PHYSICAL REVIEW D VOLUME 13, NUMBER 10 15 MAY 1976

Green's function theory of unstable particles

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(Received 10 March 1975)

A general form for the propagator $G_{ab}(x, y) = \langle 0| T(\phi_a(x)\phi_b(y))|0\rangle$ of n scalar Hermitian fields based on causality, Poincaré invariance, and the completeness of the set of states having timelike momentum is found, which is consistent with a dynamical model of the fields coupled to currents. This provides a unified theory of stable and unstable particles, which are poles of the matrix on the real axis and on the second sheet, respectively. The pole residues are in general not orthogonal, unless there is a symmetry, so the Fourier transform of the matrix, though having an exponential form, is not a semigroup. The case of symmetry breaking for $n = 2$ is interpreted as PC violation in K decay and is analyzed explicitly.

I. INTRODUCTION

It has been known for a long time that the Wigner-Weisskopf (see for example Refs. 1 and 2) approximation can be applied to describe the law of time evolution in weak nonleptonic decay of the K^0 meson, where CP invariance is violated. This technique is not rigorous in the sense that the physical states are not well defined. A more rigorous mathematical formulation has been given to this and similar problems of the decays of unstable particles by Horwitz and Marchand.[3,4] They have been able to show that the law of time evolution of unstable particles can be approximated by an exponential, a result which agrees with the Wigner-Weisskopf approximation, but they have noticed that the semigroup property of the time evolution, which is a necessary consequence of this approximation, could be violated. This question was investigated later on,[5] and it was found that the semigroup law is indeed violated in a way which does not depend on the weak coupling constant. It is valid up to $\beta^3 \sim 10^{-4}$, where β is the relative size of CP violation.

This result can be applied to nonrelativistic decays. But what about the relativistic case? Here one has to use the framework of field theory in a way which can be applied also to unstable particles. It is clear that one cannot define a field for an unstable particle because of the nonexistence of its asymptotic states. Rather one defines a fundamental field which describes a localized excitation, which produces a spectrum of masses ranging down to a lower limit characteristic of the field type. This is in fact Schwinger's[6,7] point of view of field theory and is different from the conventional one, in which to each particle is assigned a field.[8-10] This point of view is also better for the description of unstable particles, because stable particles can be viewed as poles on the real axis of the Green's function, which is the vacuum expectation value of the time-ordered product of the field. In the same way unstable particles are those poles on the second sheet of the analytically continued[11,12] Green's function. In this sense, stable and unstable particles are treated on the same footing in the Green's function, and if it is known the whole spectrum of masses can be found. This is for the case of one fundamental Hermitian field analyzed by Schwinger.[6]

But still it is possible to define several fundamental fields where the different degrees of freedom correspond to spin or to charge. (Here charge means any sort of charge, which is defined as the generator of some symmetry group, and which is a conserved or almost a conserved quantum number, used to define physical states.) In this case the Green's function has a matrix form and its analytic structure can be found from the structure of the inverse matrix. Similar works were done[13-15] along this line, but the main difference between the approach followed here and the others is first the basic assumption about the existence of fundamental fields used to describe physical phenomena; here we *do not* assign a field to the unstable particle. Moreover, here we do not use a specific dynamical model; the structure of the propagator is found only from the general assumptions of Lorentz invariance, causality, and the existence of a unique vacuum state. In the above-mentioned works, models were used to find the structure of the propagator.

From now on the rest of the proof is rather straightforward. The time evolution is found by taking the Fourier transform of the Green's function. In the pole approximation this gives an exponential law of evolution. But since the pole residues are matrices rather than numbers, the semigroup law of time evolution is violated; namely if $U(t)$ describes the time evolution then $U(t_1)U(t_2) \neq U(t_1+t_2)$, unless there is some symmetry which forces the equality. It seems that the violation for the two-dimensional case is up to β^3 as in the nonrelativistic case. Here it is important to point out that the fact that unstable particles with "complex" mass (i.e., complex poles in the analytically continued propagator) share many formal properties of true stable particles led some authors[16-18] to the assumption that unstable particles form a basis for a nonunitary representation of the Poincaré group with complex mass. This result makes the law of time evolution a semigroup, which is valid only approximately according to the result of this work.

The paper is organized as follows: In Sec. II we give a brief review of Schwinger's[6] paper but in a slightly different form, which makes the proofs simpler. In Sec. III we work out the case of n Hermitian coupled scalar fields. We prove generally that the existence of a symmetry reduces the problem to the one-dimensional case. In Sec. IV we work out the case of symmetry breaking for $n=2$, and we estimate the validity of the semigroup law.

II. THE CASE OF ONE FUNDAMENTAL FIELD

Our basic assumption here is the existence of a local scalar Hermitian field, which satisfies the causality condition (commutativity of field operators with spacelike separation). The vacuum expectation value of the time-ordered product of the field is given by

$$G(x,x') = i\langle 0|\Phi(x)\Phi(x')\theta(x_0-x_0')+\Phi(x')\Phi(x)\theta(x_0'-x_0)|0\rangle$$
$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}G(k)\,. \tag{2.1}$$

The spectral representation for $G(k)$ is

$$G(k) = \int \frac{dB(s)}{k^2 - s + i\epsilon}\,, \tag{2.2}$$

where $B(s)$ is a real non-negative function (for a Hermitian field), which tends to zero when $s\to 0$ and approaches unity for $s\to\infty$, in such a way that $\int dB(s) = 1$.[19] From (2.1) it follows that

$$dB(q^2) = \sum_n (2\pi)^3\delta^4(p_n - q)|\langle 0|\Phi(0)|n\rangle|^2 dq^2\,. \tag{2.3}$$

Therefore, the function $G(z)$ defined by

$$G(z) = \int \frac{dB(s)}{z-s} \tag{2.4}$$

can have no complex zeros because

$$G(z) = \int \frac{dB(s)(x-s)}{(x-s)^2+y^2} - iy\int \frac{dB(s)}{(s-x)^2+y^2}$$
$$=0$$

for $z=x+iy$ means that $y=0$. Therefore, the function $G^{-1}(z)-z$ is regular everywhere in the complex plane, with the exception of the positive real axis. [There are no poles in $G(z)$ for $z<0$, because such a pole becomes a particle with imaginary mass $m=\sqrt{z}$.] The positive real axis is a branch line due to the continuous spectrum of $G(z)$. Yet there might be isolated poles in $G^{-1}(z)-z$, on the positive part of the real axis, which are zeros of $G(z)$. Therefore, the function $z^{-1}[G^{-1}(z)-z]$ vanishes at infinity in the cut plane, and has a pole at the origin, so it can be written in the form

$$z^{-1}[G^{-1}(z)-z] = -\frac{\lambda^2}{z} - \int \frac{\rho(s)ds}{z-s}\,.$$

Hence

$$G^{-1}(z) = -\lambda^2 + z - z\int\frac{\rho(s)ds}{z-s}\,. \tag{2.5}$$

$G^{-1}(z)\to-\infty$ for $z\to-\infty$ [provided that $\int\rho(s)ds$ converges, which is a necessary result of (2.9) and (3.13), which is derived later] has no poles for $z<0$, and is an increasing function of z ($z<0$) because

$$\frac{dG^{-1}(z)}{dz} = 1 + \int\frac{s\rho(s)ds}{(z-s)^2} > 0\,. \tag{2.6}$$

This inequality is a result of the relation

$$\frac{1}{2iy}[G(x+iy) - G(x-iy)] < 0\,, \tag{2.7}$$

which one gets when $G(z)$ is written explicitly in terms of x, y [$dB(s)>0$]. From (2.7) one gets for any $y\neq 0$

$$\frac{1}{2iy}[G^{-1}(x+iy) - G^{-1}(x-iy)] > 0\,, \tag{2.8}$$

which by using (2.5) becomes

$$1 + \int\frac{s\rho(s)ds}{(x-s)^2+y^2} > 0\,.$$

This last relation is (2.6) for $y\to 0$, $x<0$.

$G(z)$ has no poles for $z<0$, therefore $\lambda^2>0$. Define

$$\mu_0{}^2 = \lambda^2 + \int\rho(s)ds\,, \tag{2.9}$$

so that

$$G^{-1}(z) = z - \mu_0{}^2 - \int\frac{s\rho(s)ds}{z-s}\,, \tag{2.10}$$

which means that for large k^2,

$$G(k^2) \underset{k^2\to\infty}{\sim} \frac{1}{k^2-\mu_0{}^2}\,.$$

Poles of $G(z)$ for $z>0$ are zeros of $G^{-1}(z)$ given by

$$G^{-1}(\mu_i^2)=0\,,$$

such that

$$\mu_i^2=\mu_0^2+\int\frac{s\rho(s)ds}{\mu_i^2-s}\,. \tag{2.11}$$

These poles are located between 0 and s_0, which is the beginning of the cut [the integral in (2.11) is from s_0 to ∞].

In order to find the time behavior of $G(z)$, one has to find all those terms which might give significant contributions for certain values of t. Therefore, we define the analytic continuation of the function into the second sheet. New poles which are found there become very important when they are close enough to the real axis. In order to be able to do the analytic continuation, we assume that the function $\rho(z)$ is continuous across the real axis and is analytic in the strip up to the point z.

Let us assume for the moment that there are no poles on the real axis; therefore, the analytic continuation of (2.10) becomes

$$G^{\rm II}(z)=\left[z-\mu_0^2-\int_{s_0}^{\infty}\frac{s\rho(s)ds}{z-s}+2\pi iz\rho(z)\right]^{-1}, \tag{2.12}$$

where $z\rho(z)$ is the discontinuity across the cut which begins at $s_0>0$. If $\mathrm{Im}\,z$ is small enough one can approximate

$$G^{\rm II}(z)\approx\left[\mu^2-\mu_0^2-\mathrm{P}\int_{s_0}^{\infty}\frac{s\rho(s)}{\mu^2-s}ds+\pi i\mu^2\rho(\mu^2)\right]^{-1}, \tag{2.13}$$

$$\mu^2=\mathrm{Re}z\,. \tag{2.14}$$

Poles on the second sheet are located at the points

$$z_p=\mu_0^2+\int_{s_0}^{\infty}\frac{s\rho(s)}{z_p-s}ds-2\pi iz_p\rho(z_p)\,, \tag{2.15}$$

and for small $\mathrm{Im}\,z_p$

$$\mu_p^2=\mathrm{Re}z_p\approx\mu_0^2+\mathrm{P}\int_{s_0}^{\infty}\frac{s\rho(s)}{\mu_p^2-s}ds\,, \tag{2.16}$$

$$\gamma_p=-\mathrm{Im}z_p\approx\pi\mu_p^2\rho(\mu_p^2)\,. \tag{2.17}$$

Poles on the real axis should be added, if they exist, as zero points of $G^{-1}(z)$ on the first sheet. An addition of a weak interaction may change the location of such poles from the real axis into the second sheet if the threshold s_{0W} is less than μ_i^2 (μ_i^2 is the real pole). Here $\rho(s)$ has two contributions strong (S) and weak (W):

$$\begin{aligned}&\rho(s)=\rho_S(s)+\rho_W(s)\,,\\&\rho_S(s)=0\,,\quad s\leqslant s_{0S}\\&\rho_W(s)=0\,,\quad s\leqslant s_{0W}<s_{0S}\end{aligned} \tag{2.18}$$

and

$$\rho_W(s)\ll\rho_S(s)\,,\quad s>s_{0S}\,.$$

All those $s_{0W}<\mu_i^2<s_{0S}$ which were poles on the real axis in the absence of weak interaction and given by

$$\mu_i^2=\mu_0^2+\int_{s_{0S}}^{\infty}\frac{s\rho_S(s)}{\mu_i^2-s}ds \tag{2.19}$$

lose this property due to the additional cut contributions of the weak interaction, but if $\rho_W(s)$ is small enough and sufficiently well behaved, new poles are found on the second sheet close to the original ones. They are given by

$$z_i=\mu_0^2+\int_{s_{0W}}^{\infty}\frac{s\rho_W(s)}{z_i-s}ds+\int_{s_{0S}}^{\infty}\frac{s\rho_S(s)}{z_i-s}ds-2\pi iz_i\rho_W(z_i)\,.$$

So for small $\rho_W(s)$

$$\mu_i'^2=\mathrm{Re}z_i\approx\mu_0^2+\int_{s_{0S}}^{\infty}\frac{s\rho_S(s)}{\mu_i'^2-s}ds+\mathrm{P}\int_{s_{0W}}^{\infty}\frac{s\rho_W(s)}{\mu_i'^2-s}ds\,,$$

$$\gamma_i=-\mathrm{Im}z_i\approx\pi\mu_i'^2\rho_W(\mu_i'^2). \tag{2.20}$$

Here

$$s_{0W}<\mu_i'^2<s_{0S}$$

and

$$|\mu_i'^2-\mu_i^2|\sim O(\rho_W(\mu_i^2)).$$

The Fourier transform of (2.2) gives the time dependence of the Green's function

$$G(\vec{k},t)=\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}e^{-ik_0t}\int\frac{dB(s)}{k_0^2-\vec{k}^2-s+i\epsilon}\,,$$

$$G(\vec{k},t)=\int\frac{dB(s)}{2E_s}e^{-iE_st}\,, \tag{2.21}$$

where

$$E_s=(s+\vec{k}^2)^{1/2}\,. \tag{2.22}$$

The poles on the real axis are the dominant contributions to (2.21). The contribution of the cut can be dominant for certain values of t, so by deforming the contour of integration into the second sheet, it can be approximated by the poles there, provided the contribution of the branch point is small and the integral at infinity can be neglected, see also Ref. 12. In this approximation

$G(\vec{k}, t)$ is given by

$$G(\vec{k}, t) \approx \sum_i \frac{B_i}{2E_i} e^{-iE_i t} + \sum_p \frac{B_p}{2E_p} e^{-iE_{z_p} t} . \quad (2.23)$$

B_i and B_p are the pole residues at the corresponding points μ_i^2 and z_p given in (2.11) and (2.15), respectively,

$$B_i = \frac{b_i}{1 + \int s\rho(s) ds/(\mu_i^2 - s)^2} , \quad (2.24)$$

$$B_p = \frac{b_p}{1 + \int s\rho(s) ds/(z_p - s)^2 + 2\pi i (d/dz)[z\rho(z)]|_{z=z_p}} , \quad (2.25)$$

$$E_{z_p} = (z_p + \vec{k}^2)^{1/2} \approx E_p - i\gamma_p/2E_p , \quad (2.26)$$

$$E_p = (\mu_p^2 + \vec{k}^2)^{1/2} . \quad (2.27)$$

μ_p^2 and γ_p are given in (2.16) and (2.17). E_i has a similar form to (2.27) but with μ_p^2 replaced by μ_i^2, which is given in (2.11).

The pole approximation is valid when t is large enough, and for this order of t, $G(\vec{k}, t)$ has an exponential form. The propagation of stable and unstable particles is given by $G(\vec{k}, t)$. Stable particles are poles on the real axis located at the points μ_i^2, and unstable particles are poles on the second sheet at the points z_p, with mass μ_p and width γ_p. Poles on the real axis wander to the second sheet when a weak interaction is added whose threshold for decay $s_{0W} < \mu_i^2$ for certain i. Hence stable particles become unstable, decaying under the weak interactions, and their mass is shifted due to the additional interaction. This was noticed also by Horwitz and Marchand[3] in the Lee model for the nonrelativistic case, and by Matthews and Salam[9,10] in the relativistic case.

III. THE GENERAL CASE OF n FUNDAMENTAL FIELDS

Suppose that there are n fundamental Hermitian scalar fields ($n = 2$, for example, is the case of $\pi^\pm, K_{1,2}$), whose commutators are zero for spacelike separation; their Green's function is an $n \times n$ matrix:

$$G_{ab}(x, x') = i\langle 0 | T(\phi_a(x)\phi_b(x')) | 0\rangle . \quad (3.1)$$

This is a function of $(x - x')^2$ from Lorentz and translational invariance. Its Fourier transform has the spectral representation

$$G_{ab}(k^2) = \int \frac{dB_{ab}(s)}{k^2 - s + i\epsilon} , \quad (3.2)$$

where for $s = q^2$

$$dB_{ab}(s) = \sum_n (2\pi)^3 \langle 0 | \phi_a(0) | n\rangle\langle n | \phi_b(0) | 0\rangle \times \delta^4(p_n - q) dq^2 . \quad (3.3)$$

$B_{ab}(s)$ is a Hermitian matrix, and from PCT invariance one gets

$$B^*_{ab}(s) = B_{ba}(s) = B_{ab}(s) , \quad (3.4)$$

making the matrix real symmetric with non-negative diagonal elements. Moreover, its matrix elements vanish for $s \leq 0$ because there are no physical states with $q^2 \leq 0$ (there is no known physical spinless state with $q^2 = 0$). The matrix $G_{ab}(z)$ defined by

$$G_{ab}(z) = \int \frac{dB_{ab}(s)}{z - s} \quad (3.5)$$

has a cut along the positive real axis and its diagonal matrix elements have no complex zeros. If there were complex zeros, then $\mathrm{Im}G_{aa}(z) = 0$ at the zero point; however, $\mathrm{Im}G_{aa}(z) \propto \mathrm{Im}\, z$ and therefore $\mathrm{Im}\, z = 0$. (The integral which multiplies $\mathrm{Im}\, z$ cannot vanish, because the integrand is positive.)

$G_{ab}(z)$ behaves like δ_{ab}/z for large z. To prove this, one has to take the derivative with respect to x^0 of

$$\langle 0 | [\phi_a(x), \phi_b(y)] | 0\rangle = i\Delta'_{ab}(x - y) = i \int dB_{ab}(s) \Delta(x - y, s) . \quad (3.6)$$

$\Delta(x - y, s)$ is the well known[20] Δ function defined by

$$\Delta(x - y, s) = \Delta^+(x - y, s) + \Delta^-(x - y, s) .$$

From the properties of $\Delta(x - y, s)$ and the assumptions about the equal-time commutator one gets from (3.6)

$$\langle 0 | [\partial_0\phi_a(x), \phi_b(y)] |_{x^0 = y^0} | 0\rangle = -i\delta_{ab}\delta^3(\vec{x} - \vec{y}) = -i \int dB_{ab}(s) \delta^3(\vec{x} - \vec{y})$$

or

$$\int dB_{ab}(s) = \delta_{ab} . \quad (3.7)$$

Hence

$$G_{ab}(z) \underset{z\to\infty}{\sim} \frac{1}{z} \int dB_{ab}(s) = \frac{\delta_{ab}}{z} .$$

$G_{aa}(z) \to 0^-$ for $z \to -\infty$, and is a decreasing function of z in the region $-\infty < z < 0$ because

$$-\frac{dG_{aa}(z)}{dz} = \int \frac{dB_{aa}(s)}{(z - s)^2} > 0 , \quad (3.8)$$

$$-G_{aa}(0) = \int \frac{dB_{aa}(s)}{s} > 0 ; \quad (3.9)$$

therefore, there are no zeros in $G_{aa}(z)$ for $-\infty < z < 0$.

The matrix elements of $G^{-1}(z)$ have a cut on the positive part of the real axis because of the cut in $G_{ab}(z)$. They have no poles unless z is in the region $0<z<\infty$, because if there were any, then at least one of the diagonal elements of $G(z)$ would be zero out of this region, which is not possible as we have shown.

So the matrix elements of $[G^{-1}(z)-z]z^{-1}$ have a pole at the origin and vanish at infinity in the cut plane:

$$[G^{-1}{}_{ab}(z)-z\delta_{ab}]z^{-1}=\frac{-\lambda_{ab}}{z}-\int\frac{M_{ab}(s)ds}{z-s}. \quad (3.10)$$

Hence

$$G^{-1}{}_{ab}(z)=z\delta_{ab}-\lambda_{ab}-z\int\frac{M_{ab}(s)ds}{z-s}, \quad (3.11)$$

where $\lambda_{ab}, M_{ab}(s)$ are symmetric real matrices. For high k^2, $G^{-1}{}_{ab}(k^2)\propto\delta_{ab}k^2$. Now

$$G^{-1}{}_{ab}(k^2)\underset{k^2\to\infty}{\sim}k^2\delta_{ab}-\lambda_{ab}-\int M_{ab}(s)ds,$$

and hence

$$\lambda_{ab}=-\int M_{ab}(s)ds, \quad a\neq b. \quad (3.12)$$

Let us define

$$\mu_a{}^2=\lambda_{aa}+\int M_{aa}(s)ds, \quad (3.13)$$

so that

$$G^{-1}{}_{ab}(k^2)=(k^2-\mu_a{}^2)\delta_{ab}-\int\frac{sM_{ab}(s)ds}{k^2-s+i\epsilon}. \quad (3.14)$$

The behavior for high k^2 of $G^{-1}{}_{ab}(k^2)$ is like $\delta_{ab}(k^2-\mu_a{}^2)$. Here $\mu_a{}^2$ can be interpreted as a bare mass associated with the field $\phi_a(x)$ (if we can show that $\mu_a{}^2>0$), and can be connected with the spectral function $B_{aa}(s)$:

$$G_{ab}(k^2)=\int\frac{dB_{ab}(s)}{k^2-s+i\epsilon}\underset{k^2\to\infty}{\sim}\frac{1}{k^2}\int dB_{ab}(s)\left(1+\frac{s}{k^2}\right).$$

On the other hand,

$$G_{ab}(k^2)\underset{k^2\to\infty}{\sim}\frac{\delta_{ab}}{k^2-\mu_a{}^2}\approx\frac{1}{k^2}\left(1+\frac{\mu_a{}^2}{k^2}\right)\delta_{ab}.$$

If we compare these two relations after using (3.7) we get

$$\int s\,dB_{ab}(s)=\mu_a{}^2\delta_{ab}. \quad (3.15)$$

From this equation, one finds that $\mu_a{}^2>0$, and so

$$\lambda_{aa}+\int M_{aa}(s)ds=\mu_a{}^2>0.$$

Another relation which can be found is

$$-G_{ab}(0)=\int\frac{dB_{ab}(s)}{s}=(\lambda^{-1})_{ab}, \quad (3.16)$$

which means that $\lambda^{-1}{}_{aa}>0$. This does not necessarily mean that $\lambda_{aa}>0$, unless the diagonal elements of $G_{ab}(0)$ are dominant, so that $\det G(0)>0$.

The form (3.14) for $G^{-1}(k^2)$ is not surprising, and one could get it from a dynamical model of the fields coupled to currents $J_i(x)$. But here we got it only from the assumptions of causality, Poincaré invariance, and the completeness of the set of states having timelike momentum ($p^2>0$).

The Green's function (3.2) has a cut whose threshold is $s_0>0$, with possible poles on the positive part of the real axis. The contribution of the cut can be approximated by poles on the second sheet[21] provided that the contribution of the branch point is small, and the integral at infinity can be neglected. Suppose for the moment that there are no poles on the real axis; then the analytic continuation is

$$G^{-1\,\mathrm{II}}_{ab}(z)=(z-\mu_a{}^2)\delta_{ab}-\int_{s_0}^{\infty}\frac{sM_{ab}(s)}{z-s}ds+2\pi izM_{ab}(z). \quad (3.17)$$

Poles on the second sheet are those points z_i such that

$$\det G^{-1\,\mathrm{II}}(z)\big|_{z=z_i}=0. \quad (3.18)$$

Define

$$G^{-1\,\mathrm{II}}{}_{ab}(z)=z\delta_{ab}-M^{\mathrm{II}}_{ab}(z). \quad (3.19)$$

Then if $r_i(z)$ and $h_i(z)$ are the right and left eigenvectors of $M^{\mathrm{II}}(z)$, respectively, with eigenvalues $\mu_i(z)$, such that

$$M^{\mathrm{II}}(z)r_i(z)=\mu_i(z)r_i(z),$$
$$h_i(z)M^{\mathrm{II}}(z)=\mu_i(z)h_i(z), \quad (3.20)$$

then the orthogonal idempotents defined as the direct product of these vectors,

$$R_i(z)=r_i(z)\otimes h_i(z), \quad (3.21)$$

satisfy

$$R_i(z)R_j(z)=\delta_{ij}R_i(z), \quad (3.22)$$

$$\sum_i R_i(z)=I, \quad (3.23)$$

where I is the unit matrix. From (3.22) and (3.23) it is clear that $G(z)$ can be expressed in terms of $R_i(z)$ and $\mu_i(z)$:

$$G(z)=\sum_i\frac{R_i(z)}{z-\mu_i(z)}. \quad (3.24)$$

So the poles of $G(z)$ are those points z_{p_i} such that

$$L_i(z_{p_i})=z_{p_i}-\mu_i(z_{p_i})=0, \quad (3.25)$$

and the pole residues are

$$g_{p_i}=\frac{R_i(z_{p_i})}{1-(d/dz)\mu_i(z)|_{z=z_{p_i}}}\,. \tag{3.26}$$

Clearly, each of the functions $L_i(z)$ may have more than one pole, or even none. g_{p_i} are in general not orthogonal, not even if the poles are zeros of different functions $L_i(z)$, unless they are at the same point, because the orthogonality relation in (3.22) is valid only when $R_i(z), R_j(z)$ are taken at the same point z. The pole residues are not Hermitian because $G^{-1\,\mathrm{II}}(z)$ is not Hermitian, so they are not projections on the eigenspaces of the residues of the poles.

If P_p is the projection on the range of g_p, then from the fact that z_p is a pole of $G^{\mathrm{II}}(z)$ one gets

$$\mathrm{Tr}[G^{-1\,\mathrm{II}}(z_p)P_p]=0, \tag{3.27}$$

or from (3.19)

$$z_p=\mathrm{Tr}[M^{\mathrm{II}}(z_p)P_p]. \tag{3.28}$$

If $\gamma_p=-\mathrm{Im}z_p$ and $\mu_p^{\,2}=\mathrm{Re}z_p$ satisfy $\gamma_p\ll\mu_p^{\,2}$, then from (3.17), (3.19), and (3.28)

$$\mu_p^{\,2}=\mathrm{Tr}\left[P_p\left(\mu^2+\mathrm{P}\int_{s_0}^{\infty}\frac{sM(s)}{\mu_p^{\,2}-s}ds\right)\right]\,, \tag{3.29}$$

$$\gamma_p=\pi\mu_p^{\,2}\mathrm{Tr}[M(\mu_p^{\,2})P_p], \tag{3.30}$$

where μ^2 is the diagonal matrix $\mu_{ab}^{\;2}=\mu_a^{\,2}\delta_{ab}$ and the $\mu_a^{\,2}$ are given in (3.15).

Poles on the real axis, if there are any, should be added, and their residues g_i are built from the eigenvectors of $M(z)$ on the first sheet, so they are Hermitian for real $z_i=\mu_i^{\,2}$. Here

$$\mu_i^{\,2}=\mathrm{Tr}\left[\left(\mu^2+\int_{s_0}^{\infty}\frac{sM(s)}{\mu_i^{\,2}-s}ds\right)P_i\right] \tag{3.31}$$

for $\mu_i^{\,2}<s_0$ and P_i is the projector on the eigenspace of g_i.

As for the one-dimensional case, an addition of a weak interaction, whose threshold for decay $s_{0W}<s_0$, changes the location of the real poles $\mu_i^{\,2}$ $(s_{0W}<\mu_i^{\,2}<s_0)$ and they wander to the second sheet, close to the real axis. The time-dependent Green's function in the pole approximation is given by

$$G(\vec{k}^2,t)=\sum_p\frac{ig_p}{2E_p}e^{-iE_{z_p}t}+\sum_i\frac{ig_i}{2E_i}e^{-iE_it}. \tag{3.32}$$

E_{z_p} is given in (2.26) and (2.27) for complex poles, and E_i in (2.27) for real poles. Since $g_pg_{p'}\neq0$ for $p\neq p'$, even in the pole approximation there is no semigroup, i.e.,

$$G(\vec{k}^2,t_1)G(\vec{k}^2,t_2)\neq G(\vec{k}^2,t_1+t_2). \tag{3.33}$$

The same result was also shown to be valid in the nonrelativistic case.[5]

In case there is a symmetry of the interaction, if Q_i are the generators of the transformation. then the fields $\phi_a(x)$ are supposed to transform according to a certain representation of the group

$$[Q^i,\phi_a(x)]=C^i_{ab}\phi_b(x)\quad(i=1,\ldots,m;a,b=1,\ldots,n), \tag{3.34}$$

where m is the number of generators, and C^i_{ab} is the representation of the algebra.

The unitary representation of the group elements U is given by

$$U=\exp(i\epsilon_iQ^i). \tag{3.35}$$

If the vacuum is invariant under the transformation (which means that the symmetry is not spontaneously broken)

$$U|0\rangle=|0\rangle, \tag{3.36}$$

then

$$G_{ab}(x,y)=\langle0|T[U\phi_a(x)U^{-1}U\phi_b(y)U^{-1}]|0\rangle.$$

For an infinitesimal transformation

$$G_{ab}(x,y)=\langle0|T[(1+i\epsilon_iQ^i)\phi_a(x)(1-i\epsilon_iQ^i)(1+i\epsilon_iQ^i)\phi_b(y)(1-i\epsilon_iQ^i)]|0\rangle,$$

and hence

$$G(x,y)=G(x,y)+i\epsilon_i[C^iG(x,y)+G(x,y)C^{iT}], \tag{3.37}$$

i.e.,

$$C^iG(x,y)+G(x,y)C^{iT}=0. \tag{3.38}$$

But we know that for a Hermitian basis $\{\phi_a(x)\}$,

$$C^i_{ab}=-C^{i*}_{ab}=-C^i_{ba}, \tag{3.39}$$

which can be proved by taking the Hermitian conjugate of (3.34), and using the fact that $\phi_a^\dagger(x)=\phi_a(x)$; therefore, (3.38) becomes the well-known relation

$$[C^i,G(x,y)]=0. \tag{3.40}$$

This is a very strong restriction on $G(x,y)$ because if $K(x,y)$ is the orthogonal matrix which diagonalizes $G(x,y)$ (G is symmetric), then (3.40) becomes

$$A(x,y)D^i(x,y)=D^i(x,y)A(x,y),$$

where

$$A(x,y)=K(x,y)G(x,y)K^T(x,y),$$

$$D^i(x,y)=K(x,y)C^iK^T(x,y),$$

and $A(x,y)$ is a diagonal matrix.

$$D^i_{bd}(x,y)[A_{bb}(x,y)-A_{dd}(x,y)]=0 \text{ (no summation)}$$

for all i, i.e., $G_{bb}(x,y)=G_{dd}(x,y)$ for all those indices b, d such that $D^i_{bd}(x,y)\neq 0$ $(b\neq d)$. Here there are m-independent antisymmetric matrices $D^i(x,y)$; therefore, $A_{ab}(x,y)$ and hence $G_{ab}(x,y)$ become a multiple of the unit matrix

$$G_{ab}(x,y)=a(x,y)\delta_{ab}. \quad (3.41)$$

This reduces the problem to the one-dimensional case. Since in this case $G(x,y)$ is diagonal, the pole residues are orthogonal and we have the semigroup law. An example of this case is given by Schwinger for $\pi^{\pm}$. There are two Hermitian fields which form a two-dimensional vector, transforming under a gauge transformation $\varphi(x)\to e^{iq\alpha}\varphi(x)$, where q is the charge matrix. Invariance under this transformation makes $G(x,y)$ of the form (3.41).

If the symmetry is a discrete one like $PC\equiv\Theta$,

$$\begin{aligned} G_{ab}(x,y)&=\langle 0|T[\Theta\phi_a(x)\Theta^{-1}\Theta\phi_b(y)\Theta^{-1}]|0\rangle,\\ G_{ab}(x,y)&=\eta_a\eta_b\langle 0|T[\phi_a(\bar{x})\phi_b(\bar{y})]|0\rangle, \end{aligned} \quad (3.42)$$

where $\eta_a\eta_b$ are the phases of the fields after the transformation, and $\bar{x}=(x^0,-\vec{x})$. But from Poincaré invariance and the symmetry of $G_{ab}(x,y)$ it follows that it is a function of $(x-y)^2$; therefore, for all those indices, such that $\eta_a\eta_b=-1$ $(a\neq b)$, $G_{ab}[(x-y)^2]=0$ because $G_{ab}[(x-y)^2]=-G_{ab}[(x-y)^2]$, i.e., G_{ab} is diagonal:

$$G_{ab}[(x-y)^2]=A_a[(x-y)^2]\delta_{ab}. \quad (3.43)$$

So here also the problem is reduced to the one-dimensional case. An example is the case of K_1, K_2, for which G_{ab} is 2×2, and because of PC invariance, it is diagonal. Here also we have the semigroup law, with pole residues being orthogonal.

If the symmetry is broken, the nice features we found are no longer valid. But if the symmetry breaking is small, one can estimate the validity of the semigroup law. This will be done briefly in the next section.[22]

IV. THE CASE OF SYMMETRY BREAKING

We found in the last section that the propagator is diagonal, if there is a symmetry.[22] In a broken symmetry, the off-diagonal terms complicate the problem. Here we shall try to estimate the validity of the semigroup law for $n=2$. For definiteness, let us consider the K_1, K_2, and so $\phi_a(x)$ $(a=1,2)$ are the fields with $PC=\pm 1$ respectively, and define

$$\begin{aligned} \langle 0|\phi_1(0)|q+\rangle&=gf_1^{(+)}(q^2),\\ \langle 0|\phi_2(0)|q+\rangle&=g\beta f_2^{(+)}(q^2),\\ \langle 0|\phi_1(0)|q-\rangle&=g\beta f_1^{(-)}(q^2),\\ \langle 0|\phi_2(0)|q-\rangle&=gf_2^{(-)}(q^2), \end{aligned} \quad (4.1)$$

where $f_i^{(\pm)}(q^2)$ $(i=1,2)$ are invariant functions of q^2, g is the weak coupling constant, and β is a parameter, which measures the CP violation. $|q\pm\rangle$ are CP eigenstates with $CP=\pm 1$ respectively and momentum q.

From (3.3) and (4.1) $B^W_{ab}(q^2)$, which are the contributions to the spectral functions due to the weak interaction, have the following form:

$$\begin{aligned} B^W_{aa}(q^2)&=B^{W(0)}_{aa}(q^2)+\beta^2B^{W(1)}_{aa}(q^2)\quad (a=1,2),\\ B^W_{ab}(q^2)&=\beta\tilde{B}^W_{ab}(q^2)\quad (a\neq b)\quad (a,b=1,2). \end{aligned} \quad (4.2)$$

$B^{W(0)}_{ab}(q^2), B^{W(1)}_{ab}(q^2), \tilde{B}^W_{ab}(q^2)$ are functions of the $f_i^{(\pm)}(q^2)$, which appear in (4.1); they are $O(g^2)$, and for $\beta=0$ (no symmetry breaking) $B^W(q^2)$ becomes a diagonal matrix.

The inverse of the propagator is given in (3.14). From (3.15), $\mu_a^{(st)2}$ is shifted by the weak interaction, the additional term is $O(g^2)$, and there is another term $O(g^2\beta^2)$, which measures the symmetry breaking

$$\mu_a^{W2}=\mu_a^{W(0)2}+\beta^2\mu_a^{W(1)2}, \quad (4.3)$$

where μ_a^{W2} is the shift of $\mu_a^{(st)2}$ and

$$\mu_a^{W(i)2}=\int_{s_{0W}}^{\infty} sB^{W(i)}_{aa}(s)ds\quad (i=0,1). \quad (4.4)$$

The matrix λ_{ab} given in (3.16) can be split into a strong and a weak contribution:

$$G^{-1}(0)=\lambda=\frac{1}{\det\{\int_{s_0}^{\infty}[B(s)/s]\,ds\}}\begin{pmatrix} \int_{s_0}^{\infty}\frac{B_{22}(s)}{s}ds, & -\int_{s_0}^{\infty}\frac{B_{12}(s)}{s}ds\\ -\int_{s_0}^{\infty}\frac{B_{12}(s)}{s}ds, & \int_{s_0}^{\infty}\frac{B_{11}(s)}{s}ds \end{pmatrix}. \quad (4.5)$$

The lower limit for integration is s_{0W} or s_{0S} depending on the integrand; here

$$B_{ab}(s)=B^{st}_{ab}(s)+B^{W}_{ab}(s)\,. \tag{4.6}$$

$B^{st}_{ab}(s)$ are the spectral functions contributed by the strong interactions. They form a diagonal matrix because strong interactions conserve PC.

Expansion of

$$\left[\det\left(\int_{s_0}^{\infty}\frac{B(s)}{s}ds\right)\right]^{-1}$$

to the first order in g^2 gives

$$\lambda_{ab}\approx\lambda^{st}_{ab}+\lambda^{W}_{ab}+O(g^4)\,, \tag{4.7}$$

where

$$\lambda^{st}_{ab}=\left[\int_{s_{0S}}^{\infty}\frac{B^{st}_{aa}(s)}{s}ds\right]^{-1}\delta_{ab}\ , \tag{4.8}$$

$$\lambda^{W}_{ab}=\lambda^{st}_{aa}\lambda^{st}_{bb}\int_{s_{0W}}^{\infty}\frac{B^{W}_{ab}(s)}{s}ds\,. \tag{4.9}$$

Here $a,b=1,2$.

In order to find the matrix M_{ab}, which appears in the inverse of the propagator, one has to use (3.12) and (3.13), so it is easy to find out that

$$\begin{aligned}&M_{aa}=M^{st}_{aa}+M^{W}_{aa}+O(g^4)\quad(a=1,2)\,,\\ &M_{ab}=M^{W}_{ab}+O(g^4)\quad(a\neq b)\quad(a,b=1,2)\,,\end{aligned} \tag{4.10}$$

where

$$M^{W}_{aa}\sim O(g^2)+O(g^2\beta^2)$$

and

$$M^{W}_{ab}\sim O(g^2\beta)\quad(a\neq b)\,.$$

Higher-order terms in $g^2\beta^2$ contribute but they were neglected. In the absence of weak interactions $G^{-1st}{}_{ab}(z)$ has the form

$$G^{-1st}{}_{ab}(z)=\left[z-\mu^{(st)2}_{0a}-\int_{s_{0S}}^{\infty}\frac{sM^{st}_{aa}(s)}{z-s}ds\right]\delta_{ab}\ , \tag{4.11}$$

with $\mu^{(st)2}_{0a}$ given in (3.15). The poles of the propagator are $\mu^{(st)2}_{a}$ given by

$$\mu^{(st)2}_{a}=\mu^{(st)2}_{0a}+\int_{s_{0S}}^{\infty}\frac{sM^{st}_{aa}(s)}{\mu^{(st)2}_{a}-s}ds\,. \tag{4.12}$$

Using these definitions, the full propagator can be written in the form

$$\begin{aligned}G^{-1\text{II}}{}_{ab}(z)=(z-\mu^{(st)2}_{0a})\delta_{ab}&-\int_{s_{0S}}^{\infty}\frac{sM^{S}_{ab}(s)}{z-s}ds\\ &-\int_{s_{0W}}^{\infty}\frac{sM^{W}_{ab}(s)}{z-s}ds\\ &+2\pi i z M^{W}_{ab}(z)\,.\end{aligned} \tag{4.13}$$

In order to find the pole residues on the second sheet, one has to find the eigenstates of $G^{-1\text{II}}(z)$ with zero eigenvalues. We will not give their explicit dependence on $G^{-1\text{II}}{}_{ab}(z)$; they are given in our previous paper[5] [Eqs. (2.21)–(2.24)]. We only quote the main result, which is the same for the relativistic case as for the nonrelativistic case. If $g_S(z_S), g_L(z_L)$ are the pole residues corresponding to the states $|K_S\rangle$, $|K_L\rangle$, respectively, then

$$g_S(z_S)g_L(z_L)\sim O(\beta^3)\,. \tag{4.14}$$

So the validity of the semigroup law is up to β^3, where $\beta^3\sim10^{-4}$, independent of g^2. The idempotence property of the residues is better fulfilled because[5] [Eqs. (3.15) and (3.16)]

$$g_p(z_p)g_p(z_p)=[1+O(g^2)]g_p(z_p)\quad(p=S,L)\,. \tag{4.15}$$

We find also from (4.13), (3.19), and (3.28) that the corrections to the masses in (4.12) due to the weak interactions are $O(g^2)+O(g^2\beta^2)$ and the lifetimes of the unstable particles are $O(g^2)$. Their evolution in time is almost exponential, since

$$G(\vec{k}^2,t_1)G(\vec{k}^2,t_2)=G(\vec{k}^2,t_1+t_2)+O(\beta^3)\,. \tag{4.16}$$

This effect can contribute an additional phase to the decay rate in regeneration experiments, if one considers them as successive scatterings, with free propagations in between.[5]

ACKNOWLEDGMENT

I wish to thank Professor L. P. Horwitz for the interesting discussions that stimulated this work, and for reading the manuscript and criticizing it.

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[11]Analytic continuation of the propagator was also used by Lévy (see Ref. 12) to investigate the properties of the unstable particle in the Lee model. Lévy's work differs from this work in that here, in addition to the use of the covariant formulation, there are no assumptions about the dynamical model.

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[19]This sum rule and two others will be derived later on for the case of n fundamental fields [see (3.7), (3.13), and (3.15)]; the latter guarantees the convergence of (2.2) because even $\int s\, dB(s) = \mu_0^2$ is supposed to converge.

[20]See, for example, Silvan S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, and Wetherhill, Tokyo, 1964).

[21]$M_{ab}(z)$ is assumed to be sufficiently well behaved, so that the analytic continuation is allowed. See the discussion about the analytic continuation of the propagator in the case $n = 1$.

[22]Note that in the case of a continuous symmetry the propagator is a multiple of the unit matrix and one function is enough to define it, whereas in the case of discrete symmetry it is only diagonal, so at most n functions are needed to define it.