Is massless quantum electrodynamics a free-field theory?

G. Eilam* and M. Glück
Institut für Physik, Universität Mainz, 6500 Mainz, West Germany
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We show that if the photon wave-function renormalization constant is finite, then in the limit of zero fermion mass quantum electrodynamics is a free-field theory.

Investigations of the short-distance behavior of quantum-electrodynamics (QED)[1,2] led to the possibility of finite renormalization constants. As shown by Johnson, Baker, and Willey[2] a necessary condition for finite QED is the eigenvalue equation

$$F_1(y)|_{y=\alpha_0}=0, \qquad (1)$$

where F_1 is the coefficient of the logarithmically divergent part of the one-fermion-loop contribution to the photon self-energy π, and α_0 is the unrenormalized coupling constant. Subsequently, it was shown by Adler[3] that the zero of F_1 is of infinite order (essential singularity), and the interesting possibility was raised[3] that the zero of F_1 occurs at the physical coupling constant α, i.e.,

$$\left.\frac{d^nF_1}{dy^n}\right|_{y=\alpha}=0 \quad (n\geq 0). \qquad (2)$$

Efforts[4,5] to calculate α have been unsuccessful so far, mainly[4,5] because F_1 is an infinite power series in y and very little can be deduced from a perturbative calculation.

From the scaling property of π it turns out that F_1 can be calculated in a theory in which the fermions are massless ($m=0$ where m is the renormalized mass). It is therefore clear that a better understanding of massless QED is essential for massive QED (note that "massless" and "massive" refer always to the fermion mass).

The following interesting theorem results in finite QED[2]: The $2n$-point function with only photons as external lines vanishes for $m\to 0$; thus at the eigenvalue

$$T_{n_1\gamma\to n_2\gamma}(q_1,\cdots,q_{2n};\,m)|_{m=0}=0 \quad (n_1+n_2=2n), \qquad (3)$$

where T denotes amplitudes and q_i are the photon momenta. From Eq. (3) it is obvious that in a world with massless fermions,[6] an experimentalist performing a $\gamma\gamma\to\gamma\gamma$ experiment will observe a zero cross section. Would he observe a null result for other processes such as $e^+e^-\to e^+e^-$ too?

Johnson and Baker[4] speculated that the above experimentalist will observe a null result for all possible reactions. In other words, massless QED is equivalent to a free-field theory. Therefore in the absence of fermion mass there are no interactions, in agreement with the experimental absence of charged massless fermions. Proving such an assertion along the lines leading to Eq. (3) seems impossible since the Federbush-Johnson theorem[7] used to prove Eq. (3) does not hold when external charged lines are present. We show here, relying on unitarity and crossing arguments, that massless QED is indeed a free-field theory.

Our aim is thus to show the following: If massive QED is finite (i.e., the photon wave-function renormalization is finite, and consequently all the other renormalization constants are finite[2]) then massless QED is a free-field theory. It is again important to note that one cannot attack the problem from low-order perturbation calculations; results should hold for the amplitudes as an infinite power series in the coupling constant.

Let us proceed in eight steps:

(1) Our starting point is Eq. (3), which—as stated above—is equivalent to Eq. (2).[2] Note that Eq. (3) holds for both virtual and real photons. From Eq. (3) it follows that, in particular,

$$\mathrm{Im}T_{\gamma\gamma\to\gamma\gamma}(s,\theta=0)|_{m=0}=0, \qquad (4)$$

where s is the square of the center-of-mass energy and θ is the scattering angle.

(2) From the optical theorem we then obtain

$$\sigma_{\gamma\gamma\to\text{all}}(s)|_{m=0}=0. \qquad (5)$$

Divergences may cause difficulty in applying the optical theorem to our case. However, since amplitudes that vanish for $m\to 0$ are proportional to an (even) power of m,[3] divergences in integrals over intermediate states are probably avoided.[8]

(3) Now it is obvious that each partial cross section with two photons as an incoming channel has to vanish separately; in particular,

$$T_{\gamma\gamma\to l^+l^-+k\gamma}|_{m=0}=0 \quad (k\geq 0) \qquad (6)$$

for all values of the kinematic variables, where

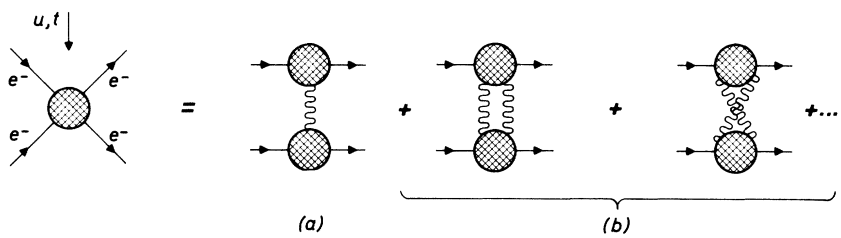

FIG. 1. Decomposition of $e^-e^- \to e^-e^-$ into an infinite sum of crossed-channels annihilation diagrams. Blobs represent n-point functions. Each diagram stands for exchanges in the t channel plus exchanges in the u channel.

l denotes a lepton.

(4) Applying crossing symmetry we can write

$$T_{(k+2)\gamma l^{\pm} \to l^{\pm}}|_{m=0} = 0 \quad (k \geq 0) \tag{7}$$

for all values of the kinematic variables. Although crossing symmetry may be difficult to apply to a massless theory, since all thresholds are open at zero energy we can apply it to the physical (finite) $m \neq 0$ case first. Then, if the amplitudes in Eqs. (6) and (7) are analytic continuations of each other for $m \neq 0$, the proportionality of the amplitude in Eq. (6) to m^2 implies the same for the amplitude in Eq. (7). Now by letting $m \to 0$ after the continuation Eq. (7) is recovered.[9]

Though the proportionality of amplitudes to a power of m is of great help, one would certainly like to have in the future a more rigorous treatment of crossing and analyticity.

(5) Let us now analyze the process $e^-e^- \to e^-e^-$.[10] The above process has, unlike $e^+e^- \to e^+e^-$, diagrams where one photon, two photons, etc., are exchanged in crossed channels (see Fig. 1).[11] But according to Eq. (7) only the one-photon crossed-channels annihilation diagrams [Fig. (1a)] survive, and we can write

$$T_{e^-e^- \to e^-e^-}(s,\theta)|_{m=0} = T^{1\gamma\,\mathrm{ann}}_{e^-e^- \to e^-e^-}(s,\theta)|_{m=0}; \tag{8}$$

note that Fig. 1(a) by itself is an infinite power series in the coupling constant.

(6) From the optical theorem

$$\mathrm{Im}T_{e^-e^- \to e^-e^-}(s,\theta=0) = s\sigma_{e^-e^- \to e^-e^-}(s). \tag{9}$$

Since the one-photon-exchange annihilation contribution [Fig. 1(a)] does not appear in Eq. (9)[12] we conclude from Eqs. (8) and (9) that

$$\sigma_{e^-e^- \to e^-e^-}(s)|_{m=0} = 0. \tag{10}$$

(7) Now from Eq. (10) each cross section with e^-e^- as an incoming channel vanishes separately, in particular,

$$T_{e^-e^- \to e^-e^-}(s,\theta)|_{m=0} = 0. \tag{11}$$

(8) Coming back to Eq. (8) and using Eq. (11) we find that the one-photon crossed-channels annihilation contribution to $e^-e^- \to e^-e^-$ [Fig. 1(a)] vanishes for all s and θ, i.e.,

$$T^{1\gamma\,\mathrm{ann}}_{e^-e^- \to e^-e^-}(s,\theta)|_{m=0} = 0. \tag{12}$$

We can now conclude from Eq. (12) that the vertex function in massless QED vanishes identically.

To summarize, it was shown that at the eigenvalue, if it exists, $m=0$ QED is a free-field theory. Thus the electromagnetic interaction of leptons is a consequence of their mass, and there are no massless charged leptons in agreement with experiment. It would be interesting to investigate whether such an imput will further simplify[13] the eigenvalue equation, thereby leading hopefully to a calculation of α.

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*Permanent address: Physics Department, Technion-Israel Institute of Technology, Haifa, Israel.

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therein] and J. Calmet and E. de Rafael [Phys. Lett. 56B, 181 (1975)].

[6]We consider here QED disconnected from other interactions. For the stability of the eigenvalue equations against strongly interacting particles see Ref. 3 and E. B. Manoukian, Phys. Rev. D 10, 1883 (1974); 10, 1894 (1974).

[7]P. G. Federbush and K. Johnson, Phys. Rev. 120, 1296 (1960).

[8]See for instance the discussion regarding convergence in Sec. III A of Ref. 3.

[9]In fact only the lepton has to be crossed to arrive at Eq. (7) since one can show—without applying crossing symmetry—that the amplitude for $(k+2)\gamma \to l^+ l^-$ $(k \geq 0)$ vanishes for $m=0$. For $k=0$ it has already been shown [see Eq. (5)]; for $k=1$ the generalized optical theorem for $\gamma\gamma\gamma \to \gamma\gamma\gamma$, which vanishes for $m=0$, includes several terms, only one of them with $\gamma\gamma\gamma \to$ all. However from Eq. (5) all those other terms vanish for $m=0$ [including a term which involves $\gamma\gamma \to \gamma +$ all, corresponding to Mueller's theorem: A. H. Mueller, Phys. Rev. D 2, 2963 (1970)]; therefore, $\gamma\gamma\gamma \to$ all vanishes for $m=0$. The proof is now easily extended to $k>1$.

[10]An identical analysis holds for the following processes with crossed-channels annihilation diagrams only: $e^\pm e^\pm \to e^\pm e^\pm$, $\mu^\pm \mu^\pm \to \mu^\pm \mu^\pm$, $e^\pm \mu^\pm \to e^\pm \mu^\pm$, and $e^\pm \mu^\mp \to e^\pm \mu^\mp$ [the eigenvalue equation and thus Eq. (3) are independent of the fermion species number (Ref. 3)].

[11]All diagrams with fermion loops vanish separately [see Eq. (3)].

[12]This is not the case for the one-photon s-channel annihilation contribution (to all orders in α) to $e^+e^- \to e^+e^-$ or to $e^+e^- \to \mu^+\mu^-$, since the vertex function may be *a priori* complex.

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