Light-cone quantization: Study of a soluble model with q-number anticommutator*

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A massless Dirac field interacting with a massive vector meson in two-dimensional space-time is considered in an attempt to gain further understanding of interacting systems quantized in light-cone coordinates. It is shown that the solution is equivalent not to the same system quantized on a spacelike surface, but rather to a single-component fermion model already known in the literature. The commutation relations among the field variables are found to be interaction dependent, with the fermion anticommutator having a q-number term in addition to the usual δ function. This latter feature may be traced to the fact that the number of degrees of freedom of the light-cone quantized system is fewer than that of the single-component fermion model to which it is equivalent, with this discrepancy in the number of independent dynamical variables being exactly compensated by the interaction-dependent terms in the commutation relations. Some possible applications to four-dimensional space-time are briefly discussed.

I. INTRODUCTION

Between the publication of Weinberg's paper on dynamics at infinite momentum[1] and the present time there have been numerous discussions of light-cone quantization. However, it is far from clear what really happens in the interacting system of quantum fields upon quantization in such coordinates, particularly since diagrammatic calculations frequently serve only to obscure the problem. Further complications arise as a consequence of the fact that the number of degrees of freedom in light-cone coordinates is only half that of the conventionally quantized theory. Despite this circumstance there remains a fairly widespread belief in the equivalence of such systems.

In an attempt to gain some insight into this problem we study here a soluble model field theory. It consists of a massless fermion field interacting with a massive vector field in a two-dimensional world. As in the case of the other soluble models[2,3] it is necessary to adopt a specific prescription for the construction of the singular current operator $j^\mu(x)$. However, in contrast with the spacelike-quantization case, the simple prescription

$$j^\mu(x)=\tfrac{1}{2}\lim_{x'\to x}\psi(x)\alpha^\mu q\psi(x') \tag{1.1}$$

leads to consistent results in light-cone coordinates. In the case of spacelike quantization the limiting definition (1.1) is not satisfactory since it implies the free-field result.[4] The reason why Eq. (1.1) avoids such a pitfall in light-cone coordinates is that the fermion field $\psi(x)$ is not independent of the vector field, and thus the current (1.1) already contains within it elements of the interaction. This stands in marked contrast to spacelike quantization, where one must explicitly introduce the interaction effect as an exponential factor according to the prescription

$$j^\mu(x)=\tfrac{1}{2}\lim_{x'\to x}\psi(x)\alpha^\mu q\,\exp\left[ieq\int_{x'}^{x}dy_\nu f^\nu(B)\right]\psi(x') \tag{1.2}$$

for the interacting system.[4] The fact that the Dirac field is not independent of the vector field means that the equal-time (x^+) (anti-) commutators are dependent on the interaction. It is because of this interaction dependence of the commutators that inconsistency problems customarily encountered when the charge and current densities commute[5] are not so severe as in the spacelike-quantization case.

In the following section the action principle and the Lagrange multiplier method are used to determine the (anti-) commutation relations. A phase transformation considered previously by Yan in the context of this model is shown not to be sufficient to maintain canonical commutation relations for the interacting system. The solution of the model is computed in Secs. III and IV, with various aspects of the solution being examined to confirm the internal consistency of the theory. The Thirring-model limit is discussed in Sec. V, and the operator properties of the model together with a proof of covariance are given in Sec. VI. The two-point functions for the free field are discussed in Appendix A. In Appendix B it is shown explicitly that the fermion Green's functions satisfy the equations of motion and that the prescription (1.1) is consistent with the solution. The Thirring model in light-cone coordinates is shown to be free in Appendix C.

Before going on to discuss quantization, a brief summary of the notation is in order. The coordinates $x^\pm$ are defined as

$$x^{\pm}=x_{\mp}=\frac{1}{\sqrt{2}}(x^1\pm x^0),$$

which implies for the metric tensor $g^{\mu\nu}$ the form

$$g^{\mu\nu}=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The scalar product is

$$x^\mu p_\mu = x^\mu p^\nu g_{\mu\nu}=x^+p^-+x^-p^+=-x^0p^0+x^1p^1,$$

with the x^+ coordinate taken to be the analog of the time coordinate. The Dirac matrices are

$$\alpha^+=\sqrt{2}\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \alpha^-=-\sqrt{2}\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \beta=\begin{pmatrix}0&-i\\i&0\end{pmatrix};$$

this displays the fact that the Dirac matrices $\alpha^{\pm}$ serve as projection operators, i.e.,

$$P_{\pm}=\pm\frac{1}{\sqrt{2}}\alpha^{\pm}.$$

We consequently use the notation

$$\psi_{\pm}(x)=P_{\pm}\psi(x)$$

and define the usual charge matrix

$$q=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

which acts in the internal charge space of the Hermitian field $\psi(x)$. Integration and summation signs are frequently suppressed so that repeated arguments (indices) mean integration (summation) in the usual fashion.

II. ACTION PRINCIPLE AND COMMUTATION RELATIONS

We consider the model field theory described by the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu B_\nu-\partial_\nu B_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}\mu_0{}^2B^\mu B_\mu + \tfrac{1}{2}i\psi\alpha^\mu\partial_\mu\psi+ej^\mu B_\mu+B^\mu J_\mu+j^\mu A_\mu, \qquad (2.1)$$

where the current operator is formally defined as

$$j^\mu(x)=\tfrac{1}{2}\psi(x)\alpha^\mu q\psi(x), \qquad (2.2)$$

and J^μ and A^μ are external classical sources. By the action principle[6] one infers the field equations

$$F^{\mu\nu}=\partial^\mu B^\nu-\partial^\nu B^\mu, \qquad (2.3)$$

$$\partial_\nu F^{\mu\nu}=ej^\mu+J^\mu-\mu_0{}^2B^\mu, \qquad (2.4)$$

$$\alpha^\mu\left(\frac{1}{i}\partial_\mu-eqB_\mu-qA_\mu\right)\psi(x)=0 \qquad (2.5)$$

and the infinitesimal generator

$$G=\frac{i}{\sqrt{2}}\int dx^-\psi_+\delta\psi_+-\tfrac{1}{2}\int dx^-(F^{+-}\delta B^+-\delta F^{+-}B^+), \qquad (2.6)$$

which gives the (anti-) commutation relations by means of the relation

$$\tfrac{1}{2}i\delta\chi=[\chi,G]. \qquad (2.7)$$

Although in Eq. (2.6) the fields ψ_+, B^+, and F^{+-} appear to be dynamically independent field variables, the actual number of dynamical variables is reduced by the constraint equation

$$\partial_-F^{+-}=ej^+-\mu_0{}^2B^+. \qquad (2.8)$$

(Here we neglect the external sources J^μ and A^μ, as they do not affect the commutation relations.)

It is immediately apparent that the (anti-) commutation relations among the fields are dependent on the quantum interaction as a consequence of the constraint equation (2.8). In this model, since the ψ field and the B^μ field are not completely independent, one finds that there is no need for the introduction of an explicit line integral factor in the current operator. One thus takes (1.1) as the definition of the current $j^\mu(x)$ where the limit is to be taken in the "spatial" (x^-) direction (thereby preserving time locality).

To determine the (anti-) commutation relations one writes Eq. (2.7) employing the Lagrange multiplier method as

$$\left[\chi(x),\frac{i}{\sqrt{2}}\int dx^{-\prime}\psi_+(x')\delta\psi_+(x')-\tfrac{1}{2}\int dx^{-\prime}[\partial_-{}^{-1}F(x')\delta B^+(x')+\partial_-{}^{-1}B^+(x')\delta F(x')]\right]$$
$$=\int dx^{-\prime}\{\tfrac{1}{2}i\delta(x^--x^{-\prime})\delta\chi(x')+\tfrac{1}{2}i\Lambda(x,x')[\delta F(x')+\mu_0{}^2\delta B^+(x')-e\delta j^+(x')]\}, \qquad (2.9)$$

where $F=\partial_-F^{+-}$ and $\delta j^+=\sqrt{2}\,\psi_+q\delta\psi_+$. All variations are now to be considered as being independent, with $\Lambda(x,x')$ to be determined such that Eq. (2.8) is satisfied.

For $\chi=\psi_+$, $x^+=x^{+\prime}$

$$\{\psi_+(x),\psi_+(x')\}=\frac{1}{\sqrt{2}}\delta(x^--x^{-\prime})+\frac{e}{2}[\Lambda^\psi(x,x'),q\psi_+(x')],$$
$$[\psi_+(x),F(x')]=-i\mu_0{}^2\partial'_-\Lambda^\psi(x,x'), \qquad (2.10)$$
$$[\psi_+(x),B^+(x')]=-i\partial'_-\Lambda^\psi(x,x'),$$

and

$$\partial'_- \Lambda^{\psi}(x,x') = \frac{ie}{2\mu_0^2}[\psi_+(x), j^+(x')]. \qquad (2.11)$$

For $\chi = B^+$, $x^+ = x^{+\prime}$

$$[B^+(x), \psi_+(x')] = e\Lambda^B(x,x')q\psi_+(x'),$$
$$[B^+(x), F(x')] = -i\partial'_-\delta(x^- - x^{-\prime}) - i\mu_0^2\partial'_-\Lambda^B(x,x'),$$
$$[B^+(x), B^+(x')] = -i\partial'_-\Lambda^B(x,x'), \qquad (2.12)$$

and

$$2i\mu_0^2\partial'_-\Lambda^B(x,x') + [B^+(x), ej^+(x')] = -i\partial'_-\delta(x^- - x^{-\prime}). \qquad (2.13)$$

For $\chi = F$, $x^+ = x^{+\prime}$

$$[F(x), \psi_+(x')] = e\Lambda^F(x,x')q\psi_+(x'),$$
$$[F(x), F(x')] = -i\mu_0^2\partial'_-\Lambda^F(x,x'), \qquad (2.14)$$
$$[F(x), B^+(x')] = -i\partial'_-\Lambda^F(x,x') - i\partial'_-\delta(x^- - x^{-\prime}),$$

and

$$2i\partial'_-\Lambda^F(x,x') + \frac{e}{\mu_0^2}[F(x), j^+(x')] = -i\partial'_-\delta(x^- - x^{-\prime}). \qquad (2.15)$$

It is convenient first to solve Eq. (2.13). To this end one notes that the singular behavior of the operator $\psi(x)\psi(x')$ for $x' \to x$ is the same as that for the free field in the standard way. Using this one can compute $\Lambda^B(x,x')$,

$$[B^+(y), ej^+(x)]$$
$$= \frac{e}{\sqrt{2}} \lim_{x'\to x} [B^+(y), \psi_+(x)q\psi_+(x')]$$
$$= \frac{e}{\sqrt{2}} \lim_{x'\to x} \{[B^+, \psi_+(x)]q\psi_+(x') + \psi_+(x)q[B^+, \psi_+(x')]\}$$
$$= \frac{e^2}{\sqrt{2}} \lim_{x'\to x} \psi_+(x)\psi_+(x')[\Lambda^B(y,x') - \Lambda^B(y,x)].$$

Interpreting the product of the operators at the same point as a vacuum expectation value, one has

$$[B^+(y), ej^+(x)] = -\frac{ie^2}{2\pi}\partial^x_-\Lambda^B(y,x), \qquad (2.16)$$

where use has been made of the fact that for the free field the two-point function is

$$i\langle T(\psi(x)\psi(x'))\rangle = -\frac{\alpha^+}{4\pi}\frac{1}{x^- - x^{-\prime} + i\epsilon\epsilon(x^+ - x^{+\prime})}, \qquad (2.17)$$

a result which is derived in Appendix A. Substituting Eq. (2.16) into Eq. (2.13) one has

$$\Lambda^B(x,x') = -\frac{1}{\mu_0^2}\frac{1}{1 - e^2/4\pi\mu_0^2}\delta(x^- - x^{-\prime}) \qquad (2.18)$$

and thus

$$[B^+(x), \psi_+(x')] = -\frac{ea}{2\mu_0^2}q\psi_+(x)\partial_-\delta(x^- - x^{-\prime}), \qquad (2.19)$$

$$[B^+(x), B^+(x')] = -\frac{ia}{2\mu_0^2}\partial_-\delta(x^- - x^{-\prime}), \qquad (2.20)$$

$$[B^+(x), F(x')] = \frac{ia}{2}(1 - e^2/2\pi\mu_0^2)\partial_-\delta(x^- - x^{-\prime}), \qquad (2.21)$$

where

$$a = \frac{1}{1 - e^2/4\pi\mu_0^2}.$$

Similarly one finds

$$[F(x), j^+(x')] = \frac{ie}{2\pi}\partial'_-\Lambda^F(x,x') \qquad (2.22)$$

and

$$\Lambda^F(x,x') = -\tfrac{1}{2}a\,\delta(x^- - x^{-\prime}), \qquad (2.23)$$

so that from Eq. (2.14) there follow

$$[F(x), \psi_+(x')] = -\tfrac{1}{2}eaq\psi_+(x)\delta(x^- - x^{-\prime}), \qquad (2.24)$$

$$[F(x), F(x')] = -\tfrac{1}{2}i\mu_0^2 a\partial_-\delta(x^- - x^{-\prime}). \qquad (2.25)$$

Solving for $\Lambda^{\psi}(x,x')$ is considerably more complicated inasmuch as $\Lambda^{\psi}(x,x')$ is an operator rather than a c-number function. However, one can use the results (2.19) and (2.24) to get

$$[j^+(x), \psi_+(x')] = -aq\psi_+(x)\delta(x^- - x^{-\prime}), \qquad (2.26)$$

which upon substitution into Eq. (2.11) yields for the anticommutator

$$\{\psi_+(x), \psi_+(x')\} = \frac{1}{\sqrt{2}}\delta(x^- - x^{-\prime}) - \frac{ie^2a}{8\mu_0^2}\epsilon(x^- - x^{-\prime})[q\psi_+(x), q\psi_+(x')]. \qquad (2.27)$$

Thus one has found the rather strange result that the commutators are dependent on the interaction and, furthermore, that the fermion anticommutator is a q-number function. As will be seen, however, this is an entirely consistent result and serves to indicate that considerable caution is necessary in inferring commutation relations in light-cone coordinates.

It is convenient at this point to derive a result which will be used later for consistency checks on the model. To this end one notes that the field equation (2.5) can be written as

$$\left(\frac{1}{i}\partial_+ - eqB_+ - qA_+\right)\psi_+(x) = 0, \qquad (2.28)$$

$$\left(\frac{1}{i}\partial_- - eqB_- - qA_-\right)\psi_-(x) = 0 \qquad (2.29)$$

since as already noted the Dirac matrices $\alpha^\pm$ serve as projection operators. Because Eq. (2.29) is a homogeneous constraint equation for $\psi_-(x)$, one immediately infers that $\psi_-(x)$ vanishes and consequently that

$$j^-(x) = 0. \qquad (2.30)$$

Despite the strange form of the result (2.30) it will subsequently be shown to be consistent by explicit reference to the solution of the model. From Eq. (2.28) one can evaluate[7]

$$\begin{aligned}\partial_+ j^+(x) &= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2}}\partial_+[\psi_+(x+\tfrac{1}{2}\epsilon)\,q\psi_+(x-\tfrac{1}{2}\epsilon)]\\ &= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2}}[\psi_+(x+\tfrac{1}{2}\epsilon)\,q\psi_+(x-\tfrac{1}{2}\epsilon)\\ &\qquad + \psi_+(x+\tfrac{1}{2}\epsilon)\,q\partial_+\psi_+(x-\tfrac{1}{2}\epsilon)]\\ &= \frac{e}{2\pi}\partial_- B^-(x) + \frac{1}{2\pi}\partial_- A^-(x), \qquad (2.31)\end{aligned}$$

which together with the field equations (2.3) and (2.4) implies the result

$$\partial_- B^- - \frac{a}{2}F^{+-} = \frac{a}{2\mu_0^{\,2}}\partial_\mu J^\mu + \frac{ea}{4\pi\mu_0^{\,2}}\partial_- A^-. \qquad (2.32)$$

It is of interest to note that because of Eq. (2.30) one can write

$$\partial_\mu j^\mu(x) = \partial_+ j^+(x)$$

so that Eq. (2.31) is expressible in the source-free limit in the covariant form

$$\partial_\mu j^\mu(x) = -\frac{ea}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}(x). \qquad (2.33)$$

Before proceeding further it might be well to note two rather remarkable properties of Eq. (2.33). First, one notes that the equation appears to be parity violating since j^μ and $F_{\mu\nu}$ are (presumably) ordinary vector and tensor, respectively. It can be shown, however, that the parity operator does not exist in light-cone coordinates by the following argument. In order that the Lagrangian and the Dirac equation

$$\alpha^\mu\partial'_\mu\psi'(x') = 0, \quad (x^{+\prime}, x^{-\prime}) = (x^+, -x^-)$$

transform covariantly under space inversion, there must exist a matrix S_P such that

$$\psi' = S_P\psi,$$
$$S_P^T\alpha^+ S_P = \alpha^+,$$
$$S_P^T\alpha^- S_P = -\alpha^-,$$

with S_P being a real matrix in order to maintain the Hermiticity of ψ'. Since α^+ and α^- are projection operators one writes the last relation as

$$\begin{aligned}S_P^T P_- S_P &= -P_-\\ &= S_P^T P_- P_- S_P\\ &= (P_- S_P)^T (P_- S_P);\end{aligned}$$

this shows that the element $(S_P)_{22}$ of S_P satisfies

$$[(S_P)_{22}]^2 = -1,$$

thereby contradicting the requirement that S_P be real.

The second point alluded to above is that Eq. (2.30) implies commutativity of the charge and current densities and therefore might lead to a well-known inconsistency. In the case of spacelike quantization,[5] for example, such commutativity, i.e.

$$[j^k(x), j^0(x')] = 0,$$

implies

$$\begin{aligned}0 &= [\partial_k j^k(x), j^0(x')]\\ &= -[\partial_0 j^0(x), j^0(x')]\end{aligned}$$

or

$$0 = [[j^0(x), H], j^0(x')],$$

where H is the Hamiltonian operator. The latter equation, upon taking its vacuum expectation value, leads to the unsatisfactory result

$$\langle 0|j^\mu(x) = 0.$$

Because of the anomaly (2.33), however, one has

$$[j^+(x), \partial_+ j^+(x')] = i\left(\frac{ea}{4\pi}\right)^2\delta(x^- - x^{-\prime}) \qquad (2.34)$$

for the light-cone quantized model and no consistency problem arises.

Before concluding this section it is worthwhile to comment on a recent attempt by Yan[8] to derive the commutation relations of the theory considered here in light-cone coordinates. His approach to the constraint equation (2.8) was to introduce an operator phase transformation

$$\psi(x) = \exp[ieq\partial_-^{-1}B^+(x)]\,\phi(x) \qquad (2.35)$$

and to identify

$$\begin{aligned}j^\mu(x) &= \tfrac{1}{2}\psi(x)\alpha^\mu q\psi(x)\\ &= \tfrac{1}{2}\phi(x)\alpha^\mu q\phi(x). \qquad (2.36)\end{aligned}$$

This, he argued, eliminates all the interaction-dependent terms in the infinitesimal generator (2.6), thereby implying canonical commutation relations in terms of the field $\phi(x)$ and the vector field. However, the identification (2.36) cannot be correct for the interacting system since from Eq. (1.1) one easily finds

$$j^+(x)=\tfrac{1}{2}\lim_{x'\to x}\psi(x)\alpha^+ q\psi(x')$$
$$=\tfrac{1}{2}\phi(x)\alpha^+ q\phi(x)+\frac{e}{2\pi}B^+(x). \tag{2.37}$$

Thus $j^\mu(x)$ contains elements of the interaction and the theory cannot be made to have simple canonical commutation relations in a two-dimensional world. Although the four-dimensional world is considerably more complicated, the results obtained for the two-dimensional model suggest that a straightforward quantization of the vector-gluon model in four-dimensional space-time is not possible in light-cone coordinates.

III. BOSON MATRIX ELEMENTS

The calculation of the Green's functions of the model is greatly facilitated by the use of the device of the external source.[2,3] Thus the action principle implies

$$\frac{\delta}{\delta e}\langle 0|0\rangle_{AJe}=i\left\langle 0\left|\int dx\, j^\mu(x)B_\mu(x)\right|0\right\rangle_{AJe}$$
$$=-i\frac{\delta}{\delta A_\mu(x)}\frac{\delta}{\delta J^\mu(x)}\langle 0|0\rangle_{AJe}.$$

Elementary integration immediately yields

$$\langle 0|0\rangle_{AJe}=\exp\left(-ie\int dx\frac{\delta}{\delta A^\mu(x)}\frac{\delta}{\delta J_\mu(x)}\right)\langle 0|0\rangle_{AJ} \tag{3.1}$$

where

$$\langle 0|0\rangle_{AJ}=\langle 0|0\rangle_{AJ,e=0}$$
$$=\langle 0|0\rangle_A\langle 0|0\rangle_J.$$

It is straightforward to evaluate $\langle 0|0\rangle_J$ using the relation

$$\delta_J\langle 0|0\rangle_J=i\langle 0|\delta J^\mu(x)B_\mu(x)|0\rangle$$

and the field equation for $e=0$

$$\partial^\mu\partial_\nu B^\nu-\partial^\nu\partial_\nu B^\mu+\mu_0{}^2B^\mu=J^\mu. \tag{3.2}$$

Thus one obtains

$$\langle 0|B^\mu|0\rangle_J=\langle 0|0\rangle_J\int dx' G_0^{\mu\nu}(x-x')J_\nu(x'), \tag{3.3}$$

where

$$G_0^{\mu\nu}(x-x')=i\langle T(B^\mu(x)B^\nu(x'))\rangle+\left\langle\frac{\delta B^\mu(x)}{\delta J_\nu(x')}\right\rangle \tag{3.4}$$

is found from (3.2) to have the form

$$G_0^{\mu\nu}(x)=\int\frac{d^2p}{(2\pi)^2}e^{ip\cdot x}\left(g^{\mu\nu}+\frac{p^\mu p^\nu}{\mu_0{}^2}\right)\frac{1}{p^2+\mu_0{}^2-i\epsilon}. \tag{3.5}$$

Integrating the functional differential equation

$$\frac{\delta}{\delta J_\mu(x)}\langle 0|0\rangle_J=i\langle 0|B^\mu(x)|0\rangle_J$$

yields the result

$$\langle 0|0\rangle_J=\exp\left(\frac{i}{2}\int dx\,dx' J_\mu(x)G_0^{\mu\nu}(x-x')J_\nu(x')\right) \tag{3.6}$$

To determine $\langle 0|0\rangle_A$ one requires a definition of the current operator $j^\mu(x)$ in the presence of an external source. A rather general form is

$$j^\mu(x)=\tfrac{1}{2}\lim_{x'\to x}\psi(x)\alpha^\mu q\exp\left(iq\int_{x'}^{x}dy_\nu f^\nu(A)\right)\psi(x') \tag{3.7}$$

with $f^\nu(A)$ linear in A^ν. The usual form for $f^\nu(A)$ used in other soluble models[2,3] is not applicable here since γ_5 is not meaningful for the massless Dirac field in light-cone coordinates. It turns out that the only definition of the current operator which leads to a covariant result is such that $f^\nu(A)$ does not contribute to the current when the limit is taken in the "spatial"(x^-) direction. Thus the current expectation value can be written as

$$\langle j^\mu(x)\rangle_A=\frac{i}{2}\lim_{x'^-\to x^-}\mathrm{Tr}\,\alpha^\mu q\frac{i\langle 0|T(\psi(x)\psi(x'))|0\rangle_A}{\langle 0|0\rangle_A}, \tag{3.8}$$

where the time-ordered product is given by

$$G(x,x')=i\frac{\langle 0|T(\psi(x)\psi(x'))|0\rangle_A}{\langle 0|0\rangle_A}+\left\langle\frac{\delta\psi_-(x)}{\delta\beta\eta(x')}\right\rangle_A. \tag{3.9}$$

The second term on the right-hand side of Eq. (3.9) comes from the fact that $\psi_-(x)$ is not an independent component[9] and is easily shown to be required if the Green's function $G(x,x')$ is to satisfy the equation of motion

$$\alpha^\mu\left(\frac{1}{i}\partial_\mu-qA_\mu\right)G(x,x')=\delta(x-x'). \tag{3.10}$$

In order to solve the differential equation (3.10) one sets, as usual,

$$G(x,x')=G_0(x,x')e^{iq(F(x)-F(x'))}, \tag{3.11}$$

where the free-field Green's function satisfies

$$\alpha^\mu\frac{1}{i}\partial_\mu G_0(x,x')=\delta(x-x') \tag{3.12}$$

thereby implying for $F(x)$ the equation

$$\alpha^\mu\partial_\mu F(x)=\alpha^\mu A_\mu(x). \tag{3.13}$$

Using the solution of Eq. (3.12) as given in Appendix A, the current expectation value (3.8) may be reduced to

$$\langle j^{\mu}(x)\rangle_A = -\frac{i}{2}\lim_{x'\to x}\mathrm{Tr}\,\alpha^{\mu}\frac{\alpha^{+}}{4\pi}\frac{1}{x^{-}-x^{-\prime}}qe^{iq(F(x)-F(x'))}, \qquad (3.14)$$

which shows

$$\langle j^{-}(x)\rangle_A = 0.$$

Since Eq. (3.13) can be written as

$$\partial^2 F(x) = 2P_{+}\partial_{-}A^{-} + 2P_{-}\partial_{+}A^{+},$$

it follows that

$$\mathrm{Tr}P_{+}F(x) = -4\int dx' D(x-x')\partial_{-}A^{-}(x'), \qquad (3.15)$$

where

$$-\partial^2 D(x) = \delta(x) \qquad (3.16)$$

subject to the usual causal boundary conditions. Substituting Eq. (3.15) into (3.14) one immediately finds

$$\langle j^{+}(x)\rangle_A = -\frac{1}{\pi}\partial^{+}\partial^{+}D(x-x')A^{-}(x')$$

or, in covariant form,

$$\langle j^{\mu}(x)\rangle_A = -\frac{1}{4\pi}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})(\partial^{\nu}+\epsilon^{\nu\beta}\partial_{\beta})D(x-x')A_{\nu}(x'). \qquad (3.17)$$

One can now easily solve the functional differential equation

$$\frac{\delta}{\delta A_{\mu}(x)}\langle 0|0\rangle_A = i\langle 0|j^{\mu}(x)|0\rangle_A,$$

obtaining the result

$$\langle 0|0\rangle_A = \exp\left(\frac{i}{2}\int dx\,dx' A_{\mu}(x)D_0^{\mu\nu}(x-x')A_{\nu}(x')\right), \qquad (3.18)$$

where

$$D_0^{\mu\nu} = -\frac{1}{4\pi}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})(\partial^{\nu}+\epsilon^{\nu\beta}\partial_{\beta})D(x). \qquad (3.19)$$

The complete result for the vacuum-to-vacuum transition amplitude is computed by substitution of Eqs. (3.6) and (3.18) into (3.1). After somewhat tedious calculations one obtains the result

$$\langle 0|0\rangle_{AJe} = C\exp\left(\frac{i}{2}\int dx dx'[J_{\mu}(x)G^{\mu\nu}(x-x')J_{\nu}(x') + A_{\mu}(x)D^{\mu\nu}(x-x')A_{\nu}(x') + 2A_{\mu}(x)M^{\mu\nu}(x-x')J_{\nu}(x')]\right), \qquad (3.20)$$

where

$$G^{\mu\nu}(x) = a\left(g^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{\mu^2}\right)\Delta(x) - \frac{e^2a}{4\pi\mu_0^2}\frac{\partial^{\mu}\partial^{\nu}}{\mu_0^2}\Delta(x) + \frac{e^2}{4\pi\mu_0^2}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})(\partial^{\nu}+\epsilon^{\nu\beta}\partial_{\beta})\frac{1}{\mu_0^2}[\Delta(x)-D(x)],$$

$$D^{\mu\nu}(x) = -\frac{1}{4\pi}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})(\partial^{\nu}+\epsilon^{\nu\beta}\partial_{\beta})\left[D(x)+\frac{e^2a}{4\pi\mu_0^2}\Delta(x)\right],$$

$$M^{\mu\nu}(x) = -\frac{e}{4\pi\mu_0^2}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})(\partial^{\nu}+\epsilon^{\nu\beta}\partial_{\beta})[D(x)-\Delta(x)] - \frac{ea}{4\pi\mu_0^2}(\partial^{\mu}+\epsilon^{\mu\alpha}\partial_{\alpha})\partial^{\nu}\Delta(x),$$

$$(\mu^2-\partial^2)\Delta(x) = \delta(x), \qquad (3.21)$$

$$\mu^2 = \mu_0^2 a = \mu_0^2/(1-e^2/4\pi\mu_0^2), \qquad (3.22)$$

and C is a constant independent of A^{μ} and J^{μ}. From Eq. (3.22) we see that the vector-meson is renormalized by a nonpolynomial form in the coupling constant just as in the case of the single-component fermion model.[3-10] In fact the vacuum transition amplitude (3.20) is equivalent to that of the single-component fermion model.

With this result it is trivial to generate matrix elements of an arbitrary number of boson operators by repeated functional differentiation of (3.20). Thus the current correlation function for vanishing external fields is

$$\frac{1}{\langle 0|0\rangle}\frac{1}{i}\frac{\delta}{\delta A_{\mu}(x)}\frac{\delta}{\delta A_{\nu}(x')}\langle 0|0\rangle\bigg|_{A=J=0} = i\langle T(j^{\mu}(x)j^{\nu}(x'))\rangle + \left\langle\frac{\delta j^{\mu}(x)}{\delta A_{\nu}(x')}\right\rangle = D^{\mu\nu}(x-x'),$$

from which one infers the equal-x^{+} commutation relation

$$[j^{+}(x), j^{+}(x')] = -\frac{ia}{2\pi}\partial_{-}\delta(x^{-}-x^{-\prime}) \qquad (3.23)$$

as well as

$$\frac{\delta j^{\mu}(x)}{\delta A_{\nu}(x')} = 0. \qquad (3.24)$$

Equation (3.24) is, of course, an eminently reasonable result in view of the fact that there is no line integral factor in the definition of the current and thus no explicit dependence on A^μ.

One similarly finds from the solution

$$[j^+(x), B^+(x')] = -i\frac{ea}{4\pi\mu_0^2}\partial_-\delta(x^- - x^{-'})$$

and (3.25)

$$[B^+(x), B^+(x')] = -i\frac{a}{2\mu_0^2}\partial_-\delta(x^- - x^{-'}),$$

results which are in agreement with those found in Sec. II by the Lagrange-multiplier technique.

Before we conclude this section it is of some interest to show explicitly that the solution (3.20) is consistent with the constraint (2.32). From (3.20) one thus computes

$$\begin{aligned}\langle\partial_- B^-(x)\rangle = &-\frac{a^2}{2}\partial_+\Delta J^+ + \frac{a}{2\mu_0^2}\partial_+\delta(x-x')J^+(x') \\ &+ a(1-\tfrac{1}{2}a)\partial_-\Delta J^- + \frac{a}{2\mu_0^2}\partial_-\delta(x-x')J^-(x') \\ &-\frac{ea^2}{4\pi}\partial_-\Delta A^- + \frac{ea}{4\pi\mu_0^2}\partial_-\delta(x-x')A^-(x')\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\langle F^{+-}(x)\rangle = &-\frac{ea}{2\pi}\partial_-\Delta A^- - a\partial_+\Delta J^+ \\ &+ a(1-e^2/2\pi\mu_0^2)\partial_-\Delta J^-,\end{aligned} \tag{3.27}$$

thereby obtaining

$$\langle\partial_- B^- - \tfrac{1}{2}aF^{+-}\rangle = \frac{a}{2\mu_0^2}\partial_\mu J^\mu + \frac{ea}{4\pi\mu_0^2}\partial_- A^-, \tag{3.28}$$

which is precisely the vacuum expectation value of Eq. (2.32). Furthermore, Eq. (3.26) implies by functional differentiation that the explicit dependence of the $B^-(x)$ field on the external source is

$$\left\langle\frac{\delta B^-(x)}{\delta A^-(x')}\right\rangle = \frac{ea}{4\pi\mu_0^2}\delta(x-x'), \tag{3.29}$$

$$\left\langle\frac{\delta B^-(x)}{\delta J^-(x')}\right\rangle = \frac{a}{2\mu_0^2}\delta(x-x'), \tag{3.30}$$

and

$$\left\langle\frac{\delta B^-(x)}{\delta J^+(x')}\right\rangle = \frac{a}{2\mu_0^2}\partial_-^{-1}\partial_+\delta(x-x') + \tfrac{1}{4}a^2\partial_-^{-2}\delta(x-x'). \tag{3.31}$$

The unusual form of Eq. (3.31) in which the dependence of B^- on J^+ is seen to involve a time derivative originates in the fact that, although $B^+(x)$ is the only independent dynamical component of the vector field, there are two field equations containing time derivatives, i.e.,

$$F^{+-} = \partial_- B^- - \partial_+ B^+ \tag{3.32}$$

and

$$-\partial_+ F^{+-} = J^- - \mu_0^2 B^-. \tag{3.33}$$

Equation (3.32) serves as an equation of motion while Eq. (3.33) leads to a secondary contraint equation (2.32) which is responsible for the time-derivative term in Eq. (3.31). It is worth noting that the results displayed in Eqs. (3.30) and (3.31) are useful for the evaluation of the time-ordered product of the vector-meson field. To this end one writes the usual covariant Green's function as the time-ordered product plus contact terms, i.e.,

$$G^{\mu\nu}(x-x') = i\langle T(B^\mu(x)B^\nu(x'))\rangle + S^{\mu\nu}(x-x'). \tag{3.34}$$

Reference to Eqs. (3.30) and (3.31) leads to the identification

$$S^{++} = 0,$$

$$S^{+-} = S^{-+} = \frac{a}{2\mu_0^2}\delta(x-x'),$$

and

$$S^{--} = \frac{a}{2\mu_0^2}\partial_-^{-1}\partial_+\delta(x-x') + \tfrac{1}{4}a^2\partial_-^{-2}\delta(x-x'),$$

which allows one to infer directly the form of the (noncovariant) time-ordered product.

IV. FERMION MATRIX ELEMENTS

The fermion Green's functions are obtained by repeated functional differentiation of the vacuum transition amplitude $\langle 0|0\rangle$ with respect to the spinor source $\eta(x)$ introduced in the Lagrangian (A6) of Appendix A. Since, however, the field $\psi_-(x)$ is not an independent dynamical variable, the Green's functions are expressed as time-ordered products of the field $\psi(x)$ plus contact terms which are local in time [e.g., Eq. (3.9)]. The general form for the Green's function is, therefore,

$$G_{AJe}(x_1\cdots x_{2n}) = i^n\frac{\langle 0|T(\psi(x_1)\cdots\psi(x_{2n}))|0\rangle_{AJe}}{\langle 0|0\rangle_{AJe}} + \text{contact terms}. \tag{4.1}$$

In order to evaluate (4.1) one extracts the effect of the quantum interaction into an exponential factor, thereby obtaining

$$G_{AJe}(x_1\cdots x_{2n}) = \frac{1}{\langle 0|0\rangle_{AJe}}\exp\left(-ie\int dx\frac{\delta}{\delta A^\mu}\frac{\delta}{\delta J_\mu}\right) \times\langle 0|0\rangle_{AJ}G_{AJ}(x_1\cdots x_{2n}). \tag{4.2}$$

Generalizing the result obtained earlier for the two-point function, one writes $G_{AJ}(x_1\cdots x_{2n})$ in the form

$$G_{AJ}(x_1\cdots x_{2n})=\exp\Big[i\sum_i q_i F(x_i)\Big]G_0(x_1\cdots x_{2n}), \tag{4.3}$$

where the free-field Green's function $G_0(x_1\cdots x_{2n})$ satisfies

$$(\alpha^\mu \frac{1}{i}\partial_\mu)_1 G_0(x_1\cdots x_{2n})=\sum_i(-1)^i\delta(x_1-x_i)G_0(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n}). \tag{4.4}$$

This implies that $F(x_i)$ satisfies Eq. (3.13), the solution of which is

$$F(x_i)=-2[P_+\partial_- D(x_i-x)A^-(x)+P_-\partial_+ D(x_i-x)A^+(x)]. \tag{4.5}$$

It is now relatively straightforward to obtain the result

$$G_{AJe}(x_1\cdots x_{2n})=\exp\Big\{i\sum_i q_i\int dx\,[A_\mu(x)N^\mu(x-x_i)+J_\mu(x)M^\mu(x-x_i)]\Big\}G_{00e}(x_1\cdots x_{2n}), \tag{4.6}$$

where

$$N^\mu(x-x_i)=P_+(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)\Big[D(x-x_i)+\frac{e^2a}{4\pi\mu_0^{\,2}}\Delta(x-x_i)\Big]$$
$$+P_-\Big\{(\partial^\mu-\epsilon^{\mu\nu}\partial_\nu)D(x-x_i)+\frac{e^2}{2\pi\mu_0^{\,2}}(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)[D(x-x_i)-\tfrac{1}{2}a(1-e^2/2\pi\mu_0^{\,2})\Delta(x-x_i)]\Big\}, \tag{4.7}$$

$$M^\mu(x-x_i)=\frac{e}{\mu_0^{\,2}}P_+\{a\partial^\mu\Delta(x-x_i)+(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)[D(x-x_i)-\Delta(x-x_i)]\}$$
$$+\frac{e}{\mu_0^{\,2}}P_-\Big\{a\partial^\mu\Delta(x-x_i)+(\partial^\mu-\epsilon^{\mu\nu}\partial_\nu)[D(x-x_i)-\Delta(x-x_i)]+\frac{e^2}{2\pi\mu_0^{\,2}}(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)[D(x-x_i)-\Delta(x-x_i)]\Big\},$$

and

$$G_{00e}(x_1\cdots x_{2n})=\exp\Big[i\frac{e^2a}{2\mu_0^{\,2}}\sum_{ij}q_iq_j\Delta(x_i-x_j)\Big]G_0(x_1\cdots x_{2n}).$$

It is to be noted here that the factors proportional to P_- in the solution are associated with the contact terms $\langle\delta\psi_-(x_i)/\delta\beta\eta(x_j)\rangle$ and therefore do not play any physical role. Since $\psi_-(x)$ and $j^-(x)$ vanish (as shown in the previous sections), it is convenient to let

$$ej^\mu(x)B_\mu(x)\to ej^+(x)B^-(x)$$

in the Lagrangian (2.1). Thus a solution physically equivalent to (4.7) is obtained by the replacement

$$e\int dx\frac{\delta}{\delta A^\mu(x)}\frac{\delta}{\delta J_\mu(x)}\to e\int dx\frac{\delta}{\delta A^-(x)}\frac{\delta}{\delta J^+(x)}$$

in Eq. (4.2). As a result of this operation, one obtains in place of (4.7)

$$N^\mu(x-x_i)=P_+(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)\Big[D(x-x_i)+\frac{e^2a}{4\pi\mu_0^{\,2}}\Delta(x-x_i)\Big]+P_-(\partial^\mu-\epsilon^{\mu\nu}\partial_\nu)D(x-x_i),$$

$$M^\mu(x-x_i)=\frac{e}{\mu_0^{\,2}}P_+\{a\partial^\mu\Delta(x-x_i)+(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)[D(x-x_i)-\Delta(x-x_i)]\}, \tag{4.8}$$

and

$$G_{00e}(x_1\cdots x_{2n})=\exp\Big[i\frac{e^2a}{2\mu_0^{\,2}}\sum_{ij}q_iq_jP_+\Delta(x_i-x_j)\Big]G_0(x_1\cdots x_{2n}).$$

It is particularly interesting to note that the two-point function $G_e(x,x')$ in the absence of any external sources is

$$G_e(x,x') = G_0(x-x')\exp\left\{-i\frac{e^2a}{\mu_0^2}P_+[\Delta(x-x')-\Delta(0)]\right\}, \tag{4.9}$$

which shows that there is no infrared divergence, just as in the case of the single-component fermion model.[3]

Upon considering the discontinuity of the Green's function

$$i^n\langle T(j^+(y)\psi(x_1)\cdots\psi(x_{2n}))\rangle\Big|_{y^+=x^+_i-\epsilon}^{y^+=x^+_i+\epsilon}$$

$$=\sum_i q_i N^*(y-x_i)\Big|_{-\epsilon}^{\epsilon} G_{AJe}(x_1\cdots x_{2n}), \tag{4.10}$$

and noting that

$$\partial_- D(x)\Big|_{x^+=-\epsilon}^{x^+=\epsilon} = -\tfrac{1}{2}\delta(x^-)$$

$$=\partial_-\Delta(x)\Big|_{x^+=-\epsilon}^{x^+=\epsilon},$$

one immediately infers the commutation relation

$$[j^+(x),\psi_+(x')]_{x^+=x^{+\prime}} = -\ aqP_+\psi(x)\delta(x^- - x^{-\prime}). \tag{4.11}$$

Similarly from the Green's function $\langle T(B^+\psi\cdots\psi)\rangle$ one obtains the commutator

$$[B^+(x),\psi_+(x')]_{x^+=x^{+\prime}} = -\frac{ea}{2\mu_0^2}q\psi_+(x)\delta(x^- - x^{-\prime}) \tag{4.12}$$

in agreement with the result of Sec. II.

As shown in the preceding section, the $B^-(x)$ field is dependent on the external sources. To find the explicit dependence of $B^-(x)$ on the spinor source $\eta(x)$ we consider the Lagrangian (A6) with $A^\mu=0=J^\mu$. Following the same line of reasoning as in Eqs. (2.31) and (2.32), one finds

$$\langle\partial_- B^- - \tfrac{1}{2}aF^{+-}\rangle = \frac{iea}{2\mu_0^2}\langle\psi(x)q\beta\eta(x)\rangle. \tag{4.13}$$

Taking two functional derivatives with respect to $\eta(x)$, one obtains

$$i\langle T^*([\partial_- B^-(x) - \tfrac{1}{2}aF^{+-}(x)]\psi(x_1)\psi(x_2))\rangle\Big|_{\eta=0}$$

$$= -\frac{ea}{2\mu_0^2}\sum_{i=1}^{2} q_i\delta(x-x_i)G_e(x_1,x_2), \tag{4.14}$$

where the T^* product is the covariant Green's function (i.e., it is defined in terms of functional derivatives). Equation (4.14) shows that

$$i^n\langle T^*(\partial_- B^-(x)\psi(x_1)\cdots\psi(x_{2n}))\rangle$$

$$= i^n\langle T(\partial_- B^-(x)\psi(x_1)\cdots\psi(x_{2n}))\rangle$$

$$-\frac{ea}{2\mu_0^2}\sum_i q_i\delta(x-x_i)G_{AJe}(x_1\cdots x_{2n}) \tag{4.15}$$

and

$$\frac{\delta B^-(x)}{\delta\beta\eta(x')} = -\frac{iea}{4\mu_0^2}q\psi_+(x')\delta(x^+ - x^{+\prime})\epsilon(x^- - x^{-\prime}). \tag{4.16}$$

Using the above result one can show that the constraint equation (2.32) is also satisfied by the fermion Green's function. From the solution (4.6) and (4.8) one obtains

$$i^n\langle T(F^{+-}(x)\psi(x_1)\cdots\psi(x_{2n}))\rangle = ea\sum_i q_i\Delta(x-x_i)G_{AJe}(x_1\cdots x_{2n}) + \langle F^{+-}(x)\rangle G_{AJe}(x_1\cdots x_{2n})$$

and

$$i^n\langle T^*(\partial_- B^-(x)\psi(x_1)\cdots\psi(x_{2n}))\rangle = \left[\frac{ea}{2\mu_0^2}\sum_i q_i\partial^2\Delta(x-x_i) + \langle\partial_- B^-(x)\rangle\right]G_{AJe}$$

$$= \left[\frac{ea^2}{2}\sum_i q_i\Delta(x-x_i) + \langle\partial_- B^-(x)\rangle\right]G_{AJe}(x_1\cdots x_{2n})$$

$$-\frac{ea}{2\mu_0^2}\sum_i q_i\delta(x-x_i)G_{AJe}(x_1\cdots x_{2n}).$$

It follows from these and Eq. (4.15) that

$$\left\langle T\left(\left(\partial_- B^- - \frac{a}{2}F^{+-} - \frac{ea}{4\pi\mu_0^2}\partial_- A^- - \frac{a}{2\mu_0^2}\partial_\mu J^\mu\right)\psi(x_1)\cdots\psi(x_{2n})\right)\right\rangle = 0 \tag{4.17}$$

in agreement with the constraint equation (2.32).

It is shown in Appendix B that the fermion Green's functions satisfy the equation

$$P_+\left(\alpha^\mu \frac{1}{i}\partial_\mu\right)_1 G_{AJe}(x_1\cdots x_{2n}) = eq_1\alpha^+ i^n\langle T^*(B^-(x_1)\psi(x_1)\cdots\psi(x_{2n}))\rangle + q_1\alpha^+A^-(x_1)G_{AJe}(x_1\cdots x_{2n}) + P_+\sum_i(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n}) . \quad (4.18)$$

Of particular interest is the equation of motion for the two-point function:

$$P_+\alpha^\mu\left(\frac{1}{i}\partial_\mu - eq\langle B_\mu\rangle - qA_\mu + ieq\frac{\delta}{\delta J^\mu(x)}\right)G_{AJe}(x,y) = P_+\delta(x-y) , \quad (4.19)$$

which is the Schwinger-Dyson equation. In the absence of external sources it can be written as

$$P_+\partial_+G_e(x) = P_+\frac{i}{\sqrt{2}}\delta(x) - \frac{ie^2a}{\mu_0^{\,2}}P_+\partial_+\Delta G_e(x) . \quad (4.20)$$

Integrating Eq. (4.20) over x^+ and taking the limit, one finds

$$\lim_{\epsilon\to 0}P_+G_e(x)\Big|_{x^+=-\epsilon}^{x^+=\epsilon} = \frac{i}{\sqrt{2}}P_+\delta(x^-) + \frac{ie^2a}{8\mu_0^{\,2}}\epsilon(x^-)P_+ \times \lim_{\epsilon\to 0}[G_e(x^-,\epsilon)+G_e(x^-,-\epsilon)] , \quad (4.21)$$

which implies the correct anticommutation relation (2.27) and is consistent with the definition of the Green's function

$$P_+G(x,x') = \begin{cases} i\langle\psi_+(x)\psi_+(x')\rangle, & x^+ > x^{+\prime} \\ -i\langle\psi_+(x')\psi_+(x)\rangle, & x^+ < x^{+\prime} . \end{cases} \quad (4.22)$$

Although such considerations show that the solution implies the correct anticommutation relation, there is some difficulty if one tries to obtain Eq. (4.21) directly from the two-point function (4.9). This arises from the fact that $\Delta(x)$ has a discontinuity at $x^+=0$ and is largely a mathematical problem rather than a physical one. In particular this difficulty disappears if one introduces a cutoff function

$$e\int dx j^\mu(x)B_\mu(x) \to \int dx\,dx'\,e(x-x')j^\mu(x)B_\mu(x') \quad (4.23)$$

in the Lagrangian (2.1), and takes the local limit

$$e(x)\to e|_{\text{const}}\,\delta(x)$$

after the calculation. The process of finding matrix elements remains unchanged except for the fact that now the coupling e is interpreted as an integral operator such that

$$ef(x) = \int dy\, e(x-y)f(y) .$$

Thus the two-point function is

$$G_e(x) = G_0(x)\exp\left\{-\frac{ie^2a}{\mu_0^{\,2}}P_+[\Delta(x)-\Delta(0)]\right\},$$

where the coupling e is an integral operator as above and serves as a smearing function. Because of this smearing function the problem referred to earlier disappears and the discontinuity of $G_e(x)$ can be written as

$$P_+G_e(x)\Big|_{x^+=-\epsilon}^{x^+=\epsilon} = \frac{i}{\sqrt{2}}P_+\delta(x^-) + \frac{ie^2a}{\mu_0^{\,2}}\frac{\alpha^+}{4\pi x^-}\Delta(x)\Big|_{x^+=-\epsilon}^{x^+=\epsilon}\exp\left\{-\frac{ie^2a}{\mu_0^{\,2}}P_+[\Delta(x)-\Delta(0)]_{x^+=0}\right\} . \quad (4.24)$$

It is easily shown that Eq. (4.24) implies the correct anticommutator (2.27) upon taking the local limit.

Having thus demonstrated in some detail the consistency of the solution, one can compare the results obtained with those of other soluble models known in the literature. One finds that the model is equivalent to the single-component fermion model,[3,10] but not to the same model [i.e., that described by the Lagrangian (2.1)] quantized on a spacelike surface.[2] We note here that the single-component fermion model has only one independent component for the Dirac field, as in the case of the model (2.1) quantized in light-cone coordinates. Whether the fermion fields have the same number of independent components seems to be directly related to the question of equivalence between two quantization schemes. This is also true in the case of the Thirring model, where there is a difference between the number of independent components of the Dirac field in the two quantization schemes, which leads to inequivalent theories.

V. THE THIRRING-MODEL LIMIT

One sees from the Lagrangian (2.1) that in the limits

$$e^2 \to \infty\,,\quad {\mu_0}^2 \to \infty\,,\quad \text{and}\quad e^2/{\mu_0}^2 \to \lambda\,, \tag{5.1}$$

the Lagrangian becomes that of the Thirring model. As shown in Appendix C, the Thirring model in light-cone coordinates is free, a result which will now be shown to follow also from the solution of the model considered in this paper.

One first notes that in the limit (5.1) the boson propagator becomes

$$\Delta(x) = \frac{1}{\mu^2 - \partial^2} \to 0\,. \tag{5.2}$$

One thus finds

$$G^{\mu\nu}(x) \to 0\,,$$

$$D^{\mu\nu}(x) \to -\frac{1}{4\pi}(\partial^\mu + \epsilon^{\mu\alpha}\partial_\alpha)(\partial^\nu + \epsilon^{\nu\beta}\partial_\beta)D(x) = D_0^{\mu\nu}(x), \tag{5.3}$$

$$M^{\mu\nu}(x) \to 0\,,$$

so that the vacuum transition amplitude becomes

$$\langle 0 | 0\rangle_{A\lambda} = \exp\left[\tfrac{1}{2}i \int dx dx'\, A_\mu(x) D_0^{\mu\nu}(x,x') A_\nu(x')\right], \tag{5.4}$$

i.e., the free-field result.

In the limit (5.1) the fermion Green's function (4.8) becomes

$$N^\mu(x - x_i) \to P_+(\partial^\mu + \epsilon^{\mu\nu}\partial_\nu)D(x - x_i) + P_-(\partial^\mu - \epsilon^{\mu\nu}\partial_\nu)D(x - x_i) = N_0^\mu(x - x_i)\,, \tag{5.5}$$

$$M^\mu(x - x_i) \to 0\,,$$

$$G_{00e}(x_1 \cdots x_{2n}) \to G_0(x_1 \cdots x_{2n})\,,$$

so that

$$G_{A\lambda}(x_1 \cdots x_{2n}) = \exp\Big(i \sum_i q_i \int dx\, A_\mu(x) N_0^\mu(x - x_i)\Big) \times G_0(x_1 \cdots x_{2n})\,. \tag{5.6}$$

Again this is the Green's function for the system with only external sources. Equations (5.4) and (5.6) thus show that the solutions in the previous sections indeed approach the expected Thirring-model limit.

On the other hand if one takes the limit (5.1) in the commutation relations found in the previous sections, they do not agree with those derived from Eqs. (5.4) and (5.6) (i.e., the free-field result). This indicates that here the operation of taking the discontinuity and the operation of taking the limit (5.1) do not commute (unlike the case of spacelike quantization[2,3]). The reason for this is that when one determines the commutation relations (as for example in Sec. II) more information about the internal properties of the model was required than in the case of spacelike quantization. This also means that the commutation relations depend on the internal structure of the system (as is apparent from the fact that the commutation relations involve details of the quantized interaction).

Although the model has the Thirring-model limit as shown above, the ${\mu_0}^2 \to 0$ limit clearly does not exist (just as it does not exist in the case of the single-component fermion model[3]). This shows that a gauge field theory in two-dimensional light-cone coordinates is not consistent, a striking confirmation of results obtained in Ref. 7 to the effect that there exists an inconsistency in the U(1) version of 't Hooft's quark-binding model.[11]

VI. OPERATOR FORMALISM

Thus far we have completed the calculation of the Green's functions and considered the consistency of the solution of the model. In this section the operator structure of the theory is examined, which provides the final set of consistency checks on the model. First, one verifies that the prescription (1.1) together with the anticommutator (2.27) allows one to derive the result (4.11) for the commutator of $j^+(x)$ with the fermion field. Thus, one writes

$$\begin{aligned}[j^+(x), \psi_+(y)] &= \frac{1}{\sqrt{2}} \lim_{x' \to x} [\psi_+(x) q\{\psi_+(x'), \psi_+(y)\} - \{\psi_+(x), \psi_+(y)\} q \psi_+(x')] \\ &= -q\psi_+(x)\delta(x^- - y^-) - i\frac{e^2 a}{8{\mu_0}^2}\frac{1}{\sqrt{2}}\lim_{x'\to x}[\psi_+(x)\psi_+(x')q\psi_+(y) + q\psi_+(y)\psi_+(x)\psi_+(x')][\epsilon(x^{-\prime} - y^-) - \epsilon(x^- - y^-)].\end{aligned} \tag{6.1}$$

Interpreting $\psi_+(x)\psi_+(x')$ as its vacuum expectation value one obtains the result given in Eq. (4.11). In Eq. (6.1) it is seen that the interaction-dependent part of Eq. (4.11) comes from the q-number part of the anticommutator $\{\psi_+(x), \psi_+(y)\}$. Thus we have a situation in which the noncanonical factor $a = 1/(1 - e^2/4\pi\mu_0^2)$ in the commutator (4.11) results from the q-number part of the anticommutator, in contrast with the single-component fermion model,[3] where the same factor arises from the exponential factor in the definition of the current operator (1.2).

We now investigate the covariance of the model by considering the Dirac-Schwinger condition.[12] By the action principle one obtains the energy-momentum tensor operators

$$T^{+-}(x) = -\tfrac{1}{2}F^{+-}(x)F^{+-}(x)\,, \qquad (6.2)$$

$$T^{++}(x) = -\frac{i}{\sqrt{2}}\psi_+(x)\partial_-\psi_+(x) - \partial_- F^{+-}(x)B^+(x)\,. \qquad (6.3)$$

It is easy to show that the tensor operators (6.2) and (6.3) satisfy

$$[P^\mu, \psi_+] = i\partial^\mu\psi_+(x) \qquad (6.4)$$

and

$$[P^\mu, B^\nu(x)] = i\partial^\mu B^\nu(x)\,, \qquad (6.5)$$

where

$$P^\mu = \int dx^- T^{+\mu}(x)\,. \qquad (6.6)$$

In order to display covariance one computes the energy-momentum density commutator

$$[T^{++}(x), T^{+-}(x')] = [T^{++}(x), -\tfrac{1}{2}F^{+-}(x')F^{+-}(x')]\,.$$

Noting

$$[\psi_+\partial_-\psi_+, F^{+-}(x')] = -\tfrac{1}{4}ea\{\psi_+\partial_-[q\psi_+(x)\epsilon(x^- - x^{-\prime})] + q\psi_+(x)\partial_-\psi_+(x')\epsilon(x^- - x^{-\prime})\}$$
$$= -\frac{ea}{\sqrt{2}}j^+(x)\delta(x^- - x^{-\prime})$$

and

$$[FB^+, F^{+-}(x')] = \tfrac{1}{2}ia(1 - e^2/2\pi\mu_0^2)\partial_- F^{+-}(x)\delta(x^- - x^{-\prime}) + \tfrac{1}{2}i\mu_0^2 aB^+(x)\delta(x^- - x^{-\prime})\,,$$

one obtains the result

$$[T^{++}(x), T^{+-}(x')] = -i[\tfrac{1}{2}\partial_- F^{+-}(x)F^{+-}(x') + \tfrac{1}{2}F^{+-}(x')\partial_- F^{+-}(x)]\delta(x^- - x^{-\prime})$$
$$= i[\partial_- T^{+-}(x)]\delta(x^- - x^{-\prime})\,. \qquad (6.7)$$

Following Schwinger[12] one can now show that Eq. (6.7) is one of the sufficiency conditions for covariance. Integrating Eq. (6.7) over $x^{-\prime}$ one finds

$$[T^{++}(x), P^-] = i\partial_- T^{+-}(x)\,, \qquad (6.8)$$

which is the assertion of local energy conservation

$$\partial_\mu T^{+\mu} = 0\,. \qquad (6.9)$$

Noting

$$J^{+-} = \int dx^-(x^+T^{+-} - x^-T^{++})$$
$$= x^+P^- - \int dx^- x^- T^{++}\,, \qquad (6.10)$$

Eq. (6.8) is seen to imply

$$[x^+P^- - J^{+-}, P^-] = i\int dx^- x^- \partial_- T^{+-}\,,$$

or (6.11)

$$[J^{+-}, P^-] = iP^-\,,$$

which is one of the commutation relations associated with the infinitesimal generators of the inhomogeneous Lorentz group.

After straightforward calculation one similarly obtains

$$[T^{++}(x), T^{++}(x')] = -i\left[-\frac{i}{\sqrt{2}}\psi_+(x)\partial_-\psi_+(x') - \frac{i}{\sqrt{2}}\psi_+(x')\partial_-\psi_+(x)\right]\partial_-\delta(x^- - x^{-\prime}) - i[-B^+(x)F(x') - F(x)B^+(x')]\partial_-\delta(x^- - x^{-\prime})\,,$$

which can be written as

$$[T^{++}(x), T^{++}(x')] = -i[T^{++}(x) + T^{++}(x')]\partial_-\delta(x^- - x^{-\prime})\,. \qquad (6.12)$$

If one integrates (6.12) over $x^{-\prime}$,

$$[T^{++}(x), P^+] = -i\partial_- T^{++}\,. \qquad (6.13)$$

Integrating this over x^- with the factor x^- one finds

$$[x^+P^- - J^{+-}, P^+] = iP^+$$

or (6.14)

$$[J^{+-}, P^+] = -iP^+\,,$$

as expected.

Thus we have shown that the model satisfies Dirac-Schwinger–type conditions (6.7) and (6.12), and that the latter are sufficient for the covariance of the model.

VII. CONCLUDING REMARKS

In the preceding sections it has been shown that the model field theory (2.1) quantized in light-cone coordinates is consistent and possesses unusual features which are at variance with some currently accepted views of light-cone quantization. In particular, since canonical quantization has been found not to be possible for the model, it is required that explicit reference be made to interaction terms in the derivation of the commutation relations. In fact, a reasonable view of the results of this paper with regard to the question of commutators is that one has found that the decrease in the number of independent dynamical variables associated with light-cone coordinates has been compensated for by the appearance of interaction-dependent terms in the commutation relations. For more realistic (i.e., four-dimensional) theories such results suggest that one has to examine most carefully these aspects, taking into account both the difference in the number of degrees of freedom as well as the need for a fully consistent definition of the singular operator $j^{\mu}(x)$.

APPENDIX A: FERMION TWO-POINT FUNCTIONS FOR THE FREE FIELD

The fermion Green's function for the free field satisfies the equation

$$\alpha^{\mu}\frac{1}{i}\,\partial_{\mu}G_0(x-x')=\delta(x-x')\ , \tag{A1}$$

the solution of which would ordinarily be found by Fourier-transform methods to be

$$G_0(x)=\int\frac{dp^+dp^-}{(2\pi)^2}\,e^{ip\cdot x}\,\frac{\alpha^+p^+ + \alpha^-p^-}{p^2-i\epsilon}\ . \tag{A2}$$

It is to be noted, however, that the $G_0^{--}(x) = P_-G_0(x)P_-$ component satisfies the time-independent equation

$$\alpha^-\frac{1}{i}\,\partial_-G_0^{--}(x)=P_-\delta(x)\ , \tag{A3}$$

so that the $i\epsilon$ precription in Eq. (A2) must be taken to be a shorthand notation for

$$\frac{p^+}{p^2-i\epsilon}=\frac{p^+}{p^2-i\epsilon}\ ,$$

$$\frac{p^-}{p^2-i\epsilon}=\frac{1}{4}\left[\frac{1}{p_-+i\epsilon}+\frac{1}{p_--i\epsilon}\right],$$

thereby implying the result

$$G_0(x)=-\frac{\alpha^+}{4\pi}\,\frac{1}{x^-+i\epsilon\epsilon(x^+)}+i\,\frac{\alpha^-}{4}\,\delta(x^+)\epsilon(x^-)\ . \tag{A4}$$

In terms of the time ordered product the two point function can be written as

$$G_0(x,x')=i\,\frac{\langle 0|T(\psi(x)\psi(x'))|0\rangle}{\langle 0|0\rangle}+\left\langle 0\left|\frac{\delta\psi(x)}{\delta\beta\eta(x')}\right|0\right\rangle, \tag{A5}$$

where the second term on the right-hand side comes from the fact that $\psi_-(x)$ is not a dynamically independent field.[9] The spinor source $\eta(x)$ has been introduced by the replacement

$$\mathcal{L}\to\mathcal{L}+\psi(x)\beta\eta(x)\ , \tag{A6}$$

which has the advantage of allowing one to define the fermion Green's functions as repeated variational derivatives with respect to $\eta(x)$. In order to evaluate the second term of Eq. (A5), one considers the field equation

$$\partial_-\psi_-(x)=\tfrac{1}{2}i\,\alpha^-\beta\eta(x) \tag{A7}$$

in the absence of $A^{\mu}(x)$. Solving the differential equation (A7) one finds

$$\left\langle\frac{\delta\psi_-(x)}{\delta\beta\eta(x')}\right\rangle=i\,\tfrac{1}{4}\,\alpha^-\delta(x^+-x^{+\prime})\epsilon(x^--x^{-\prime})\ , \tag{A8}$$

which implies

$$G_0(x,x')=i\,\langle T(\psi(x)\psi(x'))\rangle + i\,\tfrac{1}{4}\,\alpha^-\delta(x^+-x^{+\prime})\epsilon(x^--x^{-\prime}) \tag{A9}$$

for the two-point functions of the free field. Comparing Eqs. (A4) and (A9) one obtains for the time-ordered product

$$i\,\langle T(\psi(x)\psi(x'))\rangle=-\frac{\alpha^+}{4\pi}\,\frac{1}{x^--x^{-\prime}+i\epsilon\epsilon(x^+-x^{+\prime})}\ . \tag{A10}$$

APPENDIX B: EQUATION OF MOTION FOR THE GREEN'S FUNCTIONS

In this appendix it is shown that the fermion Green's function (4.6) satisfies the correct equation of motion and that the prescription (1.1) is consistent with the solution. Taking the time derivative of Eq. (4.6) one finds

$$\partial_+^1 G_{AJe}(x_1\cdots x_{2n})=\left\{\partial_+^1\exp\left[i\sum_i q_i(A_{\mu}N^{\mu}+J_{\mu}M^{\mu})\right]\right\}G_{00e}(x_1\cdots x_{2n}) + \exp\left[i\sum_i q_i(A_{\mu}N^{\mu}+J_{\mu}M^{\mu})\right]\partial_+^1G_{00e}(x_1\cdots x_{2n})\ ; \tag{B1}$$

$$\text{first term of (B1)} = iq_1[A_\mu(x)\partial_+^1 N^\mu(x-x_1) + J_\mu(x)\partial_+^1 M^\mu(x-x_1)]G_{AJe}$$

$$= iq_1\left[aA^-(x_1) - \frac{e^2a}{4\pi}\Delta(x_1-x)A^-(x) + \frac{ea}{2\mu_0^2}J^-(x_1) - \tfrac{1}{2}ea(a-2)\Delta(x_1-x)J^-(x)\right.$$

$$\left. + \frac{ea}{2\mu_0^2}\partial_-^{-1}\partial_+J^+(x_1) - \tfrac{1}{2}ea^2\partial_-^{-1}\partial_+\Delta(x_1-x)J^+(x)\right]G_{AJe} \;.$$

From Eq. (3.26) this can be written as

$$\text{first term of (B1)} = iq_1[e\langle B^-(x_1)\rangle + A^-(x_1)]G_{AJe}(x_1\cdots x_{2n}) \;, \tag{B2}$$

$$\text{second term of (B1)} = i\frac{e^2a}{2\mu_0^2}\left[\partial_+^1\sum_{ij} q_iq_jP_+\Delta(x_i-x_j)\right]G_{AJe}(x_1\cdots x_{2n})$$

$$+\exp\left[i\sum_i q_i(A_\mu N^\mu + J_\mu M^\mu)\right]\exp\left[i\frac{e^2a}{2\mu_0^2}P_+\sum_{ij} q_iq_j\Delta(x_i-x_j)\right]\partial_+^1 G_0(x_1\cdots x_{2n}) \;.$$

It follows from Eq. (4.4) that

$$\text{second term of (B1)} = \frac{ie^2a}{\mu_0^2}q_1P_+\sum_{i\neq 1} q_i\partial_+^1\Delta(x_1-x_i)G_{AJe} + \frac{i}{\sqrt{2}}\sum_{i=2}^{2n}(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n})$$

$$= i^{n+1}eq_1\langle T^*(B^-(x_1)\psi(x_1)\cdots\psi(x_{2n}))\rangle + \frac{i}{\sqrt{2}}\sum_i(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n})$$

$$-i\frac{e^2a}{\mu_0^2}P_+\partial_+\Delta(x)\Big|_{x=0}G_{AJe}(x_1\cdots x_{2n}) \;. \tag{B3}$$

The term proportional to $\partial_+\Delta(0)$ in Eq. (B3) does not contribute if we symmetrize the operator product,

$$B^\mu(x)\psi(x)\to\tfrac{1}{2}[B^\mu(x)\psi(x)+\psi(x)B^\mu(x)] \;,$$

in the Lagrangian. Therefore one can write

$$\text{second term of (B1)} = i^{n+1}eq_1\langle T^*(B^-(x_1)\psi(x_1)\cdots\psi(x_{2n}))\rangle + \frac{i}{\sqrt{2}}\sum_i(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n}) \;. \tag{B4}$$

Combining Eqs. (B2) and (B4) one obtains the desired result

$$\partial_+^1G_{AJe}(x_1\cdots x_{2n}) = iq_1\left[e\langle B^-(x_1)\rangle + A^-(x_1) + \frac{e^2a}{\mu_0^2}\sum_i q_i\partial_+^1\Delta(x_1-x_i)\right]G_{AJe}$$

$$+\frac{i}{\sqrt{2}}\sum_i(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n}) \;, \tag{B5}$$

or

$$P_+\left(\alpha^\mu\frac{1}{i}\partial_\mu\right)_1 G_{AJe}(x_1\cdots x_{2n}) = eq_1\alpha^+i^n\langle T^*(B^-(x_1)\psi(x_1)\cdots\psi(x_{2n}))\rangle + q_1\alpha^+A^-(x_1)G_{AJe}(x_1\cdots x_{2n}) \,,$$

$$+P_+\sum_i(-1)^i\delta(x_1-x_i)G_{AJe}(x_2\cdots x_{i-1},x_{i+1}\cdots x_{2n}) \;, \tag{B6}$$

where use has been made of the fact that

$$\frac{1}{i}\frac{\delta}{\delta J^+}G_{AJe} = \frac{1}{i}\frac{\delta}{\delta J^+}\frac{i^n\langle 0|T(\psi(x_1)\cdots\psi(x_{2n}))\rangle_{AJe}}{\langle 0|0\rangle_{AJe}} = i^n\langle T^*(B^-(x)\psi\cdots\psi)\rangle - \langle B^-(x)\rangle G_{AJe}$$

in the presence of the external fields.

In order to show that the prescription (1.1) is consistent with the solution one computes

$$\langle j^+(x)\rangle = \frac{i}{\sqrt{2}}\lim_{x'\to x}\mathrm{Tr}\,qP_+G_{AJe}(x,x') \;. \tag{B7}$$

From the fermion Green's function (4.6) it follows that

$$\langle j^+(x)\rangle = -\frac{1}{2\pi}\,\partial_x^+[N^\mu(x-y)A_\mu(y)+M^\mu(x-y)J_\mu(y)]\ . \tag{B8}$$

From Eq. (4.8) this can be rewritten as

$$\langle j^+(x)\rangle = -\frac{1}{\pi}\,\partial^+\left\{\partial^+\left[D(x-y)+\frac{e^2a}{4\pi\mu_0{}^2}\Delta(x-y)\right]A^-(y)+\frac{e}{\mu_0{}^2}\partial^+[D(x-y)-\Delta(x-y)]J^-(y)\right.$$
$$\left.+\frac{ea}{2\mu_0{}^2}\partial_\mu\Delta(x-y)J^\mu(y)\right\}. \tag{B9}$$

It is now easy to see that Eq. (B9) is identical to

$$\frac{1}{i}\,\frac{\delta}{\delta A^-(x)}\langle 0|0\rangle_{AJe}$$

computed from the vacuum transition amplitude (3.20); thus, it displays the consistency of the prescription (1.1) with the solution.

APPENDIX C: THIRRING MODEL IN LIGHT-CONE COORDINATES

The Thirring model is described by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}i\bar\psi\alpha^\mu\partial_\mu\psi+\frac{\lambda}{2}\,j^\mu j_\mu+j^\mu A_\mu\ , \tag{C1}$$

where $A^\mu(x)$ is an external source. By the action principle, one finds the field equation

$$\alpha^\mu\left(\frac{1}{i}\,\partial_\mu-\lambda q j_\mu-qA_\mu\right)\psi(x)=0 \tag{C2}$$

and the infinitesimal generator

$$G=\frac{i}{\sqrt{2}}\int dx^-\psi_+\,\delta\psi_+\ . \tag{C3}$$

Since the infinitesimal generator (C3) is expressed in terms of the independent dynamical field component ψ_+ alone, the anticommutator has the canonical form

$$\{\psi_+(x),\psi_+(x')\}=\frac{1}{\sqrt{2}}\,\delta(x^--x^{-\prime})\ . \tag{C4}$$

In order to obtain the solution of the model one writes the vacuum transition amplitude, extracting the effect of the interaction into an exponential form

$$\langle 0|0\rangle_{A\lambda}=\exp\left[-i\,\frac{\lambda}{2}\int dx\,\frac{\delta}{\delta A^\mu(x)}\,\frac{\delta}{\delta A_\mu(x)}\right]\langle 0|0\rangle_A\ , \tag{C5}$$

where $\langle 0|0\rangle_A=\langle 0|0\rangle_{A,\lambda=0}$ is given by Eq. (3.18) of Sec. III. It is easy to show that

$$\langle 0|0\rangle_{A\lambda}=\langle 0|0\rangle_A\ , \tag{C6}$$

i.e., the Thirring model quantized in light-cone coordinates is free. One can understand this from the fact that the $\psi_-(x)$ field and $j^-(x)$ vanish for the massless Dirac field in two-dimensional light-cone coordinates and thus

$$\lambda j^\mu j_\mu\to 0\ .$$

Following the same procedure as Sec. IV, one finds the fermion Green's function

$$G_{A\lambda}(x_1\cdots x_{2n})=\exp\left[i\sum_i q_i\int dx\,A_\mu(x)N_0^\mu(x-x_i)\right]$$
$$\times G_0(x_1\cdots x_{2n})\ , \tag{C7}$$

where

$$N_0^\mu(x-x_i)=P_+(\partial^\mu+\epsilon^{\mu\nu}\partial_\nu)D(x-x_i)$$
$$+P_-(\partial^\mu-\epsilon^{\mu\nu}\partial_\nu)D(x-x_i)\ . \tag{C8}$$

The commutation relation implied by the solution (C7) is

$$[j^+(x),\psi_+(x')]=-q\psi_+(x)\delta(x^--x^{-\prime})\ ,$$

which is identical to that of a free field. Thus the Thirring model quantized in light-cone coordinates is free, and is not equivalent to the same model quantized on a spacelike surface.[2]

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[10]There is a minor misprint in Ref. 3. The fifth line from the bottom on p. 73 of Ref. 3 should read

$$G^{\mu\nu}(x)=\frac{1}{1-e^2/4\pi\mu_0{}^2}\left(g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\mu^2}\right)\Delta(x)$$
$$-\frac{e^2}{4\pi\mu_0{}^2}\frac{\partial^\mu\partial^\nu}{\mu_0{}^2}\frac{1}{1-e^2/4\pi\mu_0{}^2}\Delta(x)$$
$$+\cdots .$$

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