivations are elaborate. Thus it is helpful to have a balanced overview in a simpler context of the various features that may arise. The two-body problem provides us with just such a simple parallel and one in which most of the theoretical problems have been resolved. So we shall briefly describe the structure of the two-body time-delay theory.

We turn first to the global time delay. The idea for this definition is found first in the work of Smith[2] where it appears in a time-independent description. Later, Goldberger and Watson[3] posed this concept in an abstract definition set in the time-dependent wave-packet formalism. In this latter approach, one writes

$$(f, Qf) = \lim_{R\to\infty} \int_{-\infty}^{\infty} [(\psi(t), \mathcal{P}(R)\psi(t)) - (\phi(t), \mathcal{P}(R)\phi(t))]\,dt, \quad (1.1)$$

where $\phi(t) = e^{-iH_0 t}f$ is the freely evolving wave packet for the Hamiltonian H_0. The wave function $\psi(t)$ is the exact solution of the time-dependent Schrödinger equation with the fully interacting Hamiltonian H that evolves from $\phi(t)$. The function f specifies the initial wave packet. Finally, $\mathcal{P}(R)$ is a projection operator that is unity for interparticle distances less than R and zero otherwise.

The physical meaning of (f, Qf) may be read off from the right-hand side of Eq. (1.1). We see that the first scalar product in Eq. (1.1) represents the probability of finding the particle described by $\psi(t)$ inside the sphere of radius R at time t. When integrated over all time, this gives the total time spent by the particle in that sphere during the scattering process described by $\psi(t)$. The second term in Eq. (1.1) shares this same meaning, but the exact wave is replaced by the free wave $\phi(t)$. Clearly the integral in Eq. (1.1) gives the time difference the waves $\psi(t)$ and $\phi(t)$ dwell in the sphere. By taking $R\to\infty$ we obtain a time delay defined for all of space. Thus for each f one determines a matrix element (f, Qf), and this quantity is an aggregate property of the scattering averaged over all space and time. Consequently, we label this phenomenon global time delay.

The problem associated with global time delay is to compute (f, Qf) in terms of fundamental properties of the scattering process. A rigorous and general solution of this problem was found by Jauch and Marchand.[4] These authors succeed in relating the time delay to an on-shell Hermitian operator. First they establish that Q is energy-conserving and may be expressed as

$$\langle \vec{p}\,|\,Q\,|\,\vec{p}\,'\rangle = \frac{\delta(E-E')}{\mu p}\,\langle \hat{p}\,|\,q(E)\,|\,\hat{p}'\rangle \;. \tag{1.2}$$

In this formula E is the kinetic energy, μ is the reduced mass, and $\vec{p}, \vec{p}\,'$ are exit and incident momenta in the directions $\hat{p}, \hat{p}'$. A similar on-shell representation may be introduced for the two-body S matrix,

$$\langle \vec{p}\,|\,S\,|\,\vec{p}\,'\rangle = \frac{\delta(E-E')}{\mu p}\,\langle \hat{p}\,|\,s(E)\,|\,\hat{p}'\rangle \;. \tag{1.3}$$

These last two equations of course define a one-parameter family of energy-dependent operators, $q(E)$ and $s(E)$, that act on the two-dimensional Hilbert space $L^2(\hat{p})$ as reduced operators. The solution to the global time-delay problem is given by the operator relation

$$q(E) = -\,i s^{\dagger}(E)\,\frac{d}{dE}\,s(E) \;. \tag{1.4}$$

The unitarity of $s(E)$ implies $q(E)$ is Hermitian. The derivation of Eq. (1.4) is valid for all physical wave packets f. Also, it is known[5] that the time-dependent definition Eq. (1.1) is in fact equivalent to Smith's original time-independent definition.[2]

Let us now consider the angular time delay. Here the problem is to assign a delay for a particle incident in the direction $\hat{p}_i$ and subsequently detected in the direction $\hat{p}$. The idea for this type of time delay apparently was present in the original work of Wigner and Eisenbud,[6] and has since been studied by Brenig and Haag,[7] Froissart, Goldberger, and Watson,[8] and recently again by Goldrich and Wigner.[9] The formal definition for this concept is obtained from the expression

$$\langle x(t)\rangle = \frac{\int_{C'} d^3x\,\vec{x}\cdot\hat{p}\,|\psi(\vec{x};t)|^2}{\int_{C'} d^3x\,|\psi(\vec{x};t)|^2}. \tag{1.5}$$

In this equation $\psi(\vec{x}, t)$ is the exact time-dependent wave function that appeared in Eq. (1.1), here expressed in coordinate space. The subscript C' on the integral sign means that the integration is to be carried out only for the interior of a cone C' whose axis points in the $\hat{p}$ direction. Mathematically the cone is the set in coordinate space given by $C'(\hat{p}, \lambda) = \{\vec{x} : \vec{x}\cdot\hat{p} \geq \lambda|\vec{x}|\}$, where $0 \leq \lambda \leq 1$. Clearly, as $\lambda \to +1$ the apex angle of the cone vanishes. Thus $\langle x(t)\rangle$ has the physical meaning of the average position at time t of the portion of the wave $\psi(t)$ found inside the cone C'. For the case where the incident momentum-space wave packet f is strongly peaked about $\vec{p}_i$, computing the right-hand side of Eq. (1.5) in the limit $\lambda \to 1$ yields

$$\langle x(t)\rangle = v_0 t - v_0\langle \hat{p}\,|\,\Delta(E)\,|\,\hat{p}_i\rangle \;. \tag{1.6}$$

The time variable t in this formula is defined such that at $t=0$ the average separation of the particles in the wave function is zero. The mean velocity of the wave is just $v_0 = p_i/\mu$. It is found that[7-9]

$$\langle \hat{p}\,|\,\Delta(E)\,|\,\hat{p}_i\rangle = \frac{\partial}{\partial E}\arg\langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle \;. \tag{1.7}$$

The physical interpretation of Eq. (1.6) is straightforward. The term $v_0 t$ is the position one expects the outgoing wave to have if it always has a mean velocity v_0. The term $\Delta(E)$ gives a correction to this position and represents the time delay in the $\hat{p}$ direction. Note that formulas (1.7) and (1.4) are quite different in structure. Equation (1.4) is an operator relationship, but (1.7) involves only the energy derivative of the argument of the reduced s-matrix element.

It is interesting to understand how these two time delays are interrelated. If one sets $f(\vec{p}) = \delta(\hat{p}-\hat{p}_i)f_i(p)$ in Eq. (1.1), then it is easy to deduce that the forward matrix element, $\langle \hat{p}_i\,|\,q(E)\,|\,\hat{p}_i\rangle$, is proportional to the global time delay for a plane wave of energy E and incident direction $\hat{p}_i$. For a final scattering direction $\hat{p}$ then $\langle \hat{p}\,|\,\Delta(E)\,|\,\hat{p}_i\rangle$ is the associated angular time delay. The likelihood of the plane wave scattering into $\hat{p}$ is $|\langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle|^2$. Thus we anticipate that if we weigh the angular time delay with its probability and integrate over all angles we will get the global time delay, viz.

$$\begin{aligned}\langle \hat{p}_i\,|\,q(E)\,|\,\hat{p}_i\rangle &= \mathrm{Re}\left[-i\int \langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle^* \times\left\langle \hat{p}\,\middle|\,\frac{d}{dE}\,s(E)\,\middle|\,\hat{p}_i\right\rangle d\hat{p}\right] \\ &= \int \langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle^*\langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle \times \mathrm{Re}\left[-i\,\frac{d}{dE}\ln\langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle\right] d\hat{p} \\ &= \int |\langle \hat{p}\,|\,s(E)\,|\,\hat{p}_i\rangle|^2\,\langle \hat{p}\,|\,\Delta(E)\,|\,\hat{p}_i\rangle\,d\hat{p} \;.\end{aligned} \tag{1.8}$$

As the last form of Eq. (1.8) shows, our anticipated result is correct. The first form of Eq. (1.8) employs just the integral version of Eq. (1.4) and the reality of the diagonal element $\langle \hat{p}_i\,|\,q(E)\,|\,\hat{p}_i\rangle$. In fact, Nussenzveig[10] found a connection between the angular and global time delays. It is not difficult to show Nussenzveig's result is equivalent to that above.

There remains one important feature of the two-body time delay we have not yet discussed. This is the spectral property. For the moment, consider the resolvents $r_0(z) = (H_0 - z)^{-1}$ and $r(z) = (H - z)^{-1}$ that are related to the free and exact Hamiltonians, H_0 and H. Then the spectral property is the statement that

$$\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{tr}[r(E+i0) - r_0(E+i0)] = \frac{1}{2\pi}\,\mathrm{tr}_{\hat{p}}\, q(E)\ , \tag{1.9}$$

where tr is the trace on $L^2(\vec{p})$ and $\mathrm{tr}_{\hat{p}}$ is on $L^2(\hat{p})$. In its mathematical guise this has been carefully studied by Birman and Krein,[11] and by Buslaev.[12] The explicit connection of the spectral property to time delay is found first in the paper of Jauch, Sinha, and Misra.[13] A very simple proof is found in Ref. 14. It is well known[13,14] that the left-hand side of Eq. (1.9) is physically equal to the change of state density at energy E produced by the interaction $v = H - H_0$. It is through this state-density meaning that the global time delay enters statistical mechanics.[15,16]

The above list of properties, summarized by Eqs. (1.1) through (1.9), form the theoretical framework of time-delay theory in the two-body problem. Let us close this summary by mentioning some other approaches and applications extant in the literature. Attempts to view the problem from a classical perspective have been worked out by Smith[17] and by Bar-Gadda.[18] Time delay for two-body scattering with absorption has been analyzed by Martin.[19] The time-delay concept has been recently extended to one-dimensional field theories by Jackiw and Woo.[20] Applications of the time-delay formalism include corrections to the Boltzmann equation derived by van Santen.[21] Causal lower bounds on global time delay have been obtained by Nowakowski and Osborn.[22] Fong has discussed some aspects of the static coupled-channel problem.[23] Finally, Dalitz and Moorehouse[24] have used the time-delay formalism to investigate resonances in the coupled-channel problems.

The purpose of our paper is to derive the multiparticle equivalents of the global and angular time delay. We have elsewhere [5] discussed the mathematical features of global time delay. Consequently, the primary emphasis of this paper is to analyze the forms of the angular time delay as it occurs in the three-body problem and to establish the connection between the global and angular forms.

In Sec. II we present a discussion of the definition and physical meaning of angular time delay. We use the two-body problem to carry out this discussion and to illustrate the general method of solution we employ. Section III describes the angular time-delay results for elastic, rearrangement, and breakup scattering in the three-body problem. Section IV gives the interconnection statement that is analogous to Eq. (1.8). Also included in this last section are some remaining difficulties in the physical interpretation and a discussion of some of the observable effects of these time delays. Finally, in the Appendix we study some aspects of the convergence of the asymptotic and exact time-dependent wave functions needed in the treatment of angular time delay.

We do not derive the three-body spectral property. Its proof requires the introduction of a new analytical method in the three-body problem based on Cayley transforms and will be presented separately.

II. ASPECTS OF TWO-BODY ANGULAR TIME DELAY

The method we use to obtain the three-body angular time-delay results is easy to understand in the context of two-body scattering. So we shall use the two-body problem as a model in which we may present the more sensitive physical and mathematical aspects of the angular time-delay derivation.

Let us begin our analysis by examining the definition of angular time delay. Suppose f^- is the incident asymptotic wave packet, then the exact time-dependent solution to the Schrödinger equation, $\psi(t)$, is determined by the condition

$$\lim_{t\to-\infty} \| \psi(t) - e^{-iH_0t} f^- \| = 0\ . \tag{2.1}$$

Equation (2.1) implies that $\psi(t)$ has the unique form

$$\psi(t) = e^{-iHt}\,\Omega^{(+)} f^-\ , \tag{2.2}$$

where $\Omega^{(+)}$ is the Møller operator satisfying the outgoing radiation condition. Long after the collision $\psi(t)$ again evolves according to the free Hamiltonian, H_0, and the asymptotic wave $\phi^+(t) = e^{-iH_0t} f^+$ converges in norm to $\psi(t)$ for large positive times; i.e.,

$$\| \psi(t) - \phi^+(t) \| \equiv \delta(t) \to 0 \text{ as } t \to +\infty\ . \tag{2.3}$$

The wave-packet function f^+ characterizes the outgoing state and is known to be[26]

$$\begin{aligned} f^+ &= \Omega^{(-)\dagger}\,\Omega^{(+)} f^- \\ &= S f^-\ . \end{aligned} \tag{2.4}$$

Here $\Omega^{(-)}$ is the Møller operator with the incoming-radiation condition and S is the S-matrix operator.

The quantity we need to compute is the average position of $\psi(t)$ in a cone $C'(\hat{n}, \lambda)$ for large positive times. We find it convenient to carry out this calculation in momentum space. The position operator $\vec{x}$ has the momentum-space form

$$\vec{x} = \frac{i}{2} \overleftrightarrow{\nabla} = \frac{i}{2} (\overrightarrow{\nabla} - \overleftarrow{\nabla}) \quad . \tag{2.5}$$

Let $C(\hat{n}, \lambda)$ be the momentum-space dual of the coordinate-space cone $C'(\hat{n}, \lambda)$, i.e.,

$$C(\hat{n}, \lambda) = \{\vec{p}: \vec{p}\cdot\hat{n} > \lambda |\vec{p}|\}, \quad 0 \leqslant \lambda \leqslant 1 \ . \tag{2.6}$$

The mean position of $\psi(t)$ in cone C is given by the expression

$$\langle x(t) \rangle_C = \frac{\int_C d^3p\, \psi^*(\vec{p}, t)\, \vec{x}\cdot\hat{n}\, \psi(\vec{p}, t)}{\int_C d^3p\, |\psi(\vec{p}, t)|^2} \quad . \tag{2.7}$$

One needs the form of Eq. (2.7) only for very large positive t. Because of the convergence property (2.3) it is plausible that we are allowed to replace $\psi(\vec{p}, t)$ in Eq. (2.7) by the simpler $\phi^+(\vec{p}, t)$. We shall take as a basic ansatz that this replacement leads to an error that vanishes as $t \to \infty$. Such an ansatz is characteristic of all the former treatments of angular time delay.[8,9] Intuitively, one might think that the convergence condition (2.3) is sufficient to prove that the ansatz is correct. This would be so if we were computing the expectation value of a bounded operator. However, $\vec{x}\cdot\hat{n}$ is unbounded and is expected to have a behavior like $At + B$, where A and B are constants. Thus $\delta(t) \to 0$ is not sufficient to show

$$\frac{\int_C d^3p\, \psi^*(\vec{p}, t)\, \vec{x}\cdot\hat{n}\, \psi(\vec{p}, t)}{\int_C d^3p\, |\psi(\vec{p}, t)|^2} - \frac{\int_C d^3p\, \phi^{+*}(\vec{p}, t)\, \vec{x}\cdot\hat{n}\, \phi^+(\vec{p}, t)}{\int_C d^3p\, |\phi^+(\vec{p}, t)|^2} \to 0 \ , \tag{2.8}$$

as $t \to +\infty$. In the Appendix we study the sufficient conditions for this convergence. We conclude that if $\delta(t) < \text{const} \times t^{-1-\epsilon}$ for large t, and where $\epsilon > 0$, then the ansatz (2.8) is valid. We also prove in the Appendix that if the potential falls off faster than x^{-2} then $\delta(t)$ satisfies the estimate $\delta(t) < \text{const} \times t^{-1-\epsilon}$.

In posing the definition of the mean position in momentum space we have implicitly assumed that the correct momentum-space cone restriction is the cone $C(\hat{n}, \lambda)$ that is identical to the coordinate-space cone $C'(\hat{n}, \lambda)$, but is set in momentum space. This reasonable assumption is given a rigorous statement in a theorem by Dollard.[27] For the freely evolving wave $\phi(t) = e^{-iH_0 t} f$, Dollard proved

$$\lim_{t \to \pm\infty} \int_{C'(\hat{n}, \lambda)} d^3x\, |(e^{-iH_0 t} f)(\vec{x})|^2 = \int_{C(\pm\hat{n}, \lambda)} d^3p\, |\tilde{f}(\vec{p})|^2 \ . \tag{2.9}$$

Here f is any L^2 function and $\tilde{f}$ its Fourier transform. The physical content of the above statement is that if at time $t = 0$ a particle is contained in the momentum-space cone $C(\hat{n}, \lambda)$, then as $t \to \infty$ it must be found in the dual coordinate-space cone $C'(\hat{n}, \lambda)$.

The final important ingredient in this derivation is the nature of the incident wave packet. The simplest structure for the incident wave packet is that of a plane wave of momentum $\vec{p}_i$. This wave function is just $f^-(\vec{p}) = \delta(\vec{p} - \vec{p}_i)$. If this form of wave function is used, then our method of derivation is meaningless. Technically, we must employ normalizable wave packets that are successively more and more like a plane wave. Specifically, our assumed wave-packet properties are the following:

P1: The incident wave packet $f^-(\vec{p})$ is almost monochromatic with an average momentum value p_i. It is also highly collimated with a direction $\hat{p}_i$. These two properties are implied if the modulus of $f^-(\vec{p})$ is sharply peaked about $\vec{p}_i$. We assume that relative to this sharp peak the t matrix is slowly varying.

P2: The phase of the wave packet $f^-(\vec{p})$, $\arg f^-(\vec{p})$, is slowly varying and vanishes in the limit as f^- approaches a plane-wave structure. Note that a plane wave is represented by $\delta(\vec{p} - \vec{p}_i)$, and since it is purely real it has no phase variation.

P3: The coordinate-space position of each incident wave packet is at the origin at time $t = 0$.

This last property P3 simply defines the origin of the time variable t. The vector position, $\vec{x}^-(t)$, of the incident wave packet f^-, is determined by

$$\begin{aligned}\vec{x}^-(t) &= \int d^3p\, \phi^{-*}(\vec{p}, t) \frac{i}{2} \overleftrightarrow{\nabla} \phi^-(\vec{p}, t) \\ &= t \int d^3p\, \frac{\vec{p}}{\mu} |f^-(\vec{p})|^2 \\ &\quad - \int d^3p\, |f^-(\vec{p})|^2\, \nabla \arg f^-(\vec{p}) \ ,\end{aligned} \tag{2.10}$$

where it is assumed that f^- is normalized to unity. The requirement P3, that $\vec{x}^-(0) = 0$, means that

$$\int d^3p\, |f^-(\vec{p})|^2\, \nabla \arg f^-(\vec{p}) = 0 \ . \tag{2.11}$$

In the plane-wave limit, property P2 implies property P3.

Let us consider the derivation of Eq. (1.7) for C in any nonforward direction. To do this we need to compute the ratio

$$\frac{\int_C d^3p\, \phi^{+*}(\vec{p},t)\,\vec{x}\cdot\hat{n}\,\phi^+(\vec{p},t)}{\int_C d^3p\,|\phi^+(\vec{p},t)|^2} \tag{2.12}$$

in the $t\to+\infty$ limit. Only the numerator requires attention. Recalling the momentum-space representation of $\vec{x}$ given by (2.5), we note the useful identity,

$$\frac{i}{2}\,\phi^{+*}(\vec{p},t)\,\vec{\nabla}\cdot\hat{n}\,\phi^+(\vec{p},t) = -|\phi^+(\vec{p},t)|^2 \times\frac{p}{\mu}\,\frac{\partial}{\partial E}\arg\phi^+(\vec{p},t)\ , \tag{2.13}$$

where E is the kinetic energy $E=p^2/2\mu$. For ϕ^+ we have the representation

$$\phi^+(\vec{p},t)=e^{-iEt}\int d\hat{p}'\,\langle\hat{p}|s(E)|\hat{p}'\rangle f^-(p\hat{p}')\ . \tag{2.14}$$

Thus the energy derivative of $\arg\phi^+(\vec{p},t)$ gives us

$$\frac{\partial}{\partial E}\arg\phi^+(\vec{p},t)=-t+\frac{\partial}{\partial E}\arg\int d\hat{p}'\,\langle\hat{p}|s(E)|\hat{p}'\rangle \times f^-(p\hat{p}')\ . \tag{2.15}$$

The evaluation of the time-independent term is carried out by employing the wave-packet property P1. For C not in the forward direction $\langle\hat{p}|s(E)|\hat{p}_i\rangle$ is a smooth function of $\hat{p}$ and $\hat{p}_i$, thus we have the approximation

$$\int d\hat{p}'\,\langle\hat{p}|s(E)|\hat{p}'\rangle f^-(p\hat{p}')\simeq\langle\hat{p}|s(E)|\hat{p}_i\rangle\,F(p)\ ,$$

where (2.16)

$$F(p)\equiv\int d\hat{p}'\,f^-(p\hat{p}')\ . \tag{2.17}$$

Now Eq. (2.15) takes the form

$$\frac{\partial}{\partial E}\arg\phi^+(\vec{p},t)=-t+\frac{\partial}{\partial E}[\arg\langle\hat{p}|s(E)|\hat{p}_i\rangle +\arg F(p)] \tag{2.18}$$

and Eq. (2.13) becomes

$$\frac{i}{2}\,\phi^{+*}(\vec{p},t)\,\vec{\nabla}\cdot\hat{n}\,\phi^+(\vec{p},t)=|\phi^+(\vec{p},t)|^2 \times\left[t-\frac{\partial}{\partial E}\arg\langle\hat{p}|s(E)|\hat{p}_i\rangle -\frac{\partial}{\partial E}\arg F(p)\right]. \tag{2.19}$$

Property P2 tells us that $|f^-(p,\hat{p}')|^2$ will be strongly peaked about $\vec{p}_i$. From representation (2.14) it follows that $|\phi^+(\vec{p},t)|^2$ is also peaked about $E=E_i$. As a consequence the energy E in the two-argument terms in Eq. (2.19) may be replaced with E_i. We also have to take the opening angle of the cone C to be small enough such that the variation of $\arg\langle\hat{p}|s(E)|\hat{p}_i\rangle$ is constant with respect to $\hat{p}$. This step incorporates the limit $\lambda\to1$. Putting this modified form of (2.19) into the ratio (2.12) gives the desired solution

$$\langle x(t)\rangle_C=\langle v\rangle_C\left[t-\frac{\partial}{\partial E_i}\arg\langle\hat{p}|s(E_i)|\hat{p}_i\rangle -\frac{\partial}{\partial E_i}\arg F(p_i)\right], \tag{2.20}$$

where $\langle v\rangle_C$ is the mean velocity of $\phi^+(\vec{p},t)$ in the cone C:

$$\langle v\rangle_C=\frac{\int_C d^3p(p/\mu)\,|f^+(\vec{p})|^2}{\int_C d^3p\,|f^+(\vec{p})|^2}\ . \tag{2.21}$$

In the plane-wave limit we expect the term with $\arg F(p_i)$ to vanish. It is an essential feature of the above derivation that we only used the wave-packet properties to assist in the computation of the constant terms in $\langle x(t)\rangle_C$. The coefficient of the linear term must be computed without approximation since any error in this coefficient is multiplied by t which goes to $+\infty$.

We close this section with a few general remarks about the nature of the result found above. It is important that property P3 states that it is the noninteracting rather than the interacting wave packet that is at $\vec{x}=0$ when $t=0$. Thus it is clear that the spatial shift of the position of the outgoing wave,

$$\langle v\rangle_C\,\frac{\partial}{\partial E_i}\arg\langle\hat{p}|s(E_i)|\hat{p}_i\rangle\ ,$$

includes effects from accelerations as the particle approaches the scattering region, as well as accelerations affecting the outgoing stage of the scattering. So, for the scattering geometry $(\hat{p},\hat{p}_i)$ Eq. (2.20) represents the total time delay. With this interpretation the connection formula (1.8) makes good physical sense.

It is of some interest to contrast our method of derivation with those employed in Refs. 8 and 9. The major difference is in fact that the above derivation, Eqs. (2.13)–(2.21), is much shorter. There is a simple reason for this. The derivations of Refs. 8 and 9 are set initially in coordinate space and then Fourier transforms to momentum space must be introduced to complete the calculation. However, the presence of the cone restriction make the Fourier transform uncommonly awkward. Our derivation is set completely in momentum space so we need only algebraic manipulations.

III. THREE-BODY ANGULAR TIME DELAY

We derive here the angular time-delay results for three-body scattering. The method we use is just a multichannel modification of that employed in Sec. II. We begin our discussion by setting up the appropriate definitions. Let $f_\beta^-(\vec{p}_\beta)$ be the momentum-space function describing the free relative motion of particle β and the cluster composed of particles α and γ. The momentum $\vec{p}_\beta$ is the relative momentum of particle β and the center of mass of the $\alpha\gamma$ cluster. The internal momentum of the $\alpha\gamma$ cluster is denoted by $\vec{q}_\beta$, and the conjugate coordinates of $\vec{p}_\beta, \vec{q}_\beta$ are $\vec{x}_\beta, \vec{y}_\beta$. A more detailed description of our Jacobi coordinates may be found in Ref. 28.

The freely evolving wave packet associated with f_β^- is given by

$$\Phi_\beta^-(\vec{p},\vec{q};t)=\psi_\beta(\vec{q}_\beta)\,e^{-i\tilde{H}_\beta t}f_\beta^-(\vec{p}_\beta)\ , \tag{3.1}$$

where

$$\begin{aligned}\tilde{H}_\beta&=\frac{\vec{p}_\beta^{\,2}}{2n_\beta}-\chi_\beta^2\\&=\tilde{p}_\beta^{\,2}-\chi_\beta^2\end{aligned} \tag{3.2}$$

and

$$(\tilde{q}_\beta^{\,2}+v_\beta)\,\psi_\beta(\vec{q}_\beta)=-\chi_\beta^2,\quad \tilde{q}_\beta^{\,2}=\frac{\vec{q}_\beta^{\,2}}{2\mu_\beta}. \tag{3.3}$$

Here $\tilde{H}_\beta$ is the β-channel Hamiltonian, $-\chi_\beta^2$ is the bound-state energy of cluster $\alpha\gamma$, and $\psi_\beta(\vec{q}_\beta)$ is the unit normalized two-body bound-state wave function. The reduced mass of cluster $\alpha\gamma$ and particle β is denoted by n_β while μ_β denotes the reduced mass of particles α and γ. The exact time-dependent state $\Psi_\beta(t)$ evolving from $\Phi_\beta^-(t)$ is defined by

$$\lim_{t\to-\infty}\|\Psi_\beta(t)-\Phi_\beta^-(t)\|=0\ . \tag{3.4}$$

As $t\to+\infty$, $\Psi_\beta(t)$ has wave-function components $\Phi_\alpha^+(t)$ satisfying four different boundary conditions that are determined by

$$\lim_{t\to+\infty}\|P_\alpha[\Psi_\beta(t)-\Phi_\alpha^+(t)]\|=0,\quad \alpha=0,1,2,3\ . \tag{3.5}$$

Here P_α is a projection operator associated with the subspace of α-channel motion, and is given by the kernels

$$\langle\vec{p}_\alpha\vec{q}_\alpha|P_\alpha|\vec{p}_\alpha'\vec{q}_\alpha'\rangle=\psi_\alpha(\vec{q}_\alpha)\,\psi_\alpha^*(\vec{q}_\alpha')\,\delta(\vec{p}_\alpha-\vec{p}_\alpha')\ ,\quad \alpha=1,2,3\ . \tag{3.6}$$

For $\alpha=0$, P_0 is the identity. The asymptotic wave functions are

$$\Phi_\alpha^+(\vec{p},\vec{q};t)=\psi_\alpha(\vec{q}_\alpha)\,e^{-i\tilde{H}_\alpha t}f_\alpha^+(\vec{p}_\alpha),\quad \alpha=1,2,3, \tag{3.7}$$

$$\Phi_0^+(\vec{p},\vec{q};t)=e^{-iH_0t}f_0^+(\vec{p},\vec{q})\ , \tag{3.8}$$

where H_0 is the Hamiltonian, $H_0=\tilde{p}_\alpha^{\,2}+\tilde{q}_\alpha^{\,2}$. The outgoing wave packets f_α^+ are related to f_β^- by the S matrix,

$$f_\alpha^+=S_{\alpha\beta}f_\beta^-,\quad \alpha=0,1,2,3\ . \tag{3.9}$$

All of the above statements have been given a rigorous proof by Faddeev.[1] To find either the elastic or rearrangement time delays we have to calculate the mean position of the outgoing wave packet in a cone. The operator $\vec{x}_\alpha$, whose matrix elements give the separation of cluster $\beta\gamma$ and particle α, is

$$\vec{x}_\alpha=\frac{i}{2}\,\vec{\nabla}_{\vec{p}_\alpha}\ . \tag{3.10}$$

We denote by $C_\alpha(\hat{n}_\alpha,\lambda)$ the cone that is related to $\vec{x}_\alpha$. This cone is the set of vectors $\vec{p}_\alpha$ satisfying

$$C_\alpha(\hat{n}_\alpha,\lambda)=\{\vec{p}_\alpha;\vec{p}_\alpha\cdot\hat{n}_\alpha>\lambda|\vec{p}_\alpha|\},\quad 0\le\lambda\le1\ . \tag{3.11}$$

The mean position $\vec{x}_{\alpha\beta}(t)$ for the state evolving from f_β^- is given by the expression

$$\langle x_{\alpha\beta}(t)\rangle_{C_\alpha}=\frac{\int d^3q_\alpha\int_{C_\alpha}d^3p_\alpha\,P_\alpha\Psi_\beta^*(\vec{p}_\alpha,\vec{q}_\alpha;t)\,\vec{x}_\alpha\cdot\hat{n}_\alpha\,P_\alpha\Psi_\beta(\vec{p}_\alpha,\vec{q}_\alpha;t)}{\int d^3q_\alpha\int_{C_\alpha}d^3p_\alpha|P_\alpha\Psi_\beta(\vec{p}_\alpha,\vec{q}_\alpha;t)|^2}\ . \tag{3.12}$$

Following the procedure used in Sec. II we shall also need

$$\langle x_{\alpha\beta}^f(t)\rangle_{C_\alpha}=\frac{\int d^3q_\alpha\int_{C_\alpha}d^3p_\alpha\,\Phi_\alpha^{+*}(\vec{p}_\alpha,\vec{q}_\alpha;t)\,\vec{x}_\alpha\cdot\hat{n}_\alpha\,\Phi_\alpha^+(\vec{p}_\alpha,\vec{q}_\alpha;t)}{\int d^3q_\alpha\int_{C_\alpha}d^3p_\alpha|\Phi_\alpha^+(\vec{p}_\alpha,\vec{q}_\alpha;t)|^2}\ . \tag{3.13}$$

In this case our fundamental ansatz is that

$$\lim_{t\to+\infty}[\langle x_{\alpha\beta}(t)\rangle_C-\langle x_{\alpha\beta}^f(t)\rangle_C]=0\ . \tag{3.14}$$

As in the two-body case, sufficient conditions for the validity of this ansatz are that the limits in Eq. (3.5) vanish faster than t^{-1}. We also assume that our wave packet f_β^- satisfies the properties

P1–P3. A few changes need to be incorporated so that these properties apply to the multichannel case. For example, the position $\vec{x}_\beta$ of the noninteracting incident wave is given by (2.10) again, with f^- replaced by f_β^-. For convenience we will also assume that f_β^- is real. It is straightforward to add the terms arising from a complex f_β^-. However, the effect of P2 is to argue that in the plane-wave limit these terms vanish.

Let us complete the computation of $\langle x_{\alpha\beta}^f(t)\rangle_C$. Using representation (3.7) of $\Phi_\alpha^+(t)$ allows us to perform the $\vec{q}_\alpha$ integrations. Thus $x_{\alpha\beta}^f(t)$ may be simplified to read

$$\langle x_{\alpha\beta}^f(t)\rangle_{C_\alpha} = \frac{\int_{C_\alpha} d^3p_\alpha\, \phi_\alpha^{+*}(\vec{p}_\alpha, t)\,\vec{x}_\alpha\cdot\hat{n}_\alpha\,\phi_\alpha^+(\vec{p}_\alpha, t)}{\int_{C_\alpha} d^3p_\alpha |\phi_\alpha^+(\vec{p}_\alpha, t)|^2}, \tag{3.15}$$

where

$$\phi_\alpha^+(\vec{p}_\alpha, t) = e^{-i(\tilde{p}_\alpha^2-\chi_\alpha^2)t} f_\alpha^+(\vec{p}_\alpha) . \tag{3.16}$$

We next introduce a reduced s matrix, defined by[25]

$$\langle\vec{p}_\alpha|S_{\alpha\beta}|\vec{p}_\beta'\rangle = \frac{\delta(E-E')}{(n_\alpha p_\alpha n_\beta p_\beta')^{1/2}}\langle\hat{p}_\alpha|s_{\alpha\beta}(E)|\hat{p}_\beta'\rangle , \tag{3.17}$$

where $E=\tilde{p}_\alpha^{\,2}-\chi_\alpha^{\,2}$, $E'=\tilde{p}_\beta'^{\,2}-\chi_\beta^{\,2}$. This leads then, together with Eq. (3.9), to the following representation for $\phi_\alpha^+(\vec{p}_\alpha, t)$:

$$\phi_\alpha^+(\vec{p}_\alpha, t) = e^{-iEt}\left(\frac{n_\beta p_\beta}{n_\alpha p_\alpha}\right)^{1/2} \times\int d\hat{p}_\beta'\langle\hat{p}_\alpha|s_{\alpha\beta}(E)|\hat{p}_\beta'\rangle f_\beta^-(p_\beta, \hat{p}_\beta') , \tag{3.18}$$

with $\tilde{p}_\beta^{\,2}=E+\chi_\beta^{\,2}=\tilde{p}_\alpha^{\,2}-\chi_\alpha^{\,2}+\chi_\beta^{\,2}$. We now use the analog of Eq. (2.13). So we have to calculate the energy derivative of the argument,

$$\frac{\partial}{\partial E}\arg\phi_\alpha^+(\vec{p}_\alpha, t) = -t+\frac{\partial}{\partial E}\arg\int d\hat{p}_\beta'\langle\hat{p}_\alpha|s_{\alpha\beta}(E)|\hat{p}_\beta'\rangle \times f_\beta^-(p_\beta, \hat{p}_\beta') . \tag{3.19}$$

We observe that the presence of the factor $(n_\beta p_\beta/n_\alpha p_\alpha)^{1/2}$ in Eq. (3.18) does not modify the above expression because this factor is real and does not alter the phase. Thus we are lead to

$$\frac{i}{2}\phi_\alpha^{+*}(\vec{p}_\alpha, t)\,\vec{\nabla}_{\vec{p}_\alpha}\cdot\hat{n}_\alpha\,\phi_\alpha^+(\vec{p}_\alpha, t) = |\phi_\alpha^+(\vec{p}_\alpha, t)|^2\left[t-\frac{\partial}{\partial E_i}\arg\langle\hat{p}_\alpha|s_{\alpha\beta}(E_i)|\hat{p}_\beta^i\rangle\right], \tag{3.20}$$

where $\hat{p}_\beta^i$ is the direction of collimation of the incident wave. Defining the mean velocity in the cone C_α to be

$$\langle v_\alpha\rangle_{C_\alpha} = \frac{\int_{C_\alpha} d^3p_\alpha(p_\alpha/n_\alpha)|f_\alpha^+(\vec{p}_\alpha)|^2}{\int_{C_\alpha} d^3p_\alpha|f_\alpha^+(\vec{p}_\alpha)|^2}, \tag{3.21}$$

we have then

$$\langle x_{\alpha\beta}(t)\rangle_{C_\alpha} = \langle v_\alpha\rangle_{C_\alpha}\left[t-\frac{\partial}{\partial E_i}\arg\langle\hat{p}_\alpha|s_{\alpha\beta}(E_i)|\hat{p}_\beta^i\rangle\right]. \tag{3.22}$$

This completes the derivation for elastic and rearrangement scattering.

The remaining case to be treated is the breakup scattering. The final state of three free particles in the center-of-mass coordinate system has five degrees of freedom after energy conservation is taken into account. We shall introduce a six-dimensional spherical coordinate system to describe this final state. Let E be the total center-of-mass kinetic energy

$$E=\tilde{p}^2+\tilde{q}^2\equiv\frac{p_0^2}{2m_0}, \qquad m_0=\left(\frac{m_1m_2m_3}{m_1+m_2+m_3}\right)^{1/2} = (n_\alpha\mu_\alpha)^{1/2}, \tag{3.23}$$

where m_1, m_2, m_3 are the individual particle masses. Equation (3.23) defines a momentum p_0 that is related only to the kinetic energy. In terms of p_0, the momenta p_α and q_α are

$$p_\alpha=\left(\frac{n_\alpha}{m_0}\right)^{1/2}p_0\cos\omega_\alpha,\quad 0\leqslant\omega_\alpha\leqslant\frac{\pi}{2} \tag{3.24}$$

$$q_\alpha=\left(\frac{\mu_\alpha}{m_0}\right)^{1/2}p_0\sin\omega_\alpha, \tag{3.25}$$

where the ratio p_α/q_α may be used to define the angle ω_α. We shall further represent the pair of vectors $\{\vec{p}_\alpha, \vec{q}_\alpha\}$ by $\vec{p}_0$. Associated with $\vec{p}_0$ is the six-dimensional gradient $\nabla_{\vec{p}_0}$ and the radial component is $\hat{n}_0\cdot\nabla_{\vec{p}_0}=\partial/\partial p_0$, where $\hat{n}_0$ is the unit vector in six dimensions. The canonically conjugate variable to p_0 is ρ_0, which is given by the expression

$$\rho_0=(2m_0)^{-1/2}(2n_\alpha x_\alpha^2+2\mu_\alpha y_\alpha^2)^{1/2},\quad \alpha=1,2,3 . \tag{3.26}$$

This coordinate ρ_0 is independent of the Jacobi coordinate system chosen. It may also be written

$$\rho_0=(2m_0)^{-1/2}(2m_1r_1^2+2m_2r_2^2+2m_3r_3^2)^{1/2}, \tag{3.27}$$

where $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the individual position vectors of particle 1, 2, 3 in the center-of-mass coordinate system, respectively.

The cone restriction in momentum space is now the set

$$C_0(\hat{n}_0, \lambda) = \{\vec{p}_0 : \vec{p}_0 \cdot \hat{n}_0 > \lambda p_0\}, \quad 0 \le \lambda \le 1 . \tag{3.28}$$

The position operator of ρ_0 in the cone has the form

$$\rho_0 = \frac{i}{2} \hat{n}_0 \cdot \vec{\nabla}_{\vec{p}_0} . \tag{3.29}$$

We must calculate the mean position of ρ_0 in C_0. The ansatz (3.14) means it is sufficient to compute

$$\langle x_{0\beta}^f(t)\rangle_{C_0} = \frac{\int_{C_0} d^6 p_0 \Phi_0^{+*}(\vec{p}_0, t) \frac{1}{2} i \hat{n}_0 \cdot \vec{\nabla}_{\vec{p}_0} \Phi_0^+(\vec{p}_0, t)}{\int_{C_0} d^6 p_0 |\Phi_0^+(\vec{p}_0, t)|^2} . \tag{3.30}$$

The outgoing breakup wave packet may be represented with the help of the S matrix

$$\Phi_0^+(\vec{p}_0, t) = \int d^3 p_\beta' \langle \vec{p}_0 | S_{0\beta} | \vec{p}_\beta' \rangle \times e^{-i(\vec{p}_\beta'^2 - \chi_\beta^2)t} f_\beta^-(\vec{p}_\beta') . \tag{3.31}$$

Introducing the reduced breakup s matrix defined by[29]

$$\langle \vec{p}_0 | S_{0\beta} | \vec{p}_\beta' \rangle = \frac{\delta(E - E')}{(m_0 p_0^4)^{1/2} (n_\beta p_\beta')^{1/2}} \times \langle \hat{p}_0 | s_{0\beta}(E) | \hat{p}_\beta' \rangle , \tag{3.32}$$

Eq. (3.31) becomes

$$\Phi_0^+(\vec{p}_0, t) = \left(\frac{n_\beta p_\beta^f}{m_0 p_0^4}\right)^{1/2} e^{-iEt} \times \int d\hat{p}_\beta' \langle \hat{p}_0 | s_{0\beta} | \hat{p}_\beta' \rangle f_\beta^-(p_\beta, \hat{p}_\beta') , \tag{3.33}$$

where

$$p_\beta^f = [2n_\beta(E + \chi_\beta^2)]^{1/2} . \tag{3.34}$$

Inserting Eq. (3.33) into Eq. (3.30) gives the result

$$\langle x_{0\beta}(t)\rangle_{C_0} = \left\langle \frac{p_0}{m_0} \right\rangle_{C_0} \left[t - \frac{\partial}{\partial E_i} \langle \hat{p}_0 | s_{0\beta}(E_i) | \hat{p}_\beta^i \rangle \right] , \tag{3.35}$$

with the average radial velocity in the cone C_0 given by

$$\left\langle \frac{p_0}{m_0} \right\rangle_{C_0} = \frac{\int_{C_0} d^6 p_0 (p_0/m_0) |f_0^+(\vec{p}_0)|^2}{\int_{C_0} d^6 p_0 |f_0^+(\vec{p}_0)|^2} . \tag{3.36}$$

It is appropriate to comment here on the generality of these results and the method of analysis. Are the type of results given in formulas (3.22) and (3.35) valid in the four- and N-body problem? It is obvious that they are. The two basic ingredients of our derivation are the convergence of the asymptotic outgoing wave to the exact wave given by Eq. (3.5) and the representations (3.18) and (3.33) of the asymptotic waves in terms of the channel S matrices. These general features should be a part of any rigorous N-body scattering theory.

IV. GLOBAL TIME DELAY

In this section we discuss global time delay in three-body scattering. We emphasize the definition of this concept and the corresponding solution. The elaborate mathematical analysis needed to obtain the solution is found in an earlier paper.[25] Here we shall also derive the multichannel equivalent of connection statement (1.8). This establishes the mutual interdependence and self-consistency of the angular time-delay results obtained in Sec. III and the global time-delay results presented in this section.

Consider the exact three-body wave packet $\Psi_\beta(t)$ that evolves from the asymptotic wave packet $\Phi_\beta^-(t)$ given by Eq. (3.1). The function $\Psi_\beta(t)$ is defined by Eq. (3.4) and is known[26] to have the form

$$\Psi_\beta(t) = e^{-iHt} U_\beta^{(-)} f_\beta . \tag{4.1}$$

In this formula and subsequent ones we drop the superscript minus that appeared on f_β in Sec. III. The operator $U_\beta^{(-)}$ is the Møller operator defined in Faddeev's work. The momentum-space matrix elements of $U_\beta^{(-)}$, $\langle \vec{p}\vec{q} | U_\beta^{(-)} | \vec{p}_\beta' \rangle$, give the exact time-independent wave function solution for a plane wave of momentum $\vec{p}_\beta'$ incident in the β channel. This wave function satisfies the outgoing radiation condition in all final channels.[28] The wave functions $U_\beta^{(-)} f_\beta$ and $\Psi_\beta(t)$ are both members of the Hilbert space with six momentum or coordinate degrees of freedom.

Let us define a six-dimensional sphere in the $(\vec{x}_\alpha, \vec{y}_\alpha)$ space. An invariant radial distance, ρ_0, is specified by Eq. (3.26). Thus for any function $f(\vec{x}, \vec{y})$ we can define a projection operator onto the sphere of radius R by

$$\mathcal{P}(R) f(\vec{x}_\alpha, \vec{y}_\alpha) = \begin{cases} f(\vec{x}_\alpha, \vec{y}_\alpha) & \text{if } \rho_0 \le R , \\ 0 & \text{if } \rho_0 > R . \end{cases} \tag{4.2}$$

The matrix element $(\Psi_\beta(t), \mathcal{P}(R)\Psi_\beta(t))$ gives the probability that the state $\Psi_\beta(t)$ is inside the sphere at time t. The integral

$$\int_{-t_0}^{t_0} (\Psi_\beta(t), \mathcal{P}(R)\Psi_\beta(t))\, dt \tag{4.3}$$

has the meaning of the fraction of time between $-t_0$ and t_0 that the state Ψ_β spends inside the sphere. At this stage we refrain from letting $t_0 \to \infty$, since it is likely that (4.3) will be infinite.

If we write down the same quantity for the asymptotic solutions in the absence of the intercluster potentials, we have the integral

$$\int_{-t_0}^{t_0} (\Phi_\beta(t), \mathcal{P}(R)\Phi_\beta(t))\,dt\ . \tag{4.4}$$

The difference of these two integrals in the compound limit as $t_0 \to \infty$ followed by $R \to \infty$ defines our time delay for channel β. For each value of β and incident wave packet f_β a related time delay is defined.

We now must develop a technique which allows us to compute these time delays. We do this by embedding this problem in a larger mathematical problem. In this larger problem we find a natural Hermitian operator that is related to the time delay defined above. The larger problem is suggested by treating the matrix elements as though they were transition probabilities rather than observable amplitudes. For example, related to the integral in (4.4) one writes

$$T^F_{\alpha\beta}(R, t_0) = \delta_{\alpha\beta}\int_{-t_0}^{t_0} (f'_\beta, e^{i\tilde{H}_\beta t}\mathcal{P}_\beta(R)\, e^{-i\tilde{H}_\beta t} f_\beta)\,dt\ , \tag{4.5}$$

where $\mathcal{P}_\beta$ is the two-body-like channel projection operator that is given by the kernel

$$\langle\vec{p}_\beta|\mathcal{P}_\beta(R)|\vec{p}'_\beta\rangle = \int \psi^*_\beta(\vec{q}_\beta)\langle\vec{p}_\beta\vec{q}_\beta|\mathcal{P}(R)|\vec{p}'_\beta\vec{q}'_\beta\rangle \times\psi_\beta(\vec{q}'_\beta)\,d^3q_\beta\,d^3q'_\beta\ . \tag{4.6}$$

When $\alpha=\beta$ and $f'_\beta = f_\beta$, then $T^F_{\alpha\beta}(R, t_0)$ is equal to the integral (4.4). For the case of the interacting waves one has the quantity $T^E_{\alpha\beta}(R, t_0)$,

$$T^E_{\alpha\beta}(R, t_0) = \int_{-t_0}^{t_0} (f'_\alpha, e^{i\tilde{H}_\alpha t}\, U^{(-)\dagger}_\alpha \mathcal{P}(R)\, U^{(-)}_\beta e^{-i\tilde{H}_\beta t} f_\beta)\,dt. \tag{4.7}$$

This quantity is equal to the integral (4.3) when $\alpha=\beta$ and $f'_\alpha = f_\beta$. Thus computing the difference $T^E_{\alpha\beta}(R, t_0) - T^F_{\alpha\beta}(R, t_0)$ will solve the global time-delay problem. We denote this difference by $(f'_\alpha, Q_{\alpha\beta}(R, t_0) f_\beta)$. The operator notation for $Q_{\alpha\beta}(R, t_0)$ is justified since the difference $T^E_{\alpha\beta}(R, t_0) - T^F_{\alpha\beta}(R, t_0)$ is a bilinear functional of f'_α and f_β. So we get for the operator form of $Q_{\alpha\beta}(R, t_0)$

$$Q_{\alpha\beta}(R, t_0) = \int_{-t_0}^{t_0} e^{i\tilde{H}_\alpha t}\,[U^{(-)\dagger}_\alpha \mathcal{P}(R)\, U^{(-)}_\beta - \delta_{\alpha\beta}\mathcal{P}(R)] \times e^{-i\tilde{H}_\beta t}\,dt\ . \tag{4.8}$$

We then define $Q_{\alpha\beta}$ as the weak limit

$$Q_{\alpha\beta} = \lim_{R\to\infty}\ \lim_{t_0\to\infty} Q_{\alpha\beta}(R, t_0)\ . \tag{4.9}$$

The operator $Q_{\alpha\beta}$ has a number of properties closely related to the three-body S matrix, viz.

$$\tilde{H}_\alpha Q_{\alpha\beta} = Q_{\alpha\beta}\tilde{H}_\beta,\quad Q^\dagger_{\alpha\beta} = Q_{\beta\alpha}\ . \tag{4.10}$$

The first property means $Q_{\alpha\beta}$ is diagonal in energy, the second property is closely related to the unitarity of the S matrix. Because of the first property one may introduce the diagonal representation

$$\langle\vec{p}_\alpha|Q_{\alpha\beta}|\vec{p}'_\beta\rangle = \frac{\delta(E-E')}{(n_\alpha p_\alpha n_\beta p'_\beta)^{1/2}}\langle\hat{p}_\alpha|q_{\alpha\beta}(E)|\hat{p}'_\beta\rangle\ ,\quad \alpha>0\ . \tag{4.11}$$

The solution of the global time-delay problem found in Ref. 25 is then

$$q_{\alpha\beta}(E) = -i\sum_{\gamma=0}^{3} s^\dagger_{\gamma\alpha}(E)\,\frac{d}{dE}\,s_{\gamma\beta}(E)\ . \tag{4.12}$$

Our discussion above means that we interpret only $q_{\alpha\alpha}(E)$ as being related to observable time delays. Naturally, because of property (4.10) all diagonal inner products of $q_{\alpha\alpha}(E)$ are real—as they must be for an observable phenomenon. The matrix elements of $q_{\alpha\beta}(E)$ are, however, complex and as far as we are aware are not related to any observable phenomenon. The only calculation these matrix elements could enter would be in computing the expectation value of $Q_{\alpha\beta}$ for an initial state that is a coherent superposition of several incoming asymptotic channel wave functions. Although it is easy to write down such a state mathematically, we cannot see how such an initial state could be prepared in a scattering experiment.

We investigate now the connection between global and angular time delay. As we have shown in formulas (3.22) and (3.35), the angular time delay may be written

$$\langle\hat{p}_\alpha|\Delta_{\alpha\beta}(E)|\hat{p}'_\beta\rangle = \frac{\partial}{\partial E}\arg\langle\hat{p}_\alpha|s_{\alpha\beta}(E)|\hat{p}'_\beta\rangle\ ,\quad \alpha=0,1,2,3,\ \beta>0. \tag{4.13}$$

For the elastic scattering channel, $\Delta_{\beta\beta}(E)$, this formula is demonstrated only in the nonforward direction $\hat{p}_\beta \neq \hat{p}'_\beta$. The integral form of the global time delay Eq. (4.12) is

$$\langle\hat{p}_\beta|q_{\beta\beta}(E)|\hat{p}_\beta\rangle = -i\sum_{\gamma=0}^{3}\int d\hat{p}'_\gamma\,\langle\hat{p}'_\gamma|s_{\gamma\beta}(E)|\hat{p}_\beta\rangle^* \times\frac{d}{dE}\,\langle\hat{p}'_\gamma|s_{\gamma\beta}(E)|\hat{p}_\beta\rangle\ . \tag{4.14}$$

Since the matrix element on the left-hand side is real, we may take the real part of Eq. (4.14) without altering the left-hand-side element. So we have [cf. Eq. (1.8)]

$$\langle \hat{p}_\beta | q_{\beta\beta}(E) | \hat{p}_\beta\rangle = \sum_{\gamma=0}^{3} \int d\hat{p}'_\gamma |\langle \hat{p}'_\gamma | s_{\gamma\beta}(E) | \hat{p}_\beta\rangle|^2 \times \langle \hat{p}'_\gamma | \Delta_{\gamma\beta}(E) | \hat{p}_\beta\rangle \ . \quad (4.15)$$

This statement has an obvious physical interpretation. The global time delay for an exact scattering state specified by a plane wave of energy E and incident direction $\hat{p}_\beta$ is seen to be the superposition of the angular time delays summed over all open channels and integrated over all directions. The weighting coefficient of the angular time delay is just the probability of that scattering—i.e., the modulus of the s matrix squared.

A second form of this connection statement may be set up. We may introduce into Eq. (4.15) the differential channel cross sections. Let us break up the right-hand side of Eq. (4.15) into two sets of terms. The first will contain the elastic term $s_{\beta\beta}(E)$; the second set will have all the remaining inelastic terms. These two cases are essentially different because the elastic scattered wave may interfere with itself in the forward direction, but the inelastic channels cannot have this feature.

Consider the $\gamma \neq \beta$ terms first. The differential cross sections may be written in terms of the reduced s matrices as[30]

$$\frac{d\sigma_{\gamma\beta}(\hat{p}'_\gamma)}{d\hat{p}'_\gamma} = \frac{(2\pi)^2}{p_\beta^{\,2}} |\langle \hat{p}'_\gamma | s_{\gamma\beta}(E) | \hat{p}_\beta\rangle|^2 \ , \quad \gamma = 0, 1, 2, 3, \ \gamma \neq \beta \ . \quad (4.16)$$

Thus all the nondiagonal terms have the form

$$\frac{p_\beta^{\,2}}{(2\pi)^2} \sum_{\gamma\neq\beta} \int d\hat{p}'_\gamma \frac{d\sigma_{\gamma\beta}(\hat{p}'_\gamma)}{d\hat{p}'_\gamma} \langle \hat{p}'_\gamma | \Delta_{\gamma\beta}(E) | \hat{p}_\beta\rangle \ . \quad (4.17)$$

In order to find the diagonal matrix element let us employ the scattering amplitude, $f_\beta(E, \hat{p}'_\beta)$, defined by

$$\frac{d\sigma_{\beta\beta}(\hat{p}'_\beta)}{d\hat{p}'_\beta} = |f_\beta(E, \hat{p}'_\beta)|^2 \ , \quad (4.18)$$

$$f_\beta(E, \hat{p}'_\beta) = -2n_\beta 2\pi^2 \mathcal{K}_{\beta\beta}(\vec{p}'_\beta, \vec{p}_\beta; \bar{p}_\beta^{\,2} - \chi_\beta^{\,2} + i0) \ , \quad (4.19)$$

where $\mathcal{K}_{\beta\beta}$ is the on-shell elastic amplitude defined by Faddeev. Using these last two equations to evaluate the elastic term in Eq. (4.15) gives the expression

$$\frac{1}{2\pi}\frac{d}{dE}\mathrm{Re}[p_\beta f_\beta(E, \hat{p}_\beta)] + \frac{1}{(2\pi)^2} p_\beta^{\,2} \int d\hat{p}'_\beta \frac{d\sigma_{\beta\beta}(\hat{p}'_\beta)}{d\hat{p}'_\beta} \frac{d}{dE} \arg f_\beta(E, \hat{p}'_\beta) \ . \quad (4.20)$$

Thus, altogether we are led to

$$\langle \hat{p}_\beta | q_{\beta\beta}(E) | \hat{p}_\beta\rangle = \frac{1}{2\pi}\frac{d}{dE}\mathrm{Re}[p_\beta f_\beta(E, \hat{p}_\beta)] + \frac{p_\beta^{\,2}}{(2\pi)^2}\left[\int d\hat{p}'_\beta \frac{d\sigma_{\beta\beta}(\hat{p}'_\beta)}{d\hat{p}'_\beta} \frac{d}{dE}\arg f_\beta(E, \hat{p}'_\beta) + \sum_{\gamma\neq\beta}\int d\hat{p}'_\gamma \frac{d\sigma_{\gamma\beta}(\hat{p}'_\gamma)}{d\hat{p}'_\gamma} \frac{d}{dE} \arg\langle \hat{p}'_\gamma | s_{\gamma\beta}(E) | \hat{p}_\beta\rangle\right]. \quad (4.21)$$

In this version of the connection statement the angular time delay is weighted by the differential cross section rather than the square modulus of the s matrix. We note that the first term on the right-hand side accounts for the interference in the forward direction between the scattered and unscattered wave. For the two-body case one may restate the connection formula (1.8), in a fashion parallel to that of Eq. (4.21). This has been done by Nussenzveig.[10] The result is just the first two terms on the right-hand side of Eq. (4.21), with all the channel labels removed.

We shall close this paper with a discussion of the observable effects of the theory presented here. The entire treatment of angular time delay has been set up so that it can be directly related to scattering observed by counters. There are, in our opinion, three distinct difficulties which need to be overcome before such a measurement could be practical. We shall comment on each of these in their order of severity. All these difficulties occur both in the two- and three-body problem, so we shall confine this discussion to the simpler two-body case. The first problem is the presence of the wave-packet structure term $(\partial/\partial E)\arg F(p)$ in formula (2.20). If this term is unknown, then in a single scattering observation, one could never obtain a determination of $(\partial/\partial E)\arg\langle \hat{p} | s(E) | \hat{p}_i\rangle$ more accurately than $(\partial/\partial E)\arg F(p)$. It seems likely that in an ensemble of similar plane-wave-like packets the function $(\partial/\partial E)\arg F(p)$ is random in sign. Thus in the average of many observations the effect of $(\partial/\partial E)\arg F(p)$ will average to zero. The next problem is the limitation on observation accuracy imposed by the uncertainty principle. Neglecting

$\arg F(p)$ in (2.20) then

$$\frac{\partial}{\partial E_i} \arg \langle \hat{p} | s(E_i) | \hat{p}_i \rangle = t - \frac{\langle x(t) \rangle_C}{\langle v \rangle_C} . \quad (4.22)$$

The physical meaning of $\langle x(t) \rangle_C$ and $\langle v \rangle_C$ is the average value of position and velocity, respectively, in the scattered wave packet. Assume for the moment that t is known. If one observes the position and the velocity for a large collection of identical wave packets, then the average of the observations converge to $\langle x(t) \rangle_C$ and $\langle v \rangle_C$. For this reason, the uncertainty principle governing $\vec{x}$ and $\vec{v}$ will not limit the accuracy to which $\langle x(t) \rangle_C / \langle v \rangle_C$ can be determined. The principal difficulty concerns determining t. Property P3 defines t as the elapsed time since the moment when the *free* asymptotic incident wave packet is at $\vec{x}=0$. How one could even in principle determine this is not clear to us. Another way of stating this problem is to say that we do not know where the point $\vec{x}=0$ is.

There is one example of observed time-delay phenomena where the problems above are overcome. This is the observation of compound nuclear resonant lifetimes using crystal blocking techniques. These techniques succeed in measuring nuclear reaction times in the range 10^{-16}–10^{-18} seconds. The theoretical study of time delay for a scattering in a crystal has been carried out by Yoshida[31] and Yazaki and Yoshida.[32] The general structure of the crystal blocking problem is quite different from that studied here since the formalism must be appropriate to the compound nuclear resonance model of scattering and the effects of the crystal medium must be explicitly taken into account. The underlying feature of this type of scattering that makes the observations feasible is that one knows the location of the scattering site, and the crystal provides, through the channel mechanism, a precise measure of the distance traveled by the scattered wave.

We believe the significant observable effects of the time-delay theory studied here will come about through indirect mechanisms. The best understood of these is the spectral property given by Eq. (1.9). For example, it is well known that the second virial coefficient for a gas of interacting quantum particles has an integral representation involving the trace of the global time delay.[10] This result can be derived as an immediate consequence of the spectral property. Another consequence of the spectral property is that it implies Levinson's theorem.[14] It is reasonable to expect that similar predictions can be obtained from the three- and N-body global time-delay theory.

ACKNOWLEDGMENTS

Both authors want to thank the theory group at the Stanford Linear Accelerator Center for its hospitality during the final stages of this work. One of us (T. A. O.) is grateful to Professor L. Castillejo, Professor K. Yazaki, and Professor C. Chandler for informative discussions of this problem.

APPENDIX

In this appendix we provide a brief discussion of the convergence features of our fundamental ansatz, Eq. (2.8). Consider the quantities

$$\begin{aligned} N_1(t) &= (\phi^+(t), \vec{x}\cdot\hat{n}\,\phi^+(t))_C , \\ N_2(t) &= (\psi(t), \vec{x}\cdot\hat{n}\,\psi(t))_C \\ D_1(t) &= \|\phi^+(t)\|_C{}^2 \\ &= \|f^+\|_C{}^2 , \\ D_2(t) &= \|\psi(t)\|_C{}^2 \end{aligned} \quad (A1)$$

where all the functions are those defined in Sec. II. The subscript C on the inner product indicates a restricted domain of integration given by the cone $C(\hat{n}, \lambda)$. The function $D_1(t)$ is a constant and $D_2(t)$ converges to $D_1(t)$ by Eq. (2.3). With this notation the ansatz reads

$$\langle x(t) \rangle_C - \langle x^f(t) \rangle_C \to 0 \text{ as } t \to +\infty , \quad (A2)$$

where

$$\begin{aligned} \langle x(t) \rangle_C &= N_2(t)/D_2(t) , \\ \langle x^f(t) \rangle_C &= N_1(t)/D_1(t) . \end{aligned} \quad (A3)$$

Since $D_1(t)$ and $D_2(t)$ converge to constants we can state the convergence problem without loss of generality as proving that

$$D_2 D_1 \left(\frac{N_1}{D_1} - \frac{N_2}{D_2} \right) = N_1(D_2 - D_1) + D_1(N_1 - N_2) \quad (A4)$$

vanishes for $t \to +\infty$. We will investigate this convergence under reasonable physical assumptions. First we note that the results of Sec. II imply that

$$\|\vec{x}\cdot\hat{n}\,\phi^+(t)\|_C < A_1 t , \quad (A5)$$

with A_1 some positive constant and t sufficiently large. We also expect a similar result holds for the wave $\psi(t)$. So we make the assumption that for sufficiently large t

$$\|\vec{x}\cdot\hat{n}\,\psi(t)\|_C < A_2 t, \quad A_2 > 0 . \quad (A6)$$

Inspecting relation (A4) we see that the term with $D_2 - D_1$ is multiplied by $N_1(t)$ which has the behavior $A_3 t, A_3 > 0$. Let

$$\Delta(t) = \phi^+(t) - \psi(t) \quad (A7)$$

be the difference between the free outgoing wave and the exact one. Then we have

$$\delta(t)=\|\Delta(t)\|\geq\|\Delta(t)\|_C\geq\big|\,\|\psi(t)\|_C-\|\phi^+(t)\|_C\big|$$
$$\geq|\sqrt{D_2}-\sqrt{D_1}|=\frac{|D_2-D_1|}{\sqrt{D_1}+\sqrt{D_2}}. \tag{A8}$$

The first term in Eq. (A4) has the estimate

$$|N_1(D_2-D_1)|\leq A_3 t(\sqrt{D_1}+\sqrt{D_2})\,\delta(t). \tag{A9}$$

Clearly, if

$$\delta(t)<\delta_0 t^{-1-\epsilon},\quad \epsilon>0,\ \delta_0=\text{const}>0 \tag{A10}$$

then

$$|N_1(D_2-D_1)|\leq A_3\,\delta_0(\sqrt{D_1}+\sqrt{D_2})\,t^{-\epsilon}\to 0,\quad t\to\infty. \tag{A11}$$

We still have to consider the second term in Eq. (A4). From the definitions (A1) we are lead to

$$\begin{aligned}|N_1(t)-N_2(t)|&=|(\Delta(t),\vec{x}\cdot\hat{n}\,\Delta(t))_C\\&\quad-2\,\mathrm{Re}(\Delta(t),\vec{x}\cdot\hat{n}\,\phi^+(t))_C|\\&\leq\|\Delta(t)\|_C\,[\,\|\vec{x}\cdot\hat{n}\,\Delta(t)\|_C\\&\qquad+2\|\vec{x}\cdot\hat{n}\,\phi^+(t)\|_C]\\&\leq\delta(t)\,[\,\|\vec{x}\cdot\hat{n}\,\psi(t)\|_C+3\|\vec{x}\cdot\hat{n}\,\phi^+(t)\|_C]\\&\leq\delta(t)\,(A_2+3A_1)\,t\to 0\ \text{ as } t\to\infty.\end{aligned} \tag{A12}$$

So again the estimate (A10) is strong enough to show the second term vanishes.

Our conclusion is that if the reasonable assumption (A6) is valid, then the rate of convergence (A10) is sufficient to prove our ansatz (A2).

In the remaining portion of this section we shall compute the restriction on the potential necessary to ensure that the rate of convergence (A10) will be satisfied. Consider two sets of functions in $L^2(\vec{x})$. The first shall be defined as those functions whose momentum-space Fourier transforms vanish in both some neighborhood of the origin and some neighborhood of infinity. The second set is the Schwartz space of C^∞ functions of fast decrease.[33] This intersection is a dense set in the L^2 Hilbert space. For $f(\vec{x})$ in this intersection Donaldson, Gibson, and Hersh[34] established the estimate

$$|e^{iH_0t}f(\vec{x})|\leq\begin{cases}C\,t^{-3(1+\epsilon)/2}|\vec{x}|^{3\epsilon}, & |\vec{x}|>1\\ C\,t^{-3(1+\epsilon)/2}, & |\vec{x}|\leq 1.\end{cases} \tag{A13}$$

Now with this information the rate of convergence may be determined from the well-known bound[35]

$$\begin{aligned}\delta(t)&=\|(\Omega^{(-)}-e^{itH}e^{-itH_0})f^+\|\\&\leq\int_t^\infty\|(H-H_0)\,e^{-it'H_0}f^+\|\,dt'.\end{aligned} \tag{A14}$$

We may estimate the integrand of the expression on the right-hand side of (A14) by substituting the relation (A13) to obtain

$$\begin{aligned}\|(H-H_0)\,e^{-itH_0}f^+\|&\leq\left[\int|v(\vec{x})\,e^{-itH_0}f(\vec{x})|^2\,d^3x\right]^{1/2}\\&\leq C_1\,t^{-3(1+\epsilon)/2},\end{aligned} \tag{A15}$$

where the constant is

$$C_1^{\,2}=C^2\int_{|\vec{x}|\leq 1}|v(\vec{x})|^2\,d^3x+C^2\int_{|\vec{x}|>1}|\vec{x}|^{3\epsilon}\,|v(\vec{x})|^2\,d^3x. \tag{A16}$$

Carrying out the integration over t in (A14) gives us

$$\|(\Omega^{(-)}-e^{itH}e^{-itH_0})f^+\|\leq\frac{2C_1}{1+3\epsilon}\,t^{-1/2-3\epsilon/2}. \tag{A17}$$

The above inequality tells us how $\delta(t)$ will vanish. If we require $\delta(t)$ go to zero faster than t^{-1}, then we need a potential such that $\epsilon>\frac{1}{3}$. From (A16) we see that if the potential falls off faster than x^{-2}, then we may have values of $\epsilon>\frac{1}{3}$ such that C_1 remains finite.

---

*Work supported in part by the U. S. Energy Research and Development Administration.

†Onderzoeker IIKW Belgium.

‡On leave from Institute for Theoretical Physics, University of Leuven, B-3030 Heverlee, Belgium.

[1]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem* (Davey, New York, 1965).

[2]F. T. Smith, Phys. Rev. 118, 349 (1960).

[3]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 485.

[4]J. M. Jauch and J. P. Marchand, Helv. Phys. Acta 40, 217 (1967).

[5]D. Bollé and T. A. Osborn, Phys. Rev. D 11, 3417 (1975).

[6]L. E. Eisenbud, Ph.D. thesis, Princeton Univ., 1948 (unpublished); E. P. Wigner, Phys. Rev. 98, 145 (1955).

[7]W. Brenig and R. Haag, Fortschr. Phys. 7, 183 (1959).

[8]M. Froissart, M. L. Goldberger, and K. M. Watson, Phys. Rev. 131, 2820 (1963).

[9]F. Goldrich and E. P. Wigner, in *Magic without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday*, edited by John R. Klauder (Freeman, San Francisco, 1972), p. 147.

[10]H. M. Nussenzveig, Phys. Rev. D 6, 1534 (1972).

[11]M. G. Krein, Mat. Sb. 33, 597 (1953); M. S. Birman and M. G. Krein, Dokl. Acad. Nauk SSSR 144, 268 (1962) [Sov. Math. Dokl. 3, 707 (1962)].

[12]V. S. Buslaev, in *Spectral Theory and Wave Process*, edited by M. S. Birman (Consultants Bureau, London, 1972).

[13]J. M. Jauch, K. B. Sinha, and B. N. Misra, Helv. Phys. Acta 45, 398 (1972).

[14]T. Y. Tsang and T. A. Osborn, Nucl. Phys. A247, 43 (1975).

[15]F. T. Smith, J. Chem. Phys. 38, 1304 (1963); Phys. Rev. 131, 2803 (1963); J. Chem. Phys. 57, 5562 (1972).

[16]R. Dashen, S. Ma, and H. Bernstein, Phys. Rev. 187, 345 (1969).

[17]F. T. Smith, Phys. Rev. 130, 394 (1963).

[18]U. Bar-Gadda, Physica 62, 321 (1972).

[19]Ph. A. Martin, Ecole Polytechnique Federale report, Lausanne, Switzerland, 1975 (unpublished).

[20]R. Jackiw and G. Woo, Phys. Rev. D 12, 1643 (1975).

[21]R. A. van Santen, J. Chem. Phys. 57, 5571 (1972).

[22]J. Nowakowski and T. A. Osborn, Nucl. Phys. A249, 301 (1975).

[23]R. Fong, Phys. Rev. 140, B762 (1965).

[24]R. H. Dalitz and R. G. Moorehouse, Proc. R. Soc. London A318, 279 (1970).

[25]T. A. Osborn and D. Bollé, Phys. Lett. 52B, 13 (1974); J. Math. Phys. 16, 1533 (1975).

[26]Ref. 1, Chap. 10.

[27]J. D. Dollard, Commun. Math. Phys. 12, 193 (1969).

[28]T. A. Osborn and D. Bollé, Phys. Rev. C 8, 1198 (1973).

[29]The reduced S-matrix element used in (3.32) differs somewhat from the one used in Ref. 25. The transformation between the two is given by

$$\langle \hat{p}_0 | s_{0\beta}(E) | \hat{p}'_\beta \rangle = \frac{\langle \omega_\beta \hat{p}_\beta \hat{q}_\beta | s_{0\beta}(E) | \hat{p}'_\beta \rangle}{\cos\omega_\beta \sin\omega_\beta} .$$

We prefer the left-hand representation of $s_{0\beta}$ since this representation is independent of the Jacobi coordinate system used to describe the free three-particle system.

[30]In Ref. 28 we give a general formula for the breakup cross section in Eq. (3.15). The relation between that breakup cross section and the one employed here is

$$\frac{d\sigma}{d\hat{p}_0} = \frac{m_0 p_0{}^4}{n_\beta p_\beta} \frac{d\sigma}{d\hat{p}_\beta d\vec{q}_\beta} .$$

[31]S. Yoshida, Phys. Lett. 45B, 324 (1973).

[32]K. Yazaki and S. Yoshida, Nucl. Phys. A232, 249 (1974).

[33]L. Schwartz, *Theorie des Distributions* (Hermann, Paris, 1951), Vol. II.

[34]J. A. Donaldson, A. G. Gibson, and R. Hersh, J. Funct. Anal. 14, 131 (1973)

[35]T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966), p. 533.