Ghost problem of quantum field theories with higher derivatives*

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Second-order theories, i.e., theories described by Lagrangians quadratic in second derivatives of the fields, are carefully examined and their ghost problems isolated and clearly exhibited. In particular, theories with gauge symmetry were shown to have precisely the same ghost problems as theories without gauge symmetry. It is also shown that massless theories of the same nature are the limit of massive theories containing ghost states.

The canonical quantization of field theories described by Lagrangians quadratic in second derivatives of the fields has been studied by several people.[1] Here we present a different but simpler treatment for these kinds of theories by using the path-integral method. We are motivated by the recently suggested gravitation theory[2] in which the square of the Riemann tensor enters in the Lagrangian rather than the first power of the curvature scalar, where the canonical quantization is difficult to implement. The immediate result is that the propagator in this theory has a double pole at $k^2=0$ suggesting the appearance of a ghost in the Hilbert space and signifying that either (a) unitarity or (b) positivity of the energy spectrum might be violated.

A careful treatment of such a theory seems to be desirable, in particular with regard to the massless limit. Although our treatment could easily be extended to theories with higher derivatives, i.e., theories which are described by quadratic forms such as $\Phi\sum_{\eta=0}^{N} a_\eta \Box^\eta \Phi$, we will restrict ourselves to the more specific form of $\Phi\Box(\Box-m^2)\Phi$ and we will also be particularly interested in the $m^2\to 0$ limit of such forms.

It is our purpose to quantize such a theory using the path-integral formalism.[1] In addition, theories with gauge symmetry will be examined in particular within the context of the Faddeev-Popov "orbit volume" prescription for the generating functional in the presence of higher derivatives. We will show that in those more complicated cases where the canonical procedure is difficult to implement, the Feynman rules for a gauge-invariant theory with higher field derivatives in the massless limit are obtained by modifying the action in the obvious way. The structure of the ghost is also shown clearly.

We begin by first examining a second-order scalar theory. The Lagrangian describing such a theory can be taken as

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu\partial_\nu\varphi)(\partial^\mu\partial^\nu\varphi)-\tfrac{1}{2}m^2(\partial_\mu\varphi)(\partial^\mu\varphi)+\mathcal{L}_I(\varphi)\,, \tag{1}$$

where $\mathcal{L}_I(\varphi)$ is an arbitrary function of φ. In particular, the form $\mathcal{L}_I(\varphi)=\frac{1}{4}\lambda\varphi^4$ is sufficient for our purposes.

Examination of the action leads to the Euler-Lagrange equation

$$\partial_\mu\partial_\nu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\varphi)}-\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}+\frac{\delta\mathcal{L}}{\delta\varphi}=0\,, \tag{2}$$

from which the equation of motion is

$$\Box(\Box-m^2)\varphi=\frac{\delta\mathcal{L}_I}{\delta\varphi}. \tag{3}$$

In the usual manner, the generating functional, from which Green's functions are obtained, is defined as

$$W(J)=\int d\varphi\exp\left[i\int(\mathcal{L}+J\varphi)dx\right]. \tag{4}$$

We would like to obtain a theory having the same generating functional but describable in terms of the first-order Lagrangian. We find that the Lagrangian

$$\hat{\mathcal{L}}=-\tfrac{1}{2}(\partial_\mu\varphi_1)^2+\tfrac{1}{2}(\partial_\mu\varphi_2)^2+\tfrac{1}{2}m^2\varphi_2{}^2+\mathcal{L}_I\left(\frac{\varphi_1-\varphi_2}{m}\right), \tag{5}$$

where a massive ghost field, φ_2, appears explicitly and couples to φ_1 through $\mathcal{L}_I$, leads to an equivalent theory.

To show that Eqs. (3) and (5) describe equivalent theories we look at the generating functional determined by Eq. (5).

$$\hat{W}(J_1,J_2)=\int d\varphi_1 d\varphi_2\exp\left[i\int(\hat{\mathcal{L}}+J_1\varphi_1+J_2\varphi_2)dx\right]. \tag{6}$$

That is, apart from an unimportant multiplicative constant factor, one can show that

$$\hat{W}(J_1,J_2)\big|_{J_1=-J_2=J/m}=W(J)\,. \tag{7}$$

Let us define

$$m\varphi=\varphi_1-\varphi_2\,,$$
$$\bar{\varphi}=\varphi_2\,,$$

and take

$$J_1 = -J_2 = J/m .$$

It is then easy to see that the path integration over the field $\bar{\varphi}$ can be done explicitly leading to Eq. (7). Equation (7), therefore, tells us that Eqs. (4) and (6) describe equivalent theories insofar as the Green's functions are identical.

Working with Eq. (6) we can show that $m^2 \to 0$ limit can be taken with impunity. This is because we can write Eq. (6) equivalently as

$$W(J) = \exp\left[i\int \mathcal{L}_I\left(\frac{\delta}{i\delta J}\right)dx\right]\int d\varphi_1 d\varphi_2 \exp\left\{i\int\left[-\tfrac{1}{2}(\partial_\mu\varphi_1)^2 + \tfrac{1}{2}(\partial_\mu\varphi_2)^2 + \tfrac{1}{2}m^2\varphi_2{}^2 + \frac{J}{m}(\varphi_1 - \varphi_2)\right]dx\right\}.$$

Integration over φ_1 and φ_2 can now be done leading to

$$W(J) = \exp\left[i\int \mathcal{L}_I\left(\frac{\delta}{i\delta J}\right)dx\right]\exp\left\{\tfrac{1}{2}\int\int dx\,dy\,\frac{J(x)}{m}[\Delta_F(x-y) - \Delta_F(x-y;m^2)]\frac{J(y)}{m}\right\}. \tag{8}$$

Going to the massless limit is now feasible because

$$\lim_{m^2\to 0}\frac{1}{m^2}[\Delta_F(x) - \Delta_F(x;m^2)] \equiv -\Delta_F'(x) ,$$

where

$$\Delta_F'(x) = \frac{\partial}{\partial m^2}\Delta_F(x;m^2)\,\|_{m^2=0} .$$

Therefore, in this limit Eq. (8) becomes

$$\lim_{m^2\to 0} W(J) = \exp\left[i\int \mathcal{L}_I\left(\frac{\delta}{i\delta J}\right)dx\right]\exp\left[-\tfrac{1}{2}\int\int dx\,dy\,J(x)\Delta_F'(x-y)J(y)\right]. \tag{9}$$

It is very easy to show that if we start with a Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\partial_\nu\varphi)(\partial^\mu\partial^\nu\varphi) + \mathcal{L}_I(\varphi) \tag{10}$$

the Green's functions of this theory are precisely those obtained by the use of the generating functional of Eq. (9).

Therefore, we conclude that a description of theories such as in Eq. (10) could be obtained in terms of the massless limit without encountering any difficulties whatsoever. That is, if we insist that theories that contain an intrinsic propagator with double poles at $k^2 = 0$ be describable by first-order Lagrangians, so that their spectrum content is clear, one would have to follow the prescription that we have outlined so far. Moreover, this prescription does not fail in cases where gauge symmetry enters as will show later.

In addition, Eq. (8) shows us very clearly the existence of the ghost coupled to the current and, therefore, the indefinite-metric nature of the Hilbert space. It can also be shown that the generating functional Eq. (8) is a continous function of m^2 at $m^2 = 0$ and, therefore, the statements made for the massive model apply equally well for the massless theory. The existence of the ghost in the Hilbert space can also be demonstrated by obtaining the canonical commutation rules for the fields φ_1 and φ_2.

This is done by requiring the independence of the functionals

$$N^{-1}\int d\varphi_1 d\varphi_1 \varphi_2 \exp\left(i\int \hat{\mathcal{L}}\,dx\right)$$

and

$$N^{-1}\int d\varphi_1 d\varphi_2 \varphi_2 \exp\left(i\int \hat{\mathcal{L}}dx\right),$$

where

$$N = \int d\varphi_1 d\varphi_2 \exp\left(i\int \hat{\mathcal{L}}\,dx\right),$$

under the transformations

$$\varphi_1 \to \varphi_1 + \delta\omega(x) ,$$
$$\varphi_2 \to \varphi_2$$

and

$$\varphi_1 \to \varphi_1 ,$$
$$\varphi_2 \to \varphi_2 + \delta\omega(x) ,$$

i.e., the variations that lead to the equations of motion for φ_1 and φ_2, respectively. Following Ref. 3, we obtain the following commutation rules:

$$[\varphi_1(x,t), \varphi_1(y,t)] = -i\delta^3(x-y)\,,$$

$$[\varphi_2(x,t), \varphi_2(y,t)] = i\delta^3(x-y)\,.$$

The rest of the commutators are zero. These commutation rules readily yield

$$[\alpha(k), \alpha^\dagger(k')] = 2\omega_k\delta^3(k-k')\,,$$

$$[A(k), A^\dagger(k')] = -2\Omega_k\delta^3(k-k')\,.$$

The interpretation of $\alpha^\dagger$ and $A^\dagger$ as creation operators is fixed by the requirement of energy positivity. Thus, the negative sign in the commutation relation of A implies a negative metric for the massive particle.

As a prototype of a second-order Lagrangian with gauge symmetry, we consider

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu F^{\mu\rho}\partial_\nu F^\nu{}_\rho - \tfrac{1}{4}m^2F^{\mu\nu}F_{\mu\nu} + J_\mu A^\mu\,, \tag{11}$$

where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. We will choose to work in the Feynman gauge $\partial_\mu A^\mu = 0$.

In a way analogous to the scalar theory, we are led to examine the equivalent Lagrangian

$$\hat{\mathcal{L}} = -\tfrac{1}{4}(G_{\mu\nu})^2 + \tfrac{1}{4}(H_{\mu\nu})^2 + \tfrac{1}{2}m^2(H_\mu)^2 + \frac{J_\mu}{m}(G^\mu - H^\mu)\,, \tag{12}$$

where $G_{\mu\nu} = \partial_\nu G_\mu - \partial_\mu G_\nu$, $H_{\mu\nu} = \partial_\nu H_\mu - \partial_\mu H_\nu$, with the gauge condition $\partial_\mu H^\mu = \partial_\mu G^\mu$.

Again, the description is in terms of two fields of which the massive field is a ghost. Now we show that (12) and (11) describe the same theory. To do this, we again define the generating functional for (12),

$$\hat{W} = \int dG_\mu dH_\mu \delta(\partial_\mu G^\mu - \partial_\mu H^\mu) \times \exp\left\{ i\int [\hat{\mathcal{L}} + J_\mu(G^\mu - H^\mu)/m]dx \right\}.$$

As was done earlier, the transformation of the fields

$$G_\mu - H_\mu = mA_\mu\,,$$

$$mH_\mu = \bar{A}_\mu$$

decouples $\bar{A}_\mu$ completely, and enables one to carry out the path integration exactly. This simple calculation gives

$$\hat{W} = \int dA_\mu \delta(\partial_\mu A^\mu) \exp\left\{ i\int [-\tfrac{1}{2}\partial_\mu F^{\mu\rho}\partial_\nu F^\nu{}_\rho - \tfrac{1}{4}m^2F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu]dx \right\},$$

i.e., the generating functional of the theory described by the Lagrangian in Eq. (11).

For completeness we note that

$$\hat{W} = \exp\left[i\int dx\, J_\mu\left(\frac{\delta}{i\delta J_\mu}\right)\right] \exp\left\{ \tfrac{1}{2}\int\int dx\,dy \frac{J^\mu(x)}{m}[D^F_{\mu\nu}(x-y) - D^F_{\mu\nu}(x-y;m^2)]\frac{J^\nu(y)}{m} \right\},$$

where $D^F_{\mu\nu} = g_{\mu\nu}\Delta^F$.

Use of the gauge condition was made in the inversion of the quadratic forms. The massless limit can easily be taken to give a theory described by

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu F^{\mu\rho}\partial_\nu F^\nu{}_\rho + J_\mu A^\mu\,.$$

It is clear from Eq. (12) that the ghost problem of such theories is completely decoupled from the gauge freedom, and, in fact, it can be shown that it persists even if one considers more complicated gauge invariance such as in gravitation theory.[4] Theories with second-order Lagrangians, whether massive or not, fail as far as unitarity is concerned.[5]

In this treatment, we have shown that theories with second-order Lagrangians and gauge symmetry can be handled in the usual way, and that nothing extraordinary emerges. The description in terms of two fields, decoupled in the quadratic form, one of which is a ghost, is consistent with previous treatments. However, their nature emerges in a much more transparent manner and their quantization rules are derived from the path-integral description.

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[5]Regarding the limit of vanishing mass we would like to point out that in the case of gravitation and non-Abelian theories [see H. van Dam and M. Veltman, Nucl. Phys. B22, 397 (1970)] there is a discontinuity as $m \rightarrow 0$. It arises because certain helicity amplitudes which are coupled in a massive theory and which are decoupled in a massless (gauge) theory do not decouple when m tends to zero. In our case even for m nonzero one deals with a gauge theory so that the helicity modes which might cause the discontinuity are decoupled at all stages, both for $m = 0$ and for $m \rightarrow 0$.