Existence of a second-order phase transition in a two-dimensional ϕ^4 field theory*

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We study the stability of the vacuum in a two-dimensional ϕ^4 field theory with a fixed (mass)^2 term and a variable ϕ^4 coupling term. We make the theory finite by normal-ordering the Hamiltonian with respect to a fixed mass. As the ϕ^4 coupling strength increases, we can show that the system will undergo a second-order phase transition from a normal vacuum to an abnormal vacuum. There is no contradiction between the existence of a second-order transition in the absence of an external field and the Simon-Griffiths theorems which forbid any possible phase transition in the presence of an external field.

I. INTRODUCTION

In a previous paper,[1] we studied the stability of the vacuum in a two-dimensional ϕ^4 field theory under the Hartree approximation. One of the main conclusions based on the Hartree approximation is that for a ϕ^4 theory with a fixed mass parameter, a change of ϕ^4 coupling strength can induce a first-order phase transition (see Fig. 1 for an illustration of the Hartree result). On the other hand, Simon and Griffiths[2] have established several rigorous results in the two-dimensional ϕ^4 field theory by considering the ϕ^4 field theory as a proper limit of a generalized Ising model.[3] According to one of their theorems, there can be no phase transition in the two-dimensional ϕ^4 theory in the presence of an external field $B \neq 0$.

The purpose of the present paper is to understand whether there is a phase transition in the two-dimensional ϕ^4 theory at $B=0$. To control the ultraviolet divergence, we normal-order our Hamiltonian according to a fixed mass. In this paper, we fix the (mass)^2 term and concentrate on the effect due to the ϕ^4 coupling alone.

Using some identities relating different normal orderings introduced by Coleman,[4] we are able to establish that there is indeed a phase transition in the two-dimensional ϕ^4 theory as we vary the ϕ^4 coupling strength.

Next, we show that there is no contradiction between the Simon-Griffiths theorems and the existence of a second-order phase transition in the two-dimensional ϕ^4 field theory at $B=0$. We demonstrate explicitly how Simon-Griffiths theorems can be satisfied near a second-order transition in the Landau-Ginzburg model as well as in models obeying a scaling law. However, according to the Simon-Griffiths theorems, a first-order phase transition such as predicted by the Hartree calculation is definitely not acceptable.[5]

The conclusion reached here is not very surprising, because Simon-Griffiths theorems are derived from results established in the Ising systems which also exhibit second-order phase transitions in the absence of an external field.

The paper is organized as follows: In Sec. II, we established the existence of a phase transition in the two-dimensional ϕ^4 field theory. We then show in Sec. III that this phase transition must be second-order in nature. In Sec. IV, we discuss various implications of Ising results for the ϕ^4 field theory. A simple derivation[6] of the Hartree effective potential in the presence of an external field is given as an appendix.

II. ϕ^4 HAMILTONIANS AND THE PHASE TRANSITION

A. Normal orderings of ϕ^4 Hamiltonians

We consider the ϕ^4 field theory in one space dimension and one time dimension described by the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4}g\phi^4 \tag{2.1}$$

with $g>0$, and an arbitrary m^2. For negative m^2, it is convenient to rewrite $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H}' &= \tfrac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{4}g(\phi^2 - c^2)^2 \\ &= \tfrac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{2}gc^2\phi^2 + \tfrac{1}{4}g\phi^4 + \text{const.}\end{aligned} \tag{2.2}$$

In the following, we shall use the Hamiltonian density (2.1) to describe a positive (mass)^2 system, and use (2.2) to describe a negative (mass)^2 system. As classical systems, the ground state associated with the Hamiltonian density (2.1) is at $\phi=0$, and the ground state associated with (2.2) is at $\phi=\pm c$. From the frequency of the small oscillation around $\phi=\pm c$, we deduce that the mass of the system (2.2) at its ground state is

$$\mu^2 = 2gc^2. \tag{2.3}$$

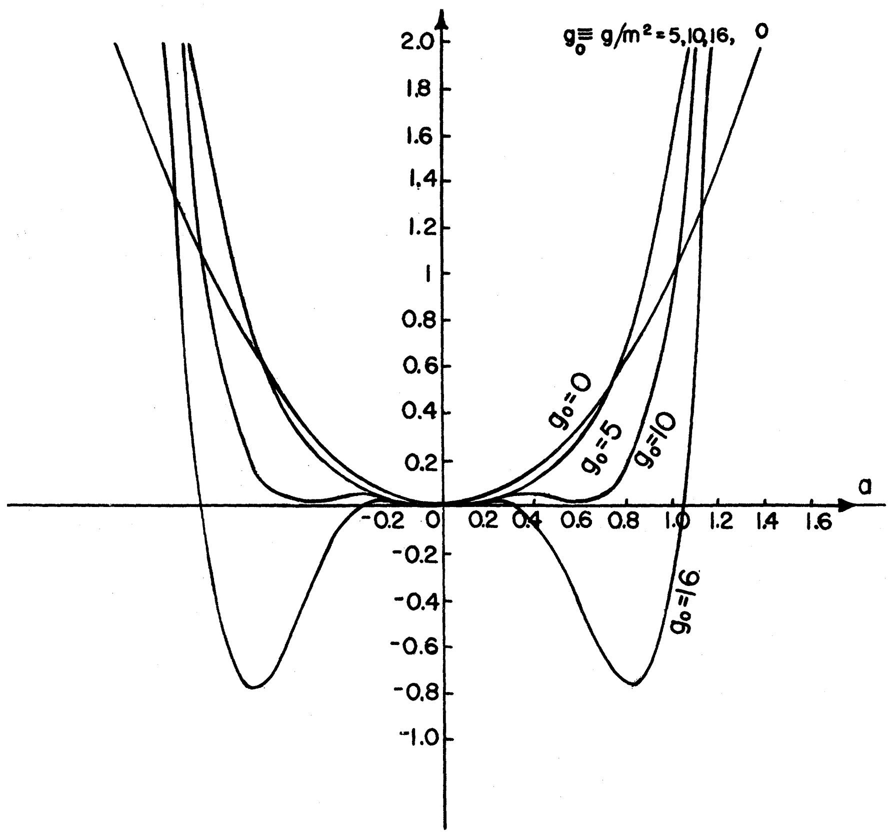

FIG. 1. The Hartree effective potential $V(a)$ as a function of the vacuum expectation value $\langle\phi\rangle = a$ $(\equiv \phi_c)$ at $g_0(\equiv g/m^2)$ $=0$, 5, 10, and 16. The calculations are based on the Hamiltonian $\mathcal{H}$ given in (2.7) with the mass parameter m. These graphs demonstrate clearly the existence of a first-order transition near $g_0 = 10$. (The exact Hartree transition point is at $g_0 = 10.211$.) This picture is taken from Fig. 4 of Ref. 1. See Ref. 1 for detailed discussions concerning the Hartree calculation.

Because this is a quantum-mechanical system, we may encounter ultraviolet divergences. For two-dimensional ϕ^4 field theory, the only irreducible divergent graph is the self-energy diagram shown in Fig. 2. It can be removed readily by normal-ordering the Hamiltonian. The method of normal-ordering the scalar fields appropriate to our application was developed by Coleman.[4] We refer the readers to this excellently written paper for details. In the following, we shall make use of a number of results derived in this paper.

To normal-order an interaction Hamiltonian, we have to specify the particle mass of the free Hamiltonian through which the free particle creation and annihilation operators are defined. Choosing a different normal-ordering mass has the same effect as choosing a different renormalization point. Coleman has proved the following identities relating two different choices of normal-ordering masses:

$$N_m\left[\tfrac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] = N_\mu\left[\tfrac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] + \frac{1}{8\pi}(\mu^2 - m^2), \qquad (2.4)$$

$$N_m(e^{i\beta\phi}) = \left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi} N_\mu(e^{i\beta\phi}). \qquad (2.5)$$

In Eqs. (2.4) and (2.5), N_m (N_μ) denotes normal ordering with respect to mass m (μ). By expanding (2.5) and equating coefficients of β^n, we have

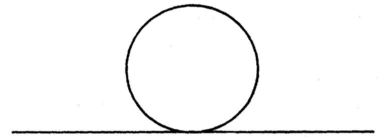

FIG. 2. The only irreducible divergent subgraph in the two-dimensional ϕ^4 field theory.

the Wick-expansion results:

$$N_m(\phi^2)=N_\mu(\phi^2)+\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}, \tag{2.6a}$$

$$N_m(\phi^3)=N_\mu(\phi^3)+3\left(\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right)N_\mu(\phi), \tag{2.6b}$$

$$N_m(\phi^4)=N_\mu(\phi^4)+6\left(\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right)N_\mu(\phi^2)+3\left(\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right)^2, \tag{2.6c}$$

etc.

Now, we go back to our Hamiltonians (2.1) and (2.2). We make the natural choice of normal-ordering them according to the masses associated with their classical ground states; i.e., we normal-order $\mathcal{H}$ in (2.1) with respect to m, and $\mathcal{H}'$ in (2.2) with respect to μ [defined as in (2.3)]:

$$\mathcal{H}=N_m\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\phi^4\right] \tag{2.7}$$

and

$$\mathcal{H}'=N_\mu\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{4}g(\phi^2-c^2)^2\right]=N_\mu\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-\tfrac{1}{4}\mu^2\phi^2+\tfrac{1}{4}g\phi^4\right]+\text{const.} \tag{2.8}$$

The advantage of this choice of normal ordering is that the systems reduce to their respective classical free particle states in the weak-coupling limit. Note that g has the dimension of (mass)^2. By weak-coupling limit, we mean g/m^2 or $g/\mu^2\to 0$. From (2.7), we see quite clearly that $\mathcal{H}$ goes to the free Hamiltonian as $g/m^2\to 0$. Hence, the ground state associated with $\mathcal{H}$ in (2.7) in the weak-coupling region $g/m^2 \ll 1$ is the normal vacuum state characterized by $\langle\phi\rangle=0$. To examine the weak-coupling limit of $\mathcal{H}'$ in (2.8), we need to specify which of the ground states $\phi=\pm c$ we are dealing with. For definiteness, we restrict ourselves to the neighborhood of $\phi=c$. Introducing

$$\phi=c+\phi',\quad c=+\left(\frac{\mu^2}{2g}\right)^{1/2}\gg 1, \tag{2.9}$$

we can express (2.8) in terms of ϕ' as

$$\mathcal{H}'=N_\mu\left[\tfrac{1}{2}\dot{\phi}'^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}\mu^2\phi'^2+\mu(\tfrac{1}{2}g)^{1/2}\phi'^3+\tfrac{1}{4}g\phi'^4\right]. \tag{2.10}$$

It is now transparent that, in the weak-coupling limit $g/\mu^2\to 0$, $\mathcal{H}'$ reduces to a free Hamiltonian with mass μ. In particular, it would imply that the ground state in the weak-coupling region $g/\mu^2=1/2c^2 \ll 1$ is characterized by

$$\langle\phi'\rangle=0,\quad \langle\phi\rangle=c\gg 1 \tag{2.11}$$

and corresponds to an abnormal vacuum state. Note also that the separation in $\langle\phi\rangle$ between these two vacuum states (i.e., $\langle\phi\rangle=c$ vs $\langle\phi\rangle=-c$) increases indefinitely as $g/\mu^2\to 0$. We expect these two vacuums to be completely decoupled in the weak-coupling limit. In short, we have shown in the weak-coupling limit that the ground state of $\mathcal{H}$ in (2.7) is a normal vacuum ($\langle\phi\rangle=0$) and that of $\mathcal{H}'$ in (2.8) is an abnormal vacuum ($\langle\phi\rangle\neq 0$). This appears to be a trivial point, but is crucial in our later analysis.

B. Equivalence relation between $\mathcal{H}$ and $\mathcal{H}'$ and the existence of a phase transition

Even though the classical systems (2.1) and (2.2) are different and are associated with different kinds of ground states; their quantum-mechanical analogs (2.7) and (2.8) may actually be identical. The additional contributions given in (2.4)–(2.6) when we transform from one normal-ordering to another can alter the appearance of the Hamiltonian and thus make their equivalence possible. Indeed, if we substitute (2.4)–(2.6) into (2.7) we have

$$\begin{aligned}\mathcal{H}&=N_\mu\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\left(\phi^4+\frac{3}{2\pi}\ln\frac{m^2}{\mu^2}\phi^2\right)\right]+\frac{1}{8\pi}(\mu^2-m^2)+\tfrac{1}{2}m^2\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}+\tfrac{3}{4}g\left(\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right)^2\\&=N_\mu\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{4}g\left(\phi^2+\frac{3}{4\pi}\ln\frac{m^2}{\mu^2}+\frac{m^2}{g}\right)^2\right]\\&\quad+\frac{1}{8\pi}(\mu^2-m^2)+\frac{m^2}{8\pi}\ln\frac{m^2}{\mu^2}+\tfrac{3}{4}g\left(\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\right)^2-\tfrac{1}{4}g\left(\frac{3}{4\pi}\ln\frac{m^2}{\mu^2}+\frac{m^2}{g}\right)^2.\end{aligned} \tag{2.12}$$

The N_μ term in (2.12) is identical to $\mathcal{H}'$ in (2.8) provided that

$$\frac{3}{4\pi}\ln\frac{m^2}{\mu^2}+\frac{m^2}{g}=-\frac{\mu^2}{2g}\ (\equiv -c^2). \tag{2.13}$$

Equation (2.13) can be rewritten as a relation between the invariant (dimensionless) coupling strengths g/m^2 of $\mathcal{H}$ and g/μ^2 of $\mathcal{H}'$:

$$\frac{m^2}{g}+\frac{3}{4\pi}\ln\frac{m^2}{g}=\frac{3}{4\pi}\ln\frac{\mu^2}{g}-\frac{\mu^2}{2g}. \tag{2.14}$$

This relation was obtained earlier in a different

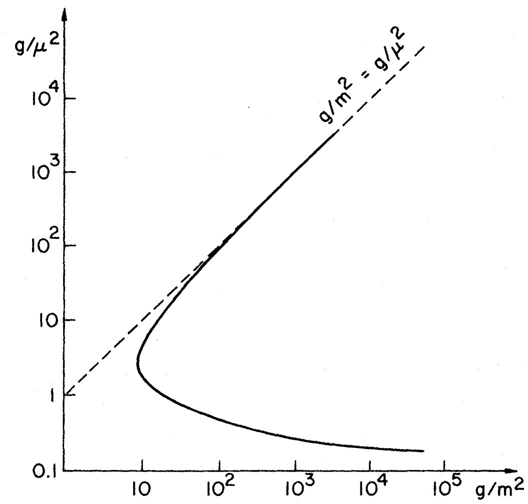

FIG. 3. Invariant coupling g/μ^2 as a function of g/m^2. The functional relation is defined in Eq. (2.14).

form under the Hartree approximation.[1,7] In Fig. 3, we plot g/μ^2 as a function of g/m^2 as defined by (2.14). This function has the following properties:

(i) For $g/m^2 < 9.045$, there is no real solution for g/μ^2.

(ii) For $g/m^2 = 9.045$ we reach the left tip of the curve with $g/\mu^2 = 2\pi/3 = 2.0944$ (i.e., $4\pi c^2 = 3$); for $g/m^2 > 9.045$ we have two solutions corresponding to the upper and the lower branches of the curve.

(iii) As $g/m^2 \to \infty$, the upper branch approaches

$$\frac{g}{\mu^2} = \frac{g}{m^2} - 2\pi + O\left(\frac{m^2}{g}\right), \tag{2.15}$$

and the lower branch approaches zero as

$$\frac{g}{\mu^2} = \frac{2\pi}{3}\left[\ln\frac{g}{m^2} + \ln\left(\ln\frac{g}{m^2}\right) + O(1)\right]^{-1}. \tag{2.16}$$

The fact that the lower branch approaches zero as $g/m^2 \to \infty$ is very important. It means that a strong-coupling theory in terms of $\mathcal{H}$ is identical to a weak-coupling theory in terms of $\mathcal{H}'$. In particular, we conclude that the ground state associated with a strong-coupling theory of $\mathcal{H}$ is an abnormal vacuum and is described by its equivalent theory in $\mathcal{H}'$ as

$$\langle\phi\rangle = \left(\frac{\mu^2}{2g}\right)^{1/2} \approx \left(\frac{3}{4\pi}\ln\frac{g}{m^2}\right)^{1/2} \neq 0. \tag{2.17}$$

On the other hand, the ground state associated with a weak-coupling theory (i.e., $g/m^2 \to 0$) is the normal vacuum described by $\langle\phi\rangle = 0$. Obviously, as we fix the mass term m and increase the coupling strength g/m^2 from 0 to ∞, at some critical value of g/m^2 the vacuum expectation value of ϕ will start developing a nonvanishing value. This proves that there must be a phase transition in the two-dimensional ϕ^4 theory with a fixed mass term but a variable coupling constant g as described by $\mathcal{H}$ in (2.7).

The connection between $\mathcal{H}$ and $\mathcal{H}'$ also supplies us a method of computing the exact behavior of the effective potential of the system $\mathcal{H}$ for both the small and the large values of g in the neighborhood of the ground state. For a small g, we use the loop expansion and obtain the effective potential:

$$V_{\text{eff}}(\phi_c) = \tfrac{1}{2}m^2\phi_c^2 + \tfrac{1}{4}g\phi_c^4 + \left[\frac{3g}{8\pi}\phi_c^2 - \frac{1}{8\pi}(m^2 + 3g\phi_c^2)\ln\frac{m^2 + 3g\phi_c^2}{m^2}\right] + \text{higher-loop corrections.} \tag{2.18}$$

In (2.18), the expression in the square brackets represents the one-loop correction and is of the order $(g/m^2)^2$. The higher-loop corrections are of $O((g/m^2)^3)$ or smaller. For a large g, we use the equivalent relation and obtain through loop expansions in $\mathcal{H}'$

$$V_{\text{eff}}(\phi_c) = \frac{1}{4\pi}gc^2 - \tfrac{1}{6}gc^4 + \tfrac{1}{4}g(\phi_c^2 - c^2)^2 + \left[\frac{3g}{8\pi}(\phi_c^2 - c^2) + \frac{g}{8\pi}(3\phi_c^2 - c^2)\ln\frac{2c^2}{3\phi_c^2 - c^2}\right] + \text{higher-loop corrections.} \tag{2.19}$$

In (2.19), the first two terms represent the large-g/m^2 limit (or, equivalently, a large-c^2 limit) of the additive constant appearing in (2.12). The expression in the square brackets denotes the one-loop corrections computed from $\mathcal{H}'$, and is of $O(1/c^2) = O(1/\ln(g/m^2))$ in comparison with the other terms. The higher-loop correction terms are of $O(1/(\ln g/m^2)^2)$ or smaller and can be ignored in the large-g limit. In the large-c limit (or equivalently, a large-g/m^2 limit), the additive constant appearing in (2.19) is large and negative. It confirms the picture of a stable vacuum at $\phi_c^2 = c^2$.

It is interesting to note that the Hartree calculation described in Ref. 1 gives rise to the correct effective potentials for both the weak-coupling ($g \to 0$) and the strong-coupling ($g \to \infty$) limits as given by Eqs. (2.18) and (2.19), respectively. (See the Appendix for the analytic expression of the Hartree effective potential, and Fig. 1 for a graphical representation.) Physically, this is not surprising. The Hartree approximation is equivalent to a variational calculation based on Gaussian fluctuations. Thus, the Hartree description is faithful when we are away from the phase-transition region as described by the limits $g \to 0$ and

$g \to \infty$. However, the Hartree approximation may not be reliable near a phase transition where the fluctuation is large and no longer described by a Gaussian distribution. In the next section, we shall study the nature of the phase transition in the ϕ^4 field theory in more detail.

III. NATURE OF THE PHASE TRANSITION

To understand the nature of the phase transition in the ϕ^4 field theory in one space dimension and one time dimension, we shall make use of several rigorous results obtained by Simon and Griffiths.[2]

A. Simon-Griffiths theorems and the nonexistence of a first-order transition

Simon and Griffiths made an important observation that the two-dimensional Euclidean ϕ^4 field theory can be obtained as a proper limit of a generalized Ising system with ferromagnetic spin-spin (pair) interactions only.[2] Using this limiting relation, they are able to establish many Ising-type results in the ϕ^4 field theory. The following results are important in our analysis:

(1) (Simon-Griffiths).[8] For a scalar field theory in two dimensions,

$$\mathcal{H}=N(\mathcal{H}_0+\tfrac{1}{4}g\,\phi^4+b\,\phi^2-B\phi),\quad g>0$$

the infinite-volume limit of the ground-state energy density exists, and is analytic for $B\neq 0$, $\mathrm{Re}B>0$ with g, b fixed.

(2) (Simon).[9] For a scalar field theory

$$\mathcal{H}=N(\mathcal{H}_0+\tfrac{1}{4}g\,\phi^4+b\,\phi^2-B\phi),\quad g>0,\quad B\neq 0$$

in one space dimension and one time dimension, the vacuum is unique.

These Simon-Griffiths theorems imply the nonexistence of any phase transition for $B\neq 0$, and also rule out the possibility of a first-order phase transition at $B=0$.

To understand the last conclusion, let us accept the contrary and assume that there is a first-order phase transition at $g/m^2=g_c/m^2\equiv G_c$ and $B=0$. By a first-order phase transition, we mean that the first derivative of the energy density with respect to g is discontinuous at $g=g_c$. This discontinuity is due to a transition between a normal vacuum characterized by a vanishing-order parameter $\langle\phi\rangle=0$ and an abnormal vacuum characterized by a finite-order parameter $\langle\phi\rangle=\phi_c\neq 0$. In the presence of a first-order transition, we find for $g/m^2<G_c$ that the energy density V associated with the normal vacuum is lower than that associated with the abnormal vacuum. For $g/m^2>G_c$, the energy density associated with the abnormal vacuum ($\phi_c\neq 0$) should be lower. Near the transition point, the energy density difference between the vacuum states various linearly as $g-g_c$. Now we turn on a constant external field B. For a small and positive B, we expect that the energy density associated with the normal vacuum is not affected to the first order in B owing to $\langle\phi\rangle=0$ and the energy density associated with the abnormal vacuum $\langle\phi\rangle=\phi_c>0$ is lowered by an amount proportional to B, $\Delta V=\phi_c B$. Thus, at sufficiently small B, this small energy difference will not change the nature of the first-order energy density crossover phenomenon between the normal and the abnormal vacuums. It only lowers slightly the transition coupling constant. The above argument can be understood most easily in a graphical representation as given in Fig. 4. Hence, we conclude that a first-order phase transition persists for a sufficiently small B (i.e., B small but $\neq 0$), and thus violates the Simon-Griffiths theorem. This contradiction implies that the original assumption of the existence of a first-order phase transition at $B=0$ is inconsistent.

In Sec. II, we proved that there definitely exists a phase transition as we increase the coupling strength in $\mathcal{H}$. Since this phase transition cannot be of the first order, it must be of second order.[10]

B. Simon-Griffiths theorems in the Landau-Ginzburg model

In the following, we would like to show that there is no contradiction between a second-order phase transition and the Simon-Griffiths theorem in the neighborhood of the phase transition. We shall use the Landau-Ginzburg model as an example to illustrate the point. This model pro-

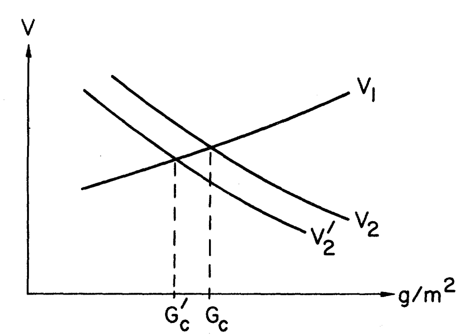

FIG. 4. Typical crossover phenomena for the effective potentials in the presence of a first-order transition. V_1 describes the effective potential associated with the normal vacuum. V_2 and V_2' describe the effective potentials associated with the abnormal vacuum for $B=0$ and for a small and positive B. Note that V_1 is not affected by B to the lowest order. The first-order transition persists for $B\neq 0$, as indicated by the crossover of V_1 and V_2' at $g/m^2=G_c'$.

vides a simple realization of a second-order phase transition.

In the neighborhood of the phase transition, we can express the effective potential $V_{\rm eff}(\phi_c)$ in Landau-Ginzburg theory as

$$V_{\rm eff}=a\phi_c{}^2+\tfrac{1}{4}b\phi_c{}^4-B\phi_c\,, \tag{3.1}$$

where parameters a and b are functions of g, m, and B^2. In the weak-external-field limit, we can ignore the B dependence in the parameters a and b. At the critical point

$$g/m^2=G_c\,,\quad B=0 \tag{3.2}$$

we have

$$a=0,\quad b>0\ \text{(critical point)}. \tag{3.3}$$

In the neighborhood of the critical point, parameter a is positive for $g/m^2<G_c$ and negative for $g/m^2>G_c$. Parameter b can be considered as a constant. Since we can always rescale the ϕ_c field in the neighborhood of the transition point (i.e., the critical region), we may choose $b=1$ without losing any generality.

For $B=0$, the effective potential $V_{\rm eff}$ has a minimum at

$$\phi_c=0\quad\text{for } a\geq 0 \tag{3.4a}$$

and has two minima at

$$\phi_c=\pm(-2a)^{1/2}\ \text{for } a<0. \tag{3.4b}$$

(Recall that we have set $b=1$.) Thus, for $a\geq 0$ the ground state is at $\phi_c=0$ and for $a<0$ the ground state is at $\phi_c=\pm(-2a)^{1/2}\neq 0$. At $a=0$, ϕ_c is continuous, but its derivative with respect to a is not. It describes a second-order phase transition as desired. Typical behavior of $V_{\rm eff}$ in the vicinity of the critical point at $B=0$ is shown in Fig. 5.

For $B>0$, the position of the minimum for $V_{\rm eff}$ is given by

$$\frac{dV_{\rm eff}}{d\phi_c}=2a\phi_c+\phi_c{}^3-B=0. \tag{3.5}$$

In terms of scaled variables α and y,

$$\alpha\equiv B^{-2/3}a\,,\quad y\equiv B^{-1/3}\phi_c\,, \tag{3.6}$$

we have

$$\frac{dV_{\rm eff}}{d\phi_c}=B(y^3+2\alpha y-1)=0. \tag{3.7}$$

Equation (3.7) defines y as a universal function of α, and this function $y=f(\alpha)$ is shown in Fig. 6. Note that y is positive for the true minimum of $V_{\rm eff}$, and this restriction selects a unique single-valued branch of $f(\alpha)$. It is easy to show that this branch of $f(\alpha)$ defines an analytic function of α, and obeys the asymptotic relations

$$\lim_{\alpha\to\infty} f(\alpha)=\frac{1}{2\alpha}+O\left(\frac{1}{\alpha^4}\right)\to 0, \tag{3.8a}$$

$$\lim_{\alpha\to-\infty} f(\alpha)=(-2\alpha)^{1/2}. \tag{3.8b}$$

In terms of a and $B>0$, the position of the ground state is described by

$$\phi_c=B^{1/3}f(aB^{-2/3}), \tag{3.9}$$

which is an analytic function of a and B as long as $B\neq 0$. Thus, there is no possible phase transition of any kind as long as $B\neq 0$, as required by the Simon-Griffiths theorem.

As $B\to 0$, however, a discontinuity in g (here

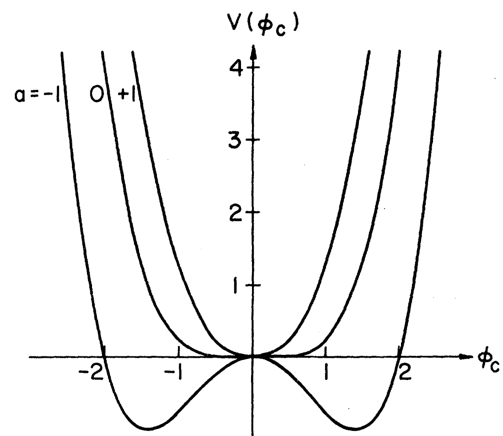

FIG. 5. Behaviors of the effective potential in the Landau-Ginzburg model near the second-order transition region. We expect to have a qualitatively similar result in the exact solution.

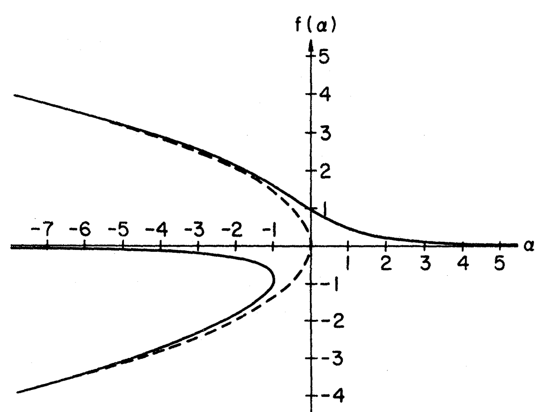

FIG. 6. Universal scaling function $f(\alpha)$ appearing in the Landau-Ginzburg model. Only the branch with $f(\alpha)>0$ is responsible for the critical behavior near the transition.

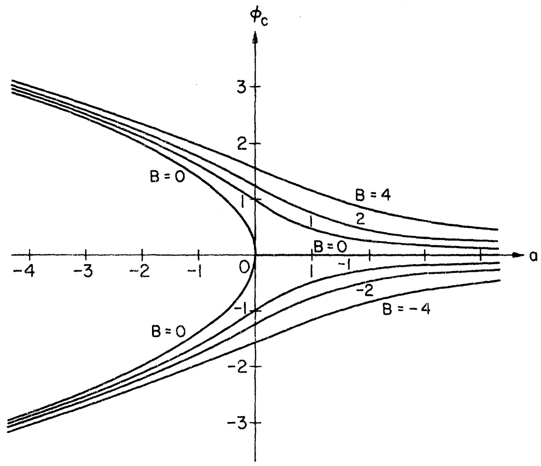

FIG. 7. The vacuum expectation value ϕ_c as a function of a at various values of B in the Landau-Ginzburg model. Note that ϕ_c is an analytic function of a for $B\neq 0$, but develops a second-order transition at $B=0$. For a fixed and negative a, ϕ_c undergoes a first-order transition when B changes from a positive to a negative value. For a positive a, ϕ_c is a continuous function of B. These properties are expected to reflect the true behavior of the ϕ^4 field theory near the transition point.

via a) can develop. To see it, we assume that $B>0$ and approaches 0 from above. (We denote this limit as $B\to 0^+$.) As $B\to 0^+$, we find from (3.8) that for $a>0$ (or equivalently $g/m^2<G_c$) the ground state is described by

$$\phi_c=\lim_{B\to 0^+} B^{1/3}f(aB^{-2/3})=\lim_{B\to 0^+}\frac{B}{2a}=0,$$

while for $a<0$ (or $g/m^2>G_c$) the ground state is described by

$$\phi_c=\lim_{B\to 0^+} B^{1/3}(-2aB^{-2/3})^{1/2}=(-2a)^{1/2}\neq 0$$

as given in (3.4). The functional dependence of ϕ_c as a function of a and B is shown in Fig. 7.

So far we have concentrated on the dependence of the system as a function of a at fixed B. In the following, we consider the dependence of the system on B. As a function of B at fixed a (or fixed g), ϕ_c is an analytic function of B for $B\neq 0$. However, as it passes through the point $B=0$, the system undergoes a first-order phase transition between $\phi_c=(-2a)^{1/2}$ and $\phi_c=-(-2a)^{1/2}$ for $g/m^2>G_c$ (i.e., $a<0$), and has no phase transition for $g/m^2<G_c$ (or $a>0$). Since we are dealing with the B dependence at $B=0$ the Simon-Griffiths theorems are not applicable here.

C. Simon-Griffiths theorems in scaling regions

In Sec. III B we used Landau-Ginzburg models to illustrate the compatability of a second-order phase transition and the Simon-Griffiths theorems. The true analyticity behavior of ϕ^4 field theory near the transition point is undoubtedly more complicated than that given by the Landau-Ginzburg model. Indeed, if we accept that the scaling law established in the theory of second-order transitions in statistical mechanics[11] is also valid in ϕ^4 field theory; we would expect that the vacuum generating functional $W(g,B)$ $[\equiv(1/i)\ln\langle 0|0\rangle^B]$ can be separated into a B-independent regular part and a singular part near the transition point $g\approx g_c$:

$$W(g,B)=W_{\rm reg}(g)+W_{\rm sing}(g,B). \qquad (3.10)$$

Only the singular part of W is responsible for the critical behavior. The scaling law suggests that for small values of B and

$$\tau\equiv g_c-g, \qquad (3.11)$$

we have

$$W_{\rm sing}=|\tau|^{2-\alpha}g_\pm(B/|\tau|^\Delta) \qquad (3.12)$$

where the subscript $\pm$ stands for the sign of τ. Constants α and Δ are known as critical exponents. Given W, the effective potential $V_{\rm eff}$ can be obtained by a Legendre transformation.

The vacuum expectation value of ϕ is given by

$$\phi_c=\frac{\partial W}{\partial B}=|\tau|^{2-\alpha-\Delta}g'_\pm(B/|\tau|^\Delta)$$
$$=B^{(2-\alpha-\Delta)/\Delta}f_\pm(|\tau|/B^{1/\Delta}) \qquad (3.13)$$

with

$$f_\pm(x)=x^{2-\alpha-\Delta}g'_\pm(x^{-\Delta}). \qquad (3.14)$$

Equation (3.13) is a natural extension of (3.9) with τ playing the role of the parameter a. [Expression (3.9) has the critical exponents $\alpha=0$, $\Delta=\frac{3}{2}$.] The Simon-Griffiths theorems are satisfied if

$$f_-(x)=f_+(-x)\equiv f(x) \qquad (3.15)$$

defines an analytic function of x. Then, ϕ_c is analytic in τ for all $B\neq 0$. If we assume further that $f(x)$ has the asymptotic behaviors

$$\lim_{x\to-\infty}\frac{f(x)}{|x|^{2-\alpha-\Delta}}=\text{finite constant}\neq 0, \qquad (3.16a)$$

$$\lim_{x\to\infty}\frac{f(x)}{|x|^{2-\alpha-\Delta}}=0, \qquad (3.16b)$$

we have for $B\to 0$

$$\phi_c\propto\tau^{2-\alpha-\Delta}\neq 0,\quad \tau<0 \text{ (or } g>g_c) \qquad (3.17a)$$

and

$$\phi_c=0,\quad \tau>0 \text{ (or } g<g_c). \qquad (3.17b)$$

Equation (3.17) describes a second-order phase transition in the absence of the external field B, and it possesses a power-dependent scaling behavior for the order parameter ϕ_c. Note that conditions (3.15)–(3.17) are satisfied in the Landau-Ginzburg model.

IV. DISCUSSION

Simon and Griffiths tell us that ϕ^4 field theory can be viewed as a proper limit of a generalized Ising theory. In particular, we would expect that the large-distance behavior of a Green's function in ϕ^4 field theory is related to the behavior of a similar Ising Green's function. We learn from the Ising model that two-point functions have the following asymptotic (i.e., $r\to\infty$) behavior in configuration space (T = temperature)[3]:

For $T\neq T_c$

$$G(r)\sim e^{-r/\xi}/r^a \tag{4.1a}$$

and for $T=T_c$

$$G(r)\sim r^{-1/4}, \tag{4.1b}$$

where r is the distance between the two lattice points, and ξ and a are constants which depend on T. Temperature T serves as a free parameter. For a given Ising system, varying T is analogous to varying the coupling constant g in the ϕ^4 theory. Parameter ξ is known as the correlation length, and is equivalent to the inverse of the physical mass in the particle theory. For the simple two-dimensional Ising model, the Onsager solution gives rise to these values of a:

$$a=\tfrac{1}{2},\quad T>T_c \tag{4.2a}$$

$$=2\quad T<T_c. \tag{4.2b}$$

Parameter a can take different values for different generalized Ising models. On the other hand, the power dependence $r^{-1/4}$ at $T=T_c$ is believed to be universal, and is the same for all generalized two-dimensional single-component (i.e., one field component) Ising models. On the basis of this information, we expect that the Green's function in two-dimensional ϕ^4 field theory should have the familiar asymptotic form (4.1a) both above and below the critical coupling but have the $r^{-1/4}$ dependence at the critical coupling.

Note that the $r^{-1/4}$ long-range dependence does not correspond to the Green's function of an isolated single-massless-particle pole. It corresponds to a coherent sum of many-massless-particle contributions. The Green's function $r^{-1/4}$ has a far more convergent large-distance behavior than the free-massless-particle Green's function $G_0\sim\ln r$. Thus, the infrared divergence due to the emission of coherent massless states in the ϕ^4 field theory at the critical point is softer than that due to the emission of a single massless particle. Hence, Coleman's theorem on the nonexistence of a massless particle in two-dimensional field theory does not apply here.[12]

In this paper, we consider a ϕ^4 Lagrange function normal-ordered according to a fixed mass parameter. Then, by varying the coupling strength, we find a phase transition in this theory. On the other hand, if we consider instead a Lagrange function with a fixed physical mass, then we can never reach a phase transition. This can be seen from the fact that the asymptotic behavior of the Green's function in a theory with a fixed physical mass m is always given by a damping exponential e^{-mr}/r^a which can never be a simple power $r^{-1/4}$ for fixed nonzero m. This point can also be understood in the language of statistical mechanics: There can never be a second-order phase transition for fixed finite correlation length ξ ($\sim 1/m$). Since the phase transition in a ϕ^4 theory is of second order, this would preclude the existence of a phase transition in a theory with a fixed physical mass.

If we consider instead a Hamiltonian with a fixed ϕ^4 coupling constant g, and vary its mass term, we obtain the expected classical results that a large and positive (mass)^2 term leads to a normal vacuum, and a large and negative (mass)^2 term leads to an abnormal vacuum. We can understand these results from the fact that in the two-dimensional ϕ^4 theory the coupling constant g has the dimension of (mass)^2. Thus, a large (mass)^2 term implies a relatively weak ϕ^4 coupling. Hence, the semiclassical picutre prevails.

In a previous paper, we have studied the stability of the vacuum in the ϕ^4 field theory under the Hartree approximation. The Hartree calculation leads to a first-order phase transition in the two-dimensional ϕ^4 theory as we increase the coupling constant. This result is in contradiction to the Simon-Griffiths theorems. We now realize that the Hartree approximation predicts correctly the existence of a phase transition and gives rise to the correct effective potential in both the weak-coupling and the strong-coupling limits, but it describes incorrectly the *nature* of the phase transition.

It is interesting to know whether the results obtained in this paper can be generalized to three- or four-dimensional ϕ^4 theory. We know that Ising models in three and four dimensions also possess a second-order phase transition, and that many of the Simon-Griffiths results are applicable to higher-dimensional theories as well. The main difficulty which prevents a straightforward generalization to three- or four-dimensional theory is how to handle the additional renormalizations appearing in the ϕ^4 field theory.

NOTE ADDED IN PROOF

(1) The method developed in this paper can be generalized straightforwardly to a three-dimensional ϕ^4 theory. Just as in a two-dimensional theory, the only divergent diagrams in a three-dimensional ϕ^4 theory are self-energy diagrams. In addition to the divergent diagram given in Fig. 2, we also encounter in the three-dimensional ϕ^4 theory a second-order divergent graph described by the splitting of a single ϕ line into three ϕ lines at one vertex and recombining them into one at another. We can define a finite ϕ^4 theory by adding a mass counterterm $(\delta m^2)_M$ in the Hamiltonian $\mathcal{H}$ to cancel these two divergent diagrams,

$$\mathcal{H}=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\phi^4-\tfrac{1}{2}(\delta m^2)_M\phi^2 .$$

The counterterm is given by

$$(\delta m^2)_M=\frac{3g}{2\pi^2}\left(\Lambda-\frac{\pi}{2}M\right)-\frac{3g^2}{16\pi^2}\ln\frac{\Lambda^2}{M^2}.$$

In $(\delta m^2)_M$, Λ stands for the ultraviolet momentum cutoff and M^2 is an arbitrary mass parameter denoting the renormalization point. The parameter M^2 is an extension of the normal-ordering mass introduced in this paper. In the small-coupling limit $(g/m \ll 1)$, it is natural to choose $M=m$. Then, it is easy to establish that the theory described by $\mathcal{H}$ possesses a normal vacuum at small g/m by the method of loop expansions.

For the ϕ^4 theory describing a broken symmetry, it is natural to introduce an alternative Hamiltonian,

$$\mathcal{H}'=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{4}\mu^2\phi^2+\tfrac{1}{4}g\phi^4-\tfrac{1}{2}(\delta m^2)_\mu\phi^2 .$$

The Hamiltonians $\mathcal{H}'$ and $\mathcal{H}$ with M set to m become identical if

$$m^2-(\delta m^2)_m=-\tfrac{1}{2}\mu^2-(\delta m^2)_\mu .$$

This relation can be expressed in a Λ-independent form,

$$\left(\frac{m}{g}\right)^2+\frac{3}{4\pi}\left(\frac{m}{g}\right)-\frac{3}{16\pi^2}\ln\frac{m^2}{g^2}=-\frac{1}{2}\left(\frac{\mu}{g}\right)^2+\frac{3}{4\pi}\left(\frac{\mu}{g}\right)-\frac{3}{16\pi^2}\ln\frac{\mu^2}{g^2}.$$

The above equation is the generalization of Eq. (2.14) to a three-dimensional ϕ^4 theory. We find from this equation that the strong-coupling limit of $\mathcal{H}'$ corresponds to the weak-coupling limit of $\mathcal{H}$. By arguments analogous to those presented in Sec. II, we can establish that a phase transition exists in a three-dimensional ϕ^4 theory with a fixed mass term and a variable coupling strength. We have also generalized the three-dimensional ϕ^4 theory to include the internal symmetry as well, and found that the phase transition exists too. (This is opposite to a two-dimensional ϕ^4 theory, which can never admit a broken continuous symmetry.) The result of this investigation will be published elsewhere [S. Magruder, Ph.D. thesis, University of Illinois at Urbana-Champaign (in preparation)].

(2) For a given g/m^2, the upper- and lower-branch solutions to Eq. (2.14) (see Fig. 3) are related by

$$\frac{3}{4\pi}\ln\frac{\mu^2}{g}-\frac{\mu^2}{2g}=\frac{3}{4\pi}\ln\frac{\mu^{*2}}{g}-\frac{\mu^{*2}}{2g}.$$

This equation defines a dual transformation between an upper-branch solution (strong coupling in $\mathcal{H}'$) and a lower-branch solution (weak coupling in $\mathcal{H}'$). This relation is reminiscent of the well-known dual transformation which appears in the two-dimensional Ising system [see, e.g., the review article by I. Syozi in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 1]. In the two-dimensional Ising system, the self-dual condition gives rise to the transition temperature. It is interesting to know whether the self-dual condition in a two-dimensional ϕ^4 theory,

$$\frac{g}{\mu^2}=\frac{g}{\mu^{*2}},\quad\text{or}\quad\frac{g}{\mu^2}=\frac{2\pi}{3}=2.0944,$$

also determines the value of the transition coupling strength. The author wishes to thank L. Kadanoff for pointing out the similarity of these two dual transformations.

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APPENDIX: HARTREE CALCULATION IN THE PRESENCE OF A CONSTANT EXTERNAL FIELD

It is known that the Hartree approximation is equivalent to a variational calculation with Gaussian trial functions. In a scalar field theory, we can identify these Gaussian trial functions as ground states associated with free-particle Hamiltonians with certain masses. Using this identification plus Coleman's normal-ordering methods, we can reproduce the previous Hartree calculation very efficiently.

Consider a normal-ordered Hamiltonian in the presence of a constant external field,

$$H=N_m\left\{\int dx\left[\tfrac{1}{2}\dot{\phi}^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\phi^4-B\phi\right]\right\}. \quad \text{(A1)}$$

To find the effective potential $V_{\text{eff}}(\phi_c)$, we first separate ϕ into a constant part and a q-number part:

$$\phi=\phi_c+\phi_q. \quad \text{(A2)}$$

We then decompose H according to the separation

$$H=H_c+H_{cq}+H_q, \quad \text{(A3)}$$

$$H_c=(\tfrac{1}{2}m^2\phi_c{}^2+\tfrac{1}{4}g\phi_c{}^4-B\phi_c)L$$

$$=\text{classical energy due to } \phi_c, \quad \text{(A4a)}$$

$$H_{cq}=\text{odd in } \phi_q, \quad \text{(A4b)}$$

$$H_q=N_m\left\{\int dx\left[\tfrac{1}{2}(\dot{\phi}_q)^2+\frac{1}{2}\left(\frac{\partial\phi_q}{\partial x}\right)^2+\tfrac{1}{2}m^2\phi_q{}^2+\tfrac{1}{4}g(\phi_q{}^4+6\phi_c{}^2\phi_q{}^2)\right]\right\}, \quad \text{(A4c)}$$

with L the total volume of the system. Then, we choose the trial state $|\ \rangle_{m'}$ as the ground state associated with a free-particle Hamiltonian (in ϕ_q) with mass m'. Then, using identities (2.4) and (2.6), we have

$$\left\langle N_m\left[\tfrac{1}{2}\phi_q{}^2+\frac{1}{2}\left(\frac{\partial\phi_q}{\partial x}\right)^2\right]\right\rangle_{m'}=\frac{1}{8\pi}(m'^2-m^2), \quad \text{(A5a)}$$

$$\langle H_{cq}\rangle_{m'}=0, \quad \text{(A5b)}$$

$$\langle N_m(\phi_q{}^2)\rangle_{m'}=\frac{1}{4\pi}\ln\frac{m^2}{m'^2}, \quad \text{(A5c)}$$

$$\langle N_m(\phi_q{}^4)\rangle_{m'}=3\left(\frac{1}{4\pi}\ln\frac{m^2}{m'^2}\right)^2. \quad \text{(A5d)}$$

By the help of (A5), we can easily compute the energy density associated with this trial state:

$$V(\phi_c,m')\equiv\langle H\rangle_{m'}/L$$

$$=\tfrac{1}{2}m^2\phi_c{}^2+\tfrac{1}{4}g\phi_c{}^4-B\phi_c+\tfrac{1}{2}(m^2+3g\phi_c{}^2)\frac{1}{4\pi}\ln\frac{m^2}{m'^2}+\frac{3g}{4}\left(\frac{1}{4\pi}\ln\frac{m^2}{m'^2}\right)^2. \quad \text{(A6)}$$

From (A6), we obtain the Hartree result by minimizing V with respect to m'^2, giving

$$\frac{\partial V}{\partial m'^2}=\frac{1}{8\pi}-\frac{3g\phi_c{}^2+m^2}{8\pi m'^2}-\frac{3g}{2(4\pi)^2m'^2}\ln\frac{m^2}{m'^2}$$

$$=\frac{1}{8\pi m'^2}\left(m'^2-m^2-3g\phi_c{}^2-\frac{3g}{4\pi}\ln\frac{m^2}{m'^2}\right)$$

$$=0. \quad \text{(A7)}$$

Equation (A7) defines m' as a function of m and ϕ_c (m' is independent of B), and (A6) and (A7) give rise to the Hartree effective potential,

$$V_{\text{eff}}(\phi_c,B)=[V(\phi_c,m')]_{m'\text{ given in (A7)}}.$$

Note that

$$V_{\text{eff}}(\phi_c,B)=V_{\text{eff}}(\phi_c)-B\phi_c, \quad \text{(A8)}$$

where $V_{\text{eff}}(\phi_c)$ is the Hartree effective potential at $B=0$. $V_{\text{eff}}(\phi_c)$ was computed earlier,[1] and its numerical results are reproduced here as Fig. 1. The Hartree calculation predicts a first-order transition at $g/m^2=10.211$ for $B=0$. The first-order transition persists for small but finite B. This prediction violates the Simon-Griffiths theorems.

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[6]This derivation is based on the normal-ordering methods invented by S. Coleman (see Ref. 4). The author wishes to thank Professor Coleman for informing him that a simpler derivation exists.

[7]The validity of Eq. (2.14) as an exact identity between the intrinsic coupling constants was first recognized

by Coleman. The author wishes to thank A. Neveu for transmitting this (unpublished) information to him.

[8]See B. Simon's book in Ref. 2, p. 342, Theorem IX.16.

[9]See B. Simon's book in Ref. 2, p. 345, Theorem IX.18.

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