Axiomatic lower bound on the slope parameter

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(Received 8 December 1975)

The axiomatic lower bound on the s- and u-channel slope parameter, denoted respectively by $B_1(s)$ and $B_2(u)$, is investigated with the help of the unitarity of the S matrix, s-u crossing symmetry, and the analyticity and polynomial upper boundedness of the scattering amplitude. The result is that at least one of $B_1(s)$ and $B_2(u)$ has the lower bound $B_1(s) \geq \text{const} \times s^{-6}(\ln s)^{-6}$ (> 0) for some sequence of $s \to \infty$ or $B_2(u) \geq \text{const} \times u^{-6}(\ln u)^{-6}$ (> 0) for some sequence of $u \to \infty$.

I. INTRODUCTION

It will be useful to know what kinds of restrictions on observable quantities can be derived from the general principles on the scattering amplitude. The celebrated restriction of this kind is the Froissart upper bound on total cross sections[1]

$$\sigma_{\text{tot}}(s) \leq \text{const} \times (\ln s)^2 \text{ as } s \to \infty. \tag{1.1}$$

On the other hand, Jin and Martin[2] obtained the lower bound on total cross sections

$$\sigma_{\text{tot}}(s) \geq \text{const} \times s^{-6}(\ln s)^{-3} \tag{1.2}$$

for at least one sequence of $s \to \infty$. In deriving (1.2), they used as general principles the unitarity of the S matrix, analyticity in s and t, s-u crossing symmetry, and the polynomial upper boundedness of the scattering amplitude.

In this paper we use the same general principles as those used by Jin and Martin, and investigate the axiomatic lower bound on the s- and u-channel slope parameter [denoted respectively by $B_1(s)$ and $B_2(u)$]. The result is that at least one of the $B_1(s)$ and $B_2(u)$ has the lower bound

$$B_1(s) \geq \text{const} \times s^{-6}(\ln s)^{-6} \ (>0) \tag{1.3}$$

for some sequence of $s \to \infty$, or

$$B_2(u) \geq \text{const} \times u^{-6}(\ln u)^{-6} \ (>0) \tag{1.4}$$

for some sequence of $u \to \infty$.

In Sec. II we formulate our general principles. In Sec. III we derive the lower bound on the slope parameter. In Sec. IV we discuss our result and compare it with the other bound.

II. FORMULATION OF THE GENERAL PRINCIPLES

For simplicity we consider the spinless elastic scattering $A+B \to A+B$ (s channel) coupled by crossing to $\bar{A}+B \to \bar{A}+B$ (u channel). If we include spin, the same result we obtain in this paper holds for the imaginary part of the helicity-nonflip scattering amplitude.

The analyticity in s and the polynomial upper boundedness of the scattering amplitude $F(s,t)$ make it possible to write the dispersion relation with two subtractions,[3]

$$F(s,t) = A(t) + B(t)(s - \sigma + \tfrac{1}{2}t) + \frac{(s-\sigma+\frac{1}{2}t)^2}{\pi}\left[\int_{(M_A+M_B)^2}^{\infty} ds' \frac{\text{Im}F_{\text{I}}(s',t)}{(s'-\sigma+\frac{1}{2}t)^2(s'-s)} + \int_{(M_A+M_B)^2}^{\infty} du' \frac{\text{Im}F_{\text{II}}(u',t)}{(u'-\sigma+\frac{1}{2}t)^2(u'-2\sigma+s+t)}\right] \tag{2.1}$$

$$= A(t) - B(t)(u - \sigma + \tfrac{1}{2}t) + \frac{(u-\sigma+\frac{1}{2}t)^2}{\pi}\left[\int_{(M_A+M_B)^2}^{\infty} ds' \frac{\text{Im}F_{\text{I}}(s',t)}{(s'-\sigma+\frac{1}{2}t)^2(s'-2\sigma+u+t)} + \int_{(M_A+M_B)^2}^{\infty} du' \frac{\text{Im}F_{\text{II}}(u',t)}{(u'-\sigma+\frac{1}{2}t)^2(u'-u)}\right], \tag{2.2}$$

where

$$\text{Im}F_{\text{I}}(s,t) = 8\pi\frac{\sqrt{s}}{k_{\text{I}}}\sum_{l=0}^{\infty}(2l+1)\text{Im}f_l^{\text{I}}(s) \times P_l(\cos\theta_{\text{I}}) \tag{2.3}$$

and

$$\text{Im}F_{\text{II}}(u,t) = 8\pi\frac{\sqrt{u}}{k_{\text{II}}}\sum_{l=0}^{\infty}(2l+1)\text{Im}f_l^{\text{II}}(u) \times P_l(\cos\theta_{\text{II}}), \tag{2.4}$$

with

$$s+t+u=2M_A{}^2+2M_B{}^2\equiv 2\sigma,$$
$$F_{\rm I}(s,t)\equiv F(s,t), \tag{2.5}$$

and

$$F_{\rm II}(u,t)\equiv F(2\sigma-u-t,t).$$

Here s (u), $k_{\rm I}$ ($k_{\rm II}$), and $\theta_{\rm I}$ ($\theta_{\rm II}$) are the center-of-mass (c.m.) energy squared, the c.m. momentum, and the c.m. scattering angle in the s channel (u channel), respectively.

The unitarity of the S matrix gives the constraints

$$0\le |f_l^{\rm I}(s')|^2\le {\rm Im}f_l^{\rm I}(s')\le 1 \tag{2.6}$$

and

$$0\le |f_l^{\rm II}(u')|^2\le {\rm Im}f_l^{\rm II}(u')\le 1,$$

so that we have from (2.3) and (2.4)

$$\begin{aligned}&{\rm Im}F_{\rm I}(s',0)\ge 0,\\ &\partial_t{\rm Im}F_{\rm I}(s',0)\ge 0,\\ &{\rm Im}F_{\rm II}(u',0)\ge 0,\end{aligned} \tag{2.7}$$

and

$$\partial_t{\rm Im}F_{\rm II}(u',0)\ge 0,$$

with the simplified notations

$$\partial_t F_{\rm I}(s,0)\equiv \frac{\partial}{\partial t}F(s,t)\bigg|_{t=0} \tag{2.8}$$

and

$$\partial_t F_{\rm II}(u,0)\equiv \frac{\partial}{\partial t}F(2\sigma-u-t,t)\bigg|_{t=0}.$$

Next, we prove that the polynomial upper boundedness of the scattering amplitude

$$|F(s,t)\le |s|^N \text{ as } |s|\to\infty \tag{2.9}$$

leads to the polynomial upper boundedness of $\partial_t F_{\rm I}(s,0)$ and $\partial_t F_{\rm II}(u,0)$. (Throughout this paper we explicitly prove relations involving $F_{\rm I}$, since relations involving $F_{\rm II}$ can be similarly obtained.) The analyticity in t of $F_{\rm I}(s,t)$ leads to the Cauchy integral formula near $t=0$.

$$F_{\rm I}(s,t)=\frac{1}{2\pi i}\int_C dt'\,\frac{F_{\rm I}(s,t')}{t'-t}, \tag{2.10}$$

where C is the circle with $|t'|=t_0<\min(4M_A{}^2, 4M_B{}^2)$. Then we have

$$\partial_t F_{\rm I}(s,0)=\frac{1}{2\pi i}\int_C dt'\frac{F_{\rm I}(s,t')}{t'^2},$$

so that

$$|\partial_t F_{\rm I}(s,0)|\le \frac{1}{t_0}|s|^N \text{ as } |s|\to\infty.$$

Similarly

$$|\partial_t F_{\rm II}(u,0)|\le \frac{1}{t_0}|u|^N \text{ as } |u|\to\infty.\ \text{Q.E.D.} \tag{2.11}$$

III. DERIVATION OF THE LOWER BOUND ON THE SLOPE PARAMETER

In this section we present the derivation of the lower bound on the slope parameter. For this purpose we prove that at least one of $\partial_t{\rm Im}F_{\rm I}(s,0)$ and $\partial_t{\rm Im}F_{\rm II}(u,0)$ has the lower bound

$$\partial_t{\rm Im}F_{\rm I}(s,0)\ge \text{const}\times s^{-5}(\ln s)^{-4}\ (>0) \tag{3.1}$$

for some sequence of $s\to\infty$, or

$$\partial_t{\rm Im}F_{\rm II}(u,0)\ge \text{const}\times u^{-5}(\ln u)^{-4}\ (>0) \tag{3.2}$$

for some sequence of $u\to\infty$.

In the case where at least one of the four conditions

$$\lim_{s\to\infty} s^2\partial_t{\rm Re}F_{\rm I}(s,0)=0, \tag{3.3}$$

$$\lim_{s\to\infty} s^3\partial_t{\rm Im}F_{\rm I}(s,0)=0, \tag{3.4}$$

$$\lim_{u\to\infty} u^2\partial_t{\rm Re}F_{\rm II}(u,0)=0, \tag{3.5}$$

and

$$\lim_{u\to\infty} u^3\partial_t{\rm Im}F_{\rm II}(u,0)=0 \tag{3.6}$$

does not hold, there exist nonvanishing constants c_1–c_4 and sequences $s_n\to\infty$ and $u_n\to\infty$ such that at least one of the following four relations holds:

$$\lim_{n\to\infty} s_n{}^2\partial_t{\rm Re}F_{\rm I}(s_n,0)=c_1\ (\ne 0), \tag{3.7}$$

$$\lim_{n\to\infty} s_n{}^3\partial_t{\rm Im}F_{\rm I}(s_n,0)=c_2\ (\ne 0), \tag{3.8}$$

$$\lim_{n\to\infty} u_n{}^2\partial_t{\rm Re}F_{\rm II}(u_n,0)=c_3\ (\ne 0), \tag{3.9}$$

and

$$\lim_{n\to\infty} u_n{}^3\partial_t{\rm Im}F_{\rm II}(u_n,0)=c_4\ (\ne 0). \tag{3.10}$$

Equation (3.8) [(3.10)] satisfies (3.1) [(3.2)]. On the other hand, (3.7) [(3.9)] is shown to lead to (3.1) [(3.2)]. From (3.7) we have

$$0<\frac{c_1^{\,2}}{2}<s^4|\partial_t F_{\rm I}(s,0)|^2\sim s^4\left|8\pi\frac{\sqrt{s}}{k_{\rm I}}\,\frac{1}{s-4}\sum_{l=0}^{L}(2l+1)(l^2+l)f_l^{\rm I}(s)\right|^2$$
$$\le(16\pi)^2s^2\left[\sum_{l=0}^{L}(2l+1)(l^2+l)\right]\left[\sum_{l=0}^{L}(2l+1)(l^2+l)|f_l^{\rm I}(s)|^2\right]\sim 8\pi s^3L^4\partial_t{\rm Im}F_{\rm I}(s,0) \tag{3.11}$$

at sufficiently high energy $s=s_n$. Here we used the Schwarz inequality, the unitarity constraint (2.6), and the fact that the existence of the nonvanishing constant $c_1^{\,2}/2$ makes it possible to make a reliable estimate of $\partial_t F_{\rm I}(s,0)$ by the partial waves up to

$$L=K\sqrt{s}\ln s \tag{3.12}$$

provided K is taken sufficiently large.[4] Substituting (3.12) into (3.11), we find (3.1) Q.E.D.

In the following we prove that *there never occurs the case when (3.3)–(3.6) hold simultaneously*. From (2.1) and (2.2) we obtain after a tedious but straightforward calculation

$$\partial_t F_{\rm I}(s,0)\left(\equiv\frac{\partial}{\partial t}F(s,t)\Big|_{t=0}\right)=D_1\omega+D_2+\frac{D_3}{\omega}+D_4(\omega)+D_5(\omega)+D_6(\omega) \tag{3.13}$$

and

$$\partial_t F_{\rm II}(u,0)\left(\equiv\frac{\partial}{\partial t}F(2\sigma-u-t,t)\Big|_{t=0}\right)=D_1\omega+\tilde D_2+\frac{D_3}{\omega}+D_6(\omega)+\tilde D_5(\omega)+D_4(\omega), \tag{3.14}$$

where

$$D_1\equiv B'(0)+\frac{1}{\pi}\int_\rho^\infty d\omega'\left[\frac{{\rm Im}F_{\rm I}(\omega'+\sigma,0)-{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'^3}-\frac{\partial_t{\rm Im}F_{\rm I}(\omega'+\sigma,0)-\partial_t{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'^2}\right],$$
$$D_2\equiv A'(0)+\frac{B(0)}{2}+\frac{1}{\pi}\int_\rho^\infty d\omega'\left[\frac{{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'^2}-\frac{\partial_t{\rm Im}F_{\rm I}(\omega'+\sigma,0)+\partial_t{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'}\right],$$
$$\tilde D_2\equiv A'(0)-\frac{B(0)}{2}+\frac{1}{\pi}\int_\rho^\infty d\omega'\left[\frac{{\rm Im}F_{\rm I}(\omega'+\sigma,0)}{\omega'^2}-\frac{\partial_t{\rm Im}F_{\rm I}(\omega'+\sigma,0)+\partial_t{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'}\right],$$
$$D_3\equiv\frac{1}{\pi}\int_\rho^\infty d\omega'\,[\partial_t{\rm Im}F_{\rm II}(\omega'+\sigma,0)-\partial_t{\rm Im}F_{\rm I}(\omega'+\sigma,0)], \tag{3.15}$$
$$D_4(\omega)\equiv-\frac{1}{\omega}\frac{1}{\pi}\int_\rho^\infty d\omega'\,\frac{\omega'\partial_t{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{\omega'+\omega},$$
$$D_5(\omega)\equiv-\frac{1}{\pi}\int_\rho^\infty d\omega'\,\frac{{\rm Im}F_{\rm II}(\omega'+\sigma,0)}{(\omega'+\omega)^2},$$
$$\tilde D_5(\omega)\equiv-\frac{1}{\pi}\int_\rho^\infty d\omega'\,\frac{{\rm Im}F_{\rm I}(\omega'+\sigma,0)}{(\omega'-\omega)^2},$$

and

$$D_6(\omega)\equiv\frac{1}{\omega}\frac{1}{\pi}\int_\rho^\infty d\omega'\,\frac{\omega'\partial_t{\rm Im}F_{\rm I}(\omega'+\sigma,0)}{\omega'-\omega},$$

with

$$\omega\equiv s-\sigma=\sigma-u.$$

In (3.15), the integration variables s' and u' have been rewritten as $\omega'+\sigma$, so that $\rho\equiv 2M_AM_B$. Considering the case where (3.4) [(3.6)] holds, we have inequalities

$$0\le\partial_t{\rm Im}F_{\rm I}(s,0)<s^{-3}\quad\text{as}\quad s\to\infty \tag{3.16}$$

$$[0\le\partial_t{\rm Im}F_{\rm II}(u,0)<u^{-3}\quad\text{as}\quad u\to\infty]. \tag{3.17}$$

Furthermore, (3.16) [(3.17)] and (2.6) give

$$0\le{\rm Im}F_{\rm I}(s,0)<16\pi+c_5s^{-2}\quad\text{as}\quad s\to\infty \tag{3.18}$$

$$[0 \leq \mathrm{Im}F_{\mathrm{II}}(u,0) < 16\pi + c_6 u^{-2} \quad \text{as} \quad u \to \infty], \tag{3.19}$$

since we have

$$s^{-3} > \partial_t \mathrm{Im}F_{\mathrm{I}}(s,0) = 8\pi \frac{\sqrt{s}}{k_{\mathrm{I}}} \frac{1}{s-4} \sum_{l=1}^{\infty}(2l+1)(l^2+l)\mathrm{Im}f_l^{\mathrm{I}}(s) \geq \frac{1}{s-4}\left[\mathrm{Im}F_{\mathrm{I}}(s,0) - 8\pi\frac{\sqrt{s}}{k_{\mathrm{I}}}\mathrm{Im}f_0^{\mathrm{I}}(s)\right]. \tag{3.20}$$

With the help of the above-obtained inequalities (3.16)–(3.19), various integrals in (3.15) can be evaluated. First, the constants D_1, D_2, $\tilde{D}_2$, and D_3 defined by (3.15) are found to be finite. Second, the uniform convergence[5] of the integrals (3.15) for $\omega^2 D_4(\omega)$ and $\omega^2 D_6(\omega)$ gives

$$\lim_{\omega\to+\infty} \omega^2 D_4(\omega) = -\frac{1}{\pi}\int_\rho^\infty d\omega'\omega'\partial_t \mathrm{Im}F_{\mathrm{II}}(\omega'+\sigma,0) \ (\equiv -f_2) \tag{3.21}$$

and

$$\lim_{\omega\to-\infty} \omega^2 D_6(\omega) = -\frac{1}{\pi}\int_\rho^\infty d\omega'\omega'\partial_t \mathrm{Im}F_{\mathrm{I}}(\omega'+\sigma,0) \ (\equiv -f_1). \tag{3.22}$$

Third, we have[5] from (3.15), (3.18), and (3.19)

$$\lim_{\omega\to+\infty} D_5(\omega) = \lim_{\omega\to-\infty} \tilde{D}_5(\omega) = 0. \tag{3.23}$$

Finally, it is proved in the Appendix that (2.11), (3.3), and (3.4) [(3.5) and (3.6)], and (3.15) lead to

$$\lim_{\omega\to+\infty} \omega D_6(\omega) = 0 \qquad \left[\lim_{\omega\to-\infty} \omega D_4(\omega) = 0\right]. \tag{3.24}$$

Then, (3.3)–(3.6), (3.13), (3.14), and (3.21)–(3.24) give

$$D_1 = D_2 = \tilde{D}_2 = 0, \tag{3.25}$$

so that

$$D_3 = -\lim_{\omega\to+\infty} \omega D_5(\omega) \ (\geq 0) = -\lim_{\omega\to-\infty} \omega\tilde{D}_5(\omega) \ (\leq 0), \tag{3.26}$$

which means

$$D_3 = 0. \tag{3.27}$$

Substituting (3.25) and (3.27) into (3.13) and (3.14), we find

$$\omega^2\partial_t F_{\mathrm{I}}(\omega+\sigma,0) = \omega^2 D_4(\omega) + \omega^2 D_5(\omega) - f_1 + g_1(\omega) \tag{3.28}$$

and

$$\omega^2\partial_t F_{\mathrm{II}}(\sigma-\omega,0) = \omega^2 D_6(\omega) + \omega^2 \tilde{D}_5(\omega) - f_2 + g_2(\omega), \tag{3.29}$$

where

$$g_1(\omega) \equiv \frac{1}{\pi}\int_\rho^\infty d\omega' \frac{\omega'^2\partial_t \mathrm{Im}F_{\mathrm{I}}(\omega'+\sigma,0)}{\omega'-\omega} = \omega^2 D_6(\omega) + f_1 \tag{3.30}$$

and

$$g_2(\omega) \equiv \frac{1}{\pi}\int_\rho^\infty d\omega' \frac{\omega'^2\partial_t \mathrm{Im}F_{\mathrm{II}}(\omega'+\sigma,0)}{\omega'+\omega} = \omega^2 D_4(\omega) + f_2. \tag{3.31}$$

At this stage, assuming that (3.3)–(3.6) are satisfied, we consider two cases in terms of α_1 and α_2 defined by

$$\alpha_1 \equiv \frac{1}{\pi}\int_\rho^\infty d\omega' \mathrm{Im}F_{\mathrm{I}}(\omega'+\sigma,0) \tag{3.32}$$

and

$$\alpha_2 \equiv \frac{1}{\pi}\int_\rho^\infty d\omega' \mathrm{Im}F_{\mathrm{II}}(\omega'+\sigma,0). \tag{3.33}$$

Case (A): $\alpha_1 = \infty$ or $\alpha_2 = \infty$.
Case (B): $\alpha_1 < \infty$ and $\alpha_2 < \infty$.

When α_2 (α_1) is infinite as in case (A), (3.28) [(3.29)] is shown to contradict (3.3) and (3.4) [(3.5) and (3.6)] by applying the Phragmén-Lindelöf theorem[6] to

$$H_1(\omega) \equiv \frac{g_1(\omega)}{\omega^2 D_5(\omega)} \tag{3.34}$$

$$\left[H_2(\omega) \equiv \frac{g_2(\omega)}{\omega^2\tilde{D}_5(\omega)}\right]. \tag{3.35}$$

Proof. First, $g_1(\omega)$ has a polynomial upper bound by (2.11), (3.15), (3.17), (3.19), and (3.28). Next, $[\omega^2 D_5(\omega)]^{-1}$ is bounded in the region $R = \{\omega \mid |\omega| \geq 1,\ 0 \leq \theta \leq \frac{1}{4}\pi\}$, and analytic since (3.15), $\alpha_2 = \infty$, and the formula

$$\frac{\omega^2}{(\omega'+\omega)^2} = \frac{1}{[(\omega'+\omega_R)^2+\omega_I^2]^2}\{[(\omega_R^2-\omega_I^2)\omega'^2 + \omega_R^4 + 2\omega'\omega_R^3 + 2\omega_I^2\omega_R(\omega'+\omega_R) + \omega_I^4] + i[2\omega_I\omega_R\omega'(\omega'+\omega_R) + 2\omega_I^3\omega']\} \tag{3.36}$$

give

$$\lim_{\substack{|\omega|\to\infty \\ 0\le\theta\le\pi/4}} [\omega^2 D_5(\omega)]^{-1} = 0, \quad (3.37)$$

with

$$\omega = \omega_R + i\omega_I = |\omega| e^{i\theta}.$$

Therefore, $H_1(\omega)$ is analytic and polynomially upper bounded in the region R. Furthermore, (3.3), (3.4), (3.21), (3.28), (3.34), and (3.37) lead to

$$\lim_{\omega\to+\infty} H_1(\omega) = -1. \quad (3.38)$$

On the other hand, we find

$$\lim_{\substack{|\omega|\to\infty \\ \theta=\pi/4}} H_1(\omega) = 0 \quad (3.39)$$

with the help of (3.37) and

$$\lim_{\substack{|\omega|\to\infty \\ \theta=\pi/4}} g_1(\omega) = 0, \quad (3.40)$$

which is obtained from (3.30) and (3.16), since the integral $\omega^2 D_6(\omega)$ is uniformly convergent.[5] However, (3.38) and (3.39) contradict the Phragmén-Lindelöf theorem.[6]

Now the remaining case is only (B). In this case $\omega^2 D_5(\omega)$ in (3.28) $[\omega^2 \tilde{D}_5(\omega)$ in (3.29)] is calculated by (3.15) as[5]

$$\lim_{\omega\to+\infty} \omega^2 D_5(\omega) = -\alpha_2 \quad (3.41)$$

$$\left[\lim_{\omega\to-\infty} \omega^2 \tilde{D}_5(\omega) = -\alpha_1\right]. \quad (3.42)$$

Then, (3.3), (3.4), (3.21), (3.28), and (3.41) [(3.5), (3.6), (3.22), (3.29), and (3.42)] give

$$\lim_{\omega\to+\infty} g_1(\omega) = f_1 + f_2 + \alpha_2 \quad (3.43)$$

$$\left[\lim_{\omega\to-\infty} g_2(\omega) = f_1 + f_2 + \alpha_1\right]. \quad (3.44)$$

On the other hand, we have (3.40) and the corresponding equation for $g_2(\omega)$. Therefore, the Phragmén-Lindelöf theorem[6] applied to $g_1(\omega)$ and $g_2(\omega)$ leads to

$$\alpha_1 = \alpha_2 = 0. \quad (3.45)$$

However, (3.45) is nothing but the physically unrealizable condition

$$\mathrm{Im}F_{\mathrm{I}}(\omega' + \sigma, 0) = \mathrm{Im}F_{\mathrm{II}}(\omega' + \sigma, 0) \equiv 0. \quad (3.46)$$

Thus, we have proved that *it never occurs that (3.3)–(3.6) hold simultaneously*.

Our conclusion is that we have at least one of (3.1) and (3.2). If we use the Froissart bound[1]

$$\mathrm{Im}F_{\mathrm{I}}(s, 0) \le \mathrm{const} \times s(\ln s)^2 \quad \text{as} \quad s\to\infty$$

and (3.47)

$$\mathrm{Im}F_{\mathrm{II}}(u, 0) \le \mathrm{const} \times u(\ln u)^2 \quad \text{as} \quad u\to\infty,$$

we obtain the lower bound on the s- and u-channel slope parameter [denoted respectively by $B_1(s)$ and $B_2(u)$]. The result is that at least one of $B_1(s)$ and $B_2(u)$ has the lower bound

$$B_1(s) \equiv \frac{\partial_t \mathrm{Im}F_{\mathrm{I}}(s, 0)}{\mathrm{Im}F_{\mathrm{I}}(s, 0)} \ge \mathrm{const} \times s^{-6}(\ln s)^{-6} \ (>0) \quad (3.48)$$

for some sequence of $s\to\infty$, or

$$B_2(u) \equiv \frac{\partial_t \mathrm{Im}F_{\mathrm{II}}(u, 0)}{\mathrm{Im}F_{\mathrm{II}}(u, 0)} \ge \mathrm{const} \times u^{-6}(\ln u)^{-6} \ (>0) \quad (3.49)$$

for some sequence of $u\to\infty$. In the special s-u crossing-even case, we have *both* (3.48) and (3.49), since $B_1(a) = B_2(a)$.

IV. DISCUSSION

In this paper the axiomatic lower bound on the s- and u-channel slope parameter [denoted respectively by $B_1(s)$ and $B_2(u)$] has been investigated with the help of the unitarity of the S matrix, s-u crossing symmetry, analyticity, and the polynomial upper boundedness of the scattering amplitude. The result is that at least one of $B_1(s)$ and $B_2(u)$ has the lower bound

$$B_1(s) \ge \mathrm{const} \times s^{-6}(\ln s)^{-6} \ (>0) \quad (4.1)$$

for some sequence of $s\to\infty$, or

$$B_2(u) \ge \mathrm{const} \times u^{-6}(\ln u)^{-6} \ (>0) \quad (4.2)$$

for some sequence of $u\to\infty$. In the special s-u crossing-even case, *both* (4.1) and (4.2) hold, since $B_1(a) = B_2(a)$.

In the process of the derivation of the lower bound, the following situation complicated the proof. The integrals (3.32) and (3.33) of the scattering amplitude happen to diverge by the contribution of the S waves, although we treat the case of the partial derivative in t of the scattering amplitude to be strongly damping with s. As a result, in order to treat $\partial_t F(s, 0)$, we need a technique which is more complicated than Simon's technique[7] in treating $F(s, 0)$.

We compare our result with the other lower bound

$$B \ge \frac{\sigma_{\mathrm{tot}}^2}{36\pi\sigma_{\mathrm{el}}} - O\left(\frac{1}{k^2}\right), \quad (4.3)$$

which has been given by MacDowell and Martin.[8] Their bound is the good bound to make a comparison between the experimental values of both sides. But unfortunately their bound (4.3) does not give a lower bound with respect to s, since their analysis allows a value $B(s)\equiv 0$. In fact, the situation $B(s)\equiv 0$ is realized when only the S wave exists in the s channel. On the other hand, our result gives the lower bound for $B_2(u)$, *even* in this special situation.

APPENDIX

In the case where (3.3) and (3.4) hold, we shall prove only

$$\lim_{\omega\to+\infty} \omega D_6(\omega)=0, \tag{A1}$$

since

$$\lim_{\omega\to-\infty} \omega D_4(\omega)=0 \tag{A2}$$

is similarly concluded from (3.5) and (3.6).

First, we define the functions $T_+(\omega)$ and $T_-(\omega)$ by

$$T_\pm(\omega)\equiv e^{\pm(i/\pi)(2+N)(\ln\omega)^2}\omega^2\partial_t F_{\rm I}(\omega+\sigma,0), \tag{A3}$$

so that $T_+(\omega)[T_-(\omega)]$ is analytic and polynomially upper bounded in the region $R_+\equiv\{\omega\,|\,|\omega|\geq 1,\ 0<\theta\leq\frac{1}{2}\pi\}$ $[R_-\equiv\{\omega\,|\,|\omega|\geq 1,\ -\frac{1}{2}\pi\leq\theta<0\}]$, where $\omega=|\omega|e^{i\theta}$. The constant N in (A3) is the constant in (2.9). Then, (2.11) gives

$$\lim_{\substack{|\omega|\to\infty\\ \theta=\pm\pi/2}} |T_\pm(\omega)|\leq\alpha, \tag{A4}$$

where α is some constant. Furthermore, (3.3) and (3.4) give

$$\lim_{\omega\to+\infty} T_\pm(\omega)=0. \tag{A5}$$

Applying the Phragmén-Lindelöf theorem[6] to $T_\pm(\omega)$ satisfying (A4) and (A5), we find that

$$|T_\pm(\omega)|\leq\beta \ (\beta \text{ is some constant}) \tag{A6}$$

holds throughout the region $R_\pm$. Therefore, we have

$$|\partial_t F_{\rm I}(\omega+\sigma,0)|\leq\beta|\omega|^{-2+[2(2+N)/\pi]|\theta|} \tag{A7}$$

provided $|\omega|\geq 1$ and $0<|\theta|\leq\frac{1}{2}\pi$.

With the help of (A7), we can prove (A1) as follows. Equation (3.15) is rewritten as

$$\begin{aligned} D_6(\omega)-i\partial_t\,{\rm Im}F_{\rm I}(\omega+\sigma,0) &= \lim_{\epsilon\to+0}\frac{1}{\pi}\left(\int_{\rho/\omega}^{1/\omega^{3/4}}dx+\int_{1/\omega^{3/4}}^{1-1/\omega}dx+\int_{1-1/\omega}^{1-\epsilon/\omega}dx+\int_{1+\epsilon/\omega}^{1+1/\omega}dx+\int_{1+1/\omega}^{\infty}dx\right)\frac{x}{x-1}\partial_t\,{\rm Im}F_{\rm I}(\omega x+\sigma,0)\\ &\equiv \delta_1(\omega)+\delta_2(\omega)+\delta_3(\omega)+\delta_4(\omega)+\delta_5(\omega). \end{aligned} \tag{A8}$$

From (3.16) we have

$$0\leq\partial_t\,{\rm Im}F_{\rm I}(\omega+\sigma,0)\leq\frac{1}{\omega^3}\ \text{ as }\ \omega\to\infty, \tag{A9}$$

so that

$$0\leq\partial_t\,{\rm Im}F_{\rm I}(\omega+\sigma,0)\leq A\ (A \text{ is some constant}) \tag{A10}$$

for any $\omega(\geq\rho)$. Then, $\delta_1(\omega)$ in (A8) is evaluated by (A10) as

$$0\leq-\delta_1(\omega)\leq\frac{A}{\pi}\omega^{-3/2}\ \text{ as }\ \omega\to\infty. \tag{A11}$$

Next, (A9) gives the estimation

$$0\leq-\delta_2(\omega)\leq\frac{1}{\omega^3}\frac{1}{\pi}\int_{1/\omega^{3/4}}^{1-1/\omega}dx\frac{1}{x^2(1-x)}\ \text{ as }\ \omega\to\infty \tag{A12}$$

and

$$0\leq\delta_5(\omega)\leq\frac{1}{\pi}\frac{1}{\omega^2}\ \text{ as }\ \omega\to\infty. \tag{A13}$$

On the other hand, $\delta_3(\omega)+\delta_4(\omega)$ in (A8) is rewritten as

$$\begin{aligned} \delta_3(\omega)+\delta_4(\omega) &= \lim_{\epsilon\to+0}\frac{1}{\pi}\int_{\epsilon/\omega}^{1/\omega}d\xi\,\frac{1}{\xi}[\partial_t{\rm Im}F_{\rm I}(\omega+\omega\xi+\sigma,0)-\partial_t{\rm Im}F_{\rm I}(\omega-\omega\xi+\sigma,0)]\\ &\quad+\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\epsilon/\omega}^{1/\omega}d\xi[\partial_t{\rm Im}F_{\rm I}(\omega+\omega\xi+\sigma,0)+\partial_t{\rm Im}F_{\rm I}(\omega-\omega\xi+\sigma,0)]\equiv d_3(\omega)+d_4(\omega). \end{aligned} \tag{A14}$$

Then, $d_4(\omega)$ in (A14) is found by (A9) to be

$$0 \le d_4(\omega) \le \frac{2}{\pi}\frac{1}{\omega^4} \quad \text{as} \quad \omega \to \infty. \tag{A15}$$

Finally, the analyticity of $\partial_t F_{\mathrm{I}}(\omega+\sigma, 0)$ in the cut ω plane gives the following expression for $d_3(\omega)$ in (A14):

$$d_3(\omega) - \partial_t \mathrm{Re} F_{\mathrm{I}}(\omega+\sigma, 0) = -\frac{1}{2\pi i}\left(\int_{\omega'=e^{i\phi},0<\phi<\pi} d\omega' + \int_{\omega'=e^{i\phi},-\pi<\phi<0} d\omega'\right)\frac{1}{\omega'}\partial_t F_{\mathrm{I}}(\omega+\omega'+\sigma, 0). \tag{A16}$$

Therefore, (A7) and (A16) lead to

$$\left|d_3(\omega) - \partial_t \mathrm{Re} F_{\mathrm{I}}(\omega+\sigma, 0)\right| \le \beta\omega^{-2+[2(2+N)/\pi]1/\omega} \quad \text{as} \quad \omega \to \infty. \tag{A17}$$

Then (A1) is concluded from (3.3), (3.4), (A8), (A11)–(A15), and (A17).

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