Collective phenomena in gauge theories. II. Renormalization in finite-temperature field theory*

M. B. Kislinger and P. D. Morley
The Enrico Fermi Institute and the Department of Physics, University of Chicago, Chicago, Illinois 60637
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A procedure is given for renormalization of finite-temperature and -density Green's functions to all orders of perturbation theory. The Lagrangian counterterms are independent of temperature, density, and all mass parameters of the Lagrangian. Thus renormalization-group equations may be written for Green's functions in a medium for which the Callan-Symanzik functions depend only on the dimensionless coupling constants. By scaling the temperature and/or chemical potential in the renormalization-group equations it is shown that asymptotic freedom in non-Abelian gauge theories occurs for large temperatures (early hot universe) and for large chemical potential (high density of the medium) as well as large momenta.

I. INTRODUCTION

This is the second paper in a series devoted to collective phenomena in gauge theories. Several authors have recently investigated the behavior of spontaneously broken gauge theories of the weak and electromagnetic interactions at finite temperature.[1] We address ourselves here to the question of renormalization of finite-temperature and/or -density field theory (FTF). A renormalization program is given here which we make use of in our investigations of the behavior of gauge theories at finite temperature and density (FTD). This program generalizes the mass-independent renormalization prescriptions of Weinberg[2] and 't Hooft[3] to a mass-, temperature-, and density-independent renormalization prescription. The advantage of this formulation is that renormalization-group equations for Green's functions at FTD have Callan-Symanzik functions depending only on the dimensionless coupling constants. This greatly facilitates the investigation of the behavior of matter at high FTD via renormalization-group equations.

We begin by reviewing the Feynman rules for a finite-temperature field theory. For simplicity we consider a field theory with one chemical potential, μ, corresponding to a conserved charge carried by the fermions, which we call fermion number. Also, for simplicity, all fermions have the same charge and antifermions have the opposite charge (for example, in a theory with fractionally charged quarks with baryon number $\frac{1}{3}$, μ corresponds to baryon number density). This paper is organized as follows: Section II is devoted to renormalization of FTF to all orders of perturbation theory; Sec. III presents the example of ϕ^4 theory at the two-loop level; in Sec. IV the renormalization-group equations for a medium are derived and discussed. There it is shown that asymptotically free gauge theories exhibit asymptotically free behavior as temperature and/or chemical potential become large.

II. FORMALIZATION

The Feynman rules are the $T=0$ (T is temperature), $\mu=0$ rules with the following replacements[4]:

$$\int \frac{d^4K}{(2\pi)^4} \to \frac{i}{\beta}\sum_N \int \frac{d^3K}{(2\pi)^3}, \quad \beta = \frac{1}{T}$$

$$K^0 = \omega_N, \quad \omega_N = 2\pi N i/\beta \quad \text{(bosons)}$$

$$K^0 = \omega_N + \mu, \quad \omega_N = (2N+1)i\pi/\beta \quad \text{(fermions)}$$

$$N = 0, \pm 1, \pm 2, \ldots .$$

A Feynman diagram in FTF has the general form

$$\left[\prod_{j=1}^{N} \sum_{\nu_N} \frac{i}{\beta} \int \frac{d^3K_j}{(2\pi)^3}\right] \frac{\mathcal{P}}{\left[\prod_{i=1}^{N}(K_i^2 - m_i^2)\right]\left\{\prod_{j=1}^{m}[(\alpha_{ji}K_i + \beta_{jl}P_l)^2 - m_j^2]\right\}} . \quad (1)$$

$\mathcal{P}$ involves γ matrices and polynomials in momenta. The P_l are external momenta with

$$P_l^0 = \omega_l', \quad \omega_l' = 2li\pi/\beta \quad \text{(bosons)}$$

$$P_l^0 = \omega_l' + \mu, \quad \omega_l' = (2l+1)i\pi/\beta \quad \text{(fermions)}$$

and m_i^2 are the squared masses.

The first step in relating renormalization of FTF to ordinary field theory is to convert frequency sums to contour integrals. For bosons,

$$\frac{i}{\beta}\sum_{-\infty}^{+\infty} f(\nu_N = 2N\pi i/\beta) = \frac{1}{2\pi}\oint_C \frac{dK^0}{\exp(\beta K^0)-1} f(K^0),$$

where contour C is given in Fig. 1. Thus for bosons

$$\frac{i}{\beta}\sum_{N=-\infty}^{+\infty} f(\nu_N) = \frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{dK^0 f(K^0)}{\exp(\beta K^0)-1} - \frac{1}{2\pi}\int_{-i\infty-\epsilon}^{+i\infty-\epsilon} \frac{dK^0 f(K^0)}{\exp(-\beta K^0)-1} + \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} dK^0 f(K^0). \tag{2}$$

The terms on the right-hand side of Eq. (2) will be denoted respectively by (2a), (2b), and (2c). For fermions

$$\frac{i}{\beta}\sum_{N=-\infty}^{+\infty} f(\nu_N+\mu,\, \nu_N=(2N+1)i\pi/\beta) = -\frac{1}{2\pi}\oint_C \frac{dK^0 f(K^0+\mu)}{\exp(\beta K^0)+1}$$

$$= -\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{dK^0 f(K^0+\mu)}{\exp(\beta K^0)+1}$$

$$- \frac{1}{2\pi}\int_{-i\infty-\epsilon}^{+i\infty-\epsilon} \frac{dK^0 f(K^0+\mu)}{\exp(-\beta K^0)+1} + \int_{-i\infty}^{+i\infty} dK^0 f(K^0+\mu).$$

Changing the integration variable, we obtain

$$\frac{i}{\beta}\sum_{N=-\infty}^{+\infty} f(\nu_N+\mu) = -\frac{1}{2\pi}\int_{-i\infty+\mu+\epsilon}^{+i\infty+\mu+\epsilon} \frac{dK^0 f(K^0)}{\exp[\beta(K^0-\mu)]+1} - \frac{1}{2\pi}\int_{-i\infty+\mu-\epsilon}^{+i\infty+\mu-\epsilon} \frac{dK^0 f(K^0)}{\exp[\beta(\mu-K^0)]+1} + \frac{1}{2\pi}\oint_{C'} dK^0 f(K^0)$$
$$+ \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} dK^0 f(K^0). \tag{3}$$

The contour C' is shown in Fig. 2. Terms on the right-hand side of Eq. (3) will be denoted respectively by (3a), (3b), (3c), and (3d). When $T,\mu\to 0$ only term (2c) in Eq. (2) and (3d) in Eq. (3) survive. Thus

$$\frac{i}{\beta}\sum_{\nu_N}\int \frac{d^3K}{(2\pi)^3} \xrightarrow[T,\mu\to 0]{} \int_{-i\infty}^{+i\infty}\frac{dK}{2\pi}\int\frac{d^3K}{(2\pi)^3},$$

which is an integral over Euclidean momenta.

The basic step in our renormalization program is to show that all infinities occur in subdiagrams in which the integrals over loop momenta are temperature and chemical-potential independent (Euclidean space integrals). That is, only terms (2c) and (3d) are involved. Then the infinities are just those which would be there if we took $\mu, T\to 0$.

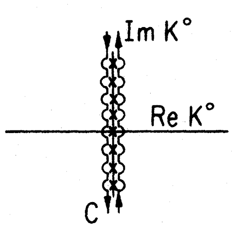

FIG. 1. C contour in complex K^0 plane, crosses are poles at $K^0=2N\pi i/\beta$ for bosons and $K^0=(2N+1)\pi i/\beta$ for fermions.

This requires demonstrating that integrals over loop momenta associated with (2a) and (2b) and (3a), (3b), and (3c) do not give rise to infinities. The demonstration relies heavily on the contour closing. We now illustrate the contour-closing method. Consider Eq. (1) and let us treat

$$\frac{i}{\beta}\sum_{\nu_1}\int\frac{d^3K_1}{(2\pi)^3}$$

first, assuming K_1 is a fermion (we find that it is easier to evaluate all fermion loop momenta before evaluating boson loop momenta). There are four terms in Eq. (3). In the first three terms we close the contours; the fourth term is left alone while we go on to the next loop momentum K_2^0. When we close the K_1^0 contours, keeping all other variables fixed, we pick up the residues of poles in K_1^0. For example, a typical denominator containing K_1^0 might be

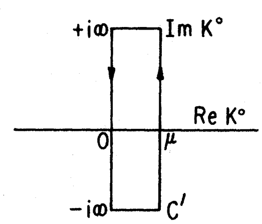

FIG. 2. Contour C'.

$$\frac{1}{(K_1-K_j)^2-m^2}=\frac{1}{\{K_1^0-K_j^0-[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}\}}\,\frac{1}{\{K_1^0-K_j^0+[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}\}},$$

with $K_j^0=2N_j\pi i/\beta$.

Upon evaluating the contour integral we get in Eq. (1) that

$$\begin{aligned}\frac{i}{\beta}\sum_{\nu_1}\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{(K_1-K_j)^2-m^2}\cdots\\
\to i\int\frac{d^3K_1}{(2\pi)^3}\,&\frac{\theta([(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}-\mu)}{2[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}[\exp\{\beta([(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}-\mu)\}+1]}\bigg|_{K_1^0-K_j^0=+[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}}\cdots\\
-i\int\frac{d^3K_1}{(2\pi)^3}\,&\frac{\theta(\mu-[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2})}{2[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}[\exp\{\beta(\mu-[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2})\}+1]}\bigg|_{K_1^0-K_j^0=+[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}}\cdots\\
+i\int\frac{d^3K_1}{(2\pi)^3}\,&\frac{1}{2[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}[\exp\{\beta(\mu+[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2})\}+1]}\bigg|_{K_1^0-K_j^0=-[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}}\cdots\\
+i\int\frac{d^3K_1}{(2\pi)^3}\,&\frac{\theta(\mu-[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2})}{2[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}}\bigg|_{K_1^0-K_j^0=+[(\vec{K}_1-\vec{K}_j)^2+m^2]^{1/2}}\cdots\\
+\frac{1}{(2\pi)^4}\int_{-i\infty}^{+i\infty}dK^0\int d^3K_1\,&\frac{1}{(K_1-K_j)^2-m^2}\cdots+\text{other pole terms}\end{aligned}\tag{4}$$

The first five terms on the right-hand side of Eq. (4) will be denoted respectively by (4a), (4b), (4b), (4c), and (4d). Terms (4a)–(4d) are respectively from (3a)–(3d). In expression (4) the dots stand for the remaining factors in Eq. (1) and we have suppressed the summation and integration symbols for the other variables. By each term we show the value which $K_1^0-K_j^0$ takes everywhere it appears in the factors not shown explicitly. The other pole terms not shown came from other propagator denominators containing K_1^0. We have also used the fact that $\exp(\beta K_j^0)=1$ since $K_j^0=2N_j\pi i/\beta$. Before going on, it is easiest to change variables in the above terms, so that $\vec{K}_1-\vec{K}_j=\vec{K}_1'$ and $\int d^3K_1\to\int d^3K_1'$. Now we see that closing the contours has resulted in various terms which have absolutely convergent three-momentum integrals over K_1'. Ultraviolet infinities due to large K_1 can occur only in the term containing the T- and μ-independent Euclidean space integral. We shall refer to the terms (4a)–(4c) as "finite K_1"; term (4d) will be called "temperature-independent K_1" or TIK_1 for short.

Next, we treat the loop momentum K_2. Closing contours proceeds as for K_1, that is, we close the temperature-dependent K_2 contours and leave the TIK_2 contours alone. There are now some new complications which we must deal with. These occur in the terms involving the TIK_1 and the temperature-dependent K_2 contours (TDK_2). A possible factor in these terms is a denominator of the form $[(K_2-K_1-K_l)^2-m^2]$ and closing the TDK_2 contour around the poles of this denominator (let us assume that K_2 is a boson momentum) will give rise to factors such as

$$\int_{-i\infty}^{+i\infty}dK_1^0\int d^3K_1\int\frac{d^3K_2}{(2\pi)^3}\,\frac{1}{2[(\vec{K}_2-\vec{K}_1-\vec{K}_l)^2+m^2]^{1/2}}\times\frac{1}{\exp\{\beta([(\vec{K}_2-\vec{K}_1-\vec{K}_l)^2+m^2]^{1/2}+K_1^0)\}-1}\bigg|_{K_2^0-K_1^0-K_l^0=[(\vec{K}_2-\vec{K}_1-\vec{K}_l)^2+m^2]^{1/2}}\cdots$$

(K_l is actually irrelevant to the discussion). Since we are dealing with $\int_{-i\infty}^{+i\infty}dK_1^0$, the K_1^0 in the argument of the exponential complicates things. We first redefine $K_2-K_1-K_l=K_2'$ with $K_2^{0\prime}=[(\vec{K}_2-\vec{K}_1-\vec{K}_l)^2+m^2]^{1/2}$, so we have terms involving

$$\int_{-i\infty}^{+i\infty}dK_1^0\int d^3K_1\int\frac{d^3K_2'}{(2\pi)^3}\,\frac{1}{2[(\vec{K}_2')^2+m^2]^{1/2}}\,\frac{1}{\exp\{\beta([(\vec{K}_2')^2+m^2]^{1/2}+K_1^0)\}-1}\bigg|_{K_2^{0\prime}=[(\vec{K}_2')^2+m^2]^{1/2}}\cdots.$$

Now we must close the K_1^0 contour to the right (thus avoiding any poles due to the vanishing of $\exp\{\beta([(\vec{K}_2')^2+m^2]^{1/2}+K_1^0)\}-1$. In doing so we will pick up poles from denominator factors such as

$$(K_1^2-m^2),\quad (K_1+K_j)^2-m^2\,|_{K_j\neq K_2},\quad\text{and}\quad (K_2'+K_1-K_m)^2-m^2\,.$$

In each case the resulting integration is a term involving absolutely convergent integrals over both K_1 and K_2. For example, the first denominator above results in a factor $(\exp\{\beta[(\vec{K}_2')^2+m^2)^{1/2}+(\vec{K}_1^2+m^2)^{1/2}]\}-1)^{-1}$

which gives the convergence. The second denominator above results in a similar convergence factor, while the third gives

$$\int \frac{d^3K_1}{(2\pi)^3} \frac{\theta([(\vec{K}_1+\vec{K}_2'-\vec{K}_m)^2+m^2]^{1/2}-[(\vec{K}_2')^2+m^2]^{1/2})}{2[(\vec{K}_1+\vec{K}_2'-\vec{K}_m)^2+m^2]^{1/2}[\exp\{\beta([(\vec{K}_1+\vec{K}_2'-\vec{K}_m)^2+m^2]^{1/2})\}-1]}\,.$$

For this last case we change variables to $\vec{K}_1'=\vec{K}_1+\vec{K}_2'-\vec{K}_m$, $K_1^0=[(\vec{K}_1')^2+m^2]^{1/2}$, so we have

$$\int \frac{d^3K_1'}{(2\pi)^3} \frac{\theta([(\vec{K}_1')^2+m^2]^{1/2}-[(\vec{K}_2')^2+m^2]^{1/2})}{2[(\vec{K}_1')^2+m^2]^{1/2}[\exp\{\beta([(\vec{K}_1')^2+m^2]^{1/2})\}-1]}\,.$$

These are all convergent integrations. Of course, there will be contributions to TIK_1 – TDK_2 in which the K_2 contour is closed around poles not involving K_1. In such cases we leave the TIK_1 alone. Terms involving finite K_1 and TDK_2 result in absolutely convergent K_1 and K_2 integrals and we will call such terms finite K_1–finite K_2. We leave the terms involving TIK_2 alone for now. Proceeding in this way we convert expression (1) to expression (5) below, which is a sum of terms, each of which will have some loop momenta whose integral over d^3q_l will be absolutely convergent (as in the case finite K_1–finite K_2 above). The remaining loop momenta are associated with temperature- and density-independent Euclidean momentum integrals

$$\pm\frac{\left[\prod_{l=1}^{s} i\int\frac{d^3q_l}{(2\pi)^3}\frac{1}{2(\vec{q}_l^{\,2}+m_l^{\,2})^{1/2}}\right]\left[\prod_i \theta(\gamma_{ij}'q_j^0+\delta_i'\mu)\right]\left[\prod_{l=s+1}^{N}\int_{-i\infty}^{+i\infty}\frac{dq_l^0\,d^3q_l}{(2\pi)^3}\right]\mathcal{P}}{\left\{\prod_{i=1}^{i_{\max}}(\exp[\beta(\gamma_{ij}q_j^0+\delta_i\mu)]+(-1)^{1+\delta_i})\right\}\left\{\prod_{j=1}^{m+N-s}[(\alpha_{jr}'q_r+\beta_{jl'}'P_{l'})^2-m_j^{\,2}]\right\}}\,, \tag{5}$$

where

$$\alpha_{jr}'=\pm1,0,\quad \beta_{jl'}'=\pm1,0$$

$$\gamma_{ij},\gamma_{ij}'=\begin{cases}\pm1,0, & 1\le j\le s\\ 0, & j\ge s+1\end{cases},\quad i_{\max}\le s,\quad 1\le r\le N$$

$$\delta_i,\delta_i'=\pm1,0,\quad q_l^0=\alpha_l(\vec{q}^{\,2}+m_l^{\,2})^{1/2},\quad \text{with } 1\le l\le s$$

and $\alpha_l=\pm1$.

The q_l ($1\le l\le s$) correspond to a change of integration variables from $\{k_i\}$ to $\{q_i\}$; the q's are N linearly independent propagator momenta from Eq. (1) which have been redefined. The γ_{ij},γ_{ij}' are such that the integrals over d^3q_l are absolutely convergent, owing to the θ and exponential functions. For the purpose of identifying infinities we may regard the q_l, $1\le l\le s$, as fixed and consider the rest of the integrations:

$$\frac{\left[\prod_{l=s+1}^{N}\int_{-i\infty}^{+i\infty}\frac{dq_l^0}{(2\pi)}\int\frac{d^3q_l}{(2\pi)^3}\,\mathcal{P}\right]}{\left\{\prod_{j=1}^{N+m-s}[(\alpha_{jr}'q_r+\beta_{jl'}'P_{l'})^2-m_j^{\,2}]\right\}}\,. \tag{6}$$

In Eq. (6) we have a subdiagram whose external momenta are q_l, $1\le l\le s$, and p_l'. The integrals over q_l, $s+1\le l\le N$, are ordinary vacuum integrals (although in Euclidean space), hence the infinities which arise are T, $\mu=0$ infinities ($T\mu$I) of subdiagrams which are removed by $T\mu$I counterterms of lower order. The only new infinities which arise are due to the term with $S=0$ in Eq. (6), i.e., in Eq. (1) let

$$\frac{i}{\beta}\sum_{\mu_l}\int\frac{d^3K_l}{(2\pi)^3}\to\int_{-i\infty}^{+i\infty}\frac{dK_l^0}{2\pi}\int\frac{d^3K_l}{(2\pi)^3}\ \text{for all } l\,.$$

However, this is just canceled by the new $T\mu$I vacuum counterterm which appears at this order.

III. ϕ^4 TO TWO LOOPS

As an example we will consider ϕ^4 theory to the two-loop level. The unrenormalized Lagrangian is

$$\mathcal{L}_0=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_R^{\,2}\phi^2\,,$$

$$\mathcal{L}_I=-\tfrac{1}{2}(m_B^{\,2}-m_R^{\,2})\phi^2-g_B\phi^4/4!\,.$$

For regularizing infinities we will use dimensional regularization.[5] The free propagator is $i/(P^2-m_R^{\,2})$. The one-loop self-energy diagram and counterterm diagrams are shown in Figs. 3(a) and 3(b). From Fig. 3(a) we get (using the contour-closing method of the preceding section)

$$-\frac{ig_R}{2}\mu^{4-N}\int\frac{d^NK}{(2\pi)^N}\frac{i}{K^2-m_l^{\,2}}=-\mu^{4-N}\frac{ig_R}{2(2\pi)^N}\pi^{N/2}\Gamma(1-\tfrac{1}{2}N)(m_R^{\,2})^{N/2-1}\frac{-ig_R}{2}\mu^{4-N}\int\frac{d^3K}{(2\pi)^3}\frac{1}{(\vec{K}^2+m_R^{\,2})^{1/2}}\times\frac{1}{\exp[\beta(\vec{K}^2+m_R^{\,2})^{1/2}]-1}$$

$$=-\frac{g_Rim_R^{\,2}}{(4\pi)^2(N-4)}\frac{-g_Ri\Gamma_{\text{finite}}(1-\tfrac{1}{2}N)m_R^{\,2}}{2(4\pi)^2}\frac{-ig_Rm_R^{\,2}}{32\pi^2}\ln\!\left(\frac{m_R^{\,2}}{4\pi\mu^2}\right)-\frac{ig_R}{2}\int\frac{d^3K}{(2\pi)^3}\frac{1}{(\vec{K}^2+m_R^{\,2})^{1/2}}\frac{1}{\exp[\beta(\vec{K}^2+m_R^{\,2})^{1/2}]-1}\,.$$

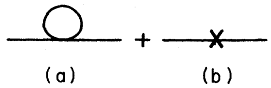

FIG. 3. (a) One-loop self-energy in ϕ^4. (b) Counterterm diagram for one loop.

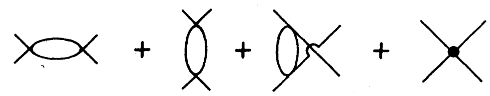

FIG. 4. Vertex corrections in two loops.

From Fig. 3(b) $-i(m_B{}^2 - m_R{}^2) = i g_R m_R{}^2/(4\pi)^2(N-4)$. This complete the one-loop calculation. There arose no temperature-dependent infinities to this order. Note that at finite temperature we have a finite-temperature-dependent radiative correction to (mass)^2 as

$$g_R \int \frac{d^3K}{(2\pi)^3} \frac{1}{2(\vec{K}^2 + m_R{}^2)^{1/2}} \frac{1}{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1} .$$

Going to two loops we will encounter temperature-dependent infinities. We have vertex corrections from Fig. 4. The infinities are all the same, so we may do just (4a),

$$\frac{(-ig_R)(-ig_R)}{2} \int \frac{d^NK}{(2\pi)^N} \frac{i}{K^2 - m_R{}^2} \frac{i}{(P-K)^2 - m_R{}^2} = \frac{g_R{}^2}{2} \frac{i\pi^2}{(2\pi)^4} \Gamma(2 - \tfrac{1}{2}N) + \text{finite terms} ,$$

so

$$g_B = \mu^{4-N}\left[g_R - \frac{3g_R{}^2}{16\pi^2(N-4)}\right] .$$

The two-loop contributions to the propagator and their counterterms are shown in Fig. 5. Figure 5(a) gives

$$\frac{-i}{2}\mu^{4-N}\left[\frac{-3g_R{}^2}{16\pi^2(N-4)}\right] \int \frac{d^NK}{(2\pi)^N} \frac{i}{K^2 - m_R{}^2}$$
$$= \frac{3}{2} \frac{\mu^{4-N} g_R{}^2}{16\pi^2(N-4)} \left\{ \frac{+i\pi^{N/2}}{(m_R{}^2)^{1-N/2}} \frac{\Gamma(1-\frac{1}{2}N)}{(2\pi)^N} + i \int \frac{d^3K}{(2\pi)^3} \frac{1}{(\vec{K}^2 + m_R{}^2)^{1/2}} \frac{1}{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1} \right\} . \tag{7}$$

Figure 5(b) gives

$$\frac{i g_R \mu^{4-N} m_R{}^2}{(4\pi)^2(N-4)} \left(\frac{-ig}{2}\right) \int \frac{d^NK}{(2\pi)^N} \left(\frac{i}{K^2 - m_R{}^2}\right)^2 = -\frac{g_R{}^2 m_R{}^2 \mu^{4-N}}{2(4\pi)^2(N-4)}$$
$$\times \left\{ \frac{i\pi^{N/2}}{(m_R{}^2)^{2-N/2}} \frac{\Gamma(2-\frac{1}{2}N)}{2(2\pi)^N} - i \int \frac{d^3K}{(2\pi)^3} \left(\frac{\beta \exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}]}{\{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1\}^2} \frac{1}{2(\vec{K}^2 + m_R{}^2)} + \frac{1}{\{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1\}} \frac{1}{2(\vec{K}^2 + m_R{}^2)^{3/2}} \right) \right\} . \tag{8}$$

Figure 5(c) gives

$$\mu^{8-2N} \frac{(-ig_R)^2}{4} \int \frac{d^NK}{(2\pi)^N} \left(\frac{i}{K^2 - m_R{}^2}\right)^2 \int \frac{d^Nl}{(2\pi)^N} \frac{i}{l^2 - m_R{}^2}$$
$$= i\mu^{8-2N} \frac{g_R{}^2}{4} \left[\frac{i\pi^{N/2}\Gamma(2-\frac{1}{2}N)}{(m_R{}^2)^{2-N/2}(2\pi)^N} - i \int \frac{d^3K}{(2\pi)^3} \left(\frac{\beta \exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}]}{\{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1\}^2} \frac{1}{2(\vec{K}^2 + m_R{}^2)} + \frac{1}{2\{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1\}} \frac{1}{(\vec{K}^2 + m_R{}^2)^{3/2}} \right) \right]$$
$$\times \left\{ -i\pi^{N/2} \frac{\Gamma(1-\frac{1}{2}N)(m_R{}^2)^{N/2-1}}{(2\pi)^N} - i \int \frac{d^3K}{(2\pi)^3} \frac{1}{(\vec{K}^2 + m_R{}^2)^{1/2}} \frac{1}{\exp[\beta(\vec{K}^2 + m_R{}^2)^{1/2}] - 1} \right\} . \tag{9}$$

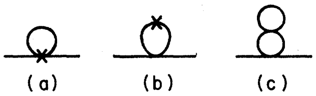

FIG. 5. Two-loop corrections and counterterms to the propagator.

FIG. 6. Overlapping two-loop contribution to propagator.

The overlapping divergence in two loops is the last diagram to evaluate, Fig. 6. This gives

$$
\begin{aligned}
&(-ig)^2\,\frac{(i)^3}{6}\int\frac{d^NK_2}{(2\pi)^N}\int\frac{d^NK_1}{(2\pi)^N}\,\frac{1}{K_2{}^2-m_R{}^2}\,\frac{1}{K_1{}^2-m_R{}^2}\,\frac{1}{(K_2-K_1-P)^2-m_R{}^2}\\
&\quad=(-ig)^2\,\frac{(i)^3}{6}\left[\frac{i}{\beta}\sum_{\nu_2}\int\frac{d^3K_2}{(2\pi)^3}\right]\\
&\quad\times\Bigg\{-i\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{2(\vec{K}_1{}^2+m_R{}^2)^{1/2}}\,\frac{1}{\exp[\beta(\vec{K}_1{}^2+m_R{}^2)^{1/2}]-1}\,\frac{1}{K_2{}^2-m_R{}^2}\,\frac{1}{(K_2-K_1-P)^2-m_R{}^2}\Bigg|_{K_1^0=\pm(\vec{K}_1{}^2+m_R{}^2)^{1/2}}\\
&\qquad-i\int\frac{d^3K_1'}{(2\pi)^3}\,\frac{1}{2[(\vec{K}_1')^2+m_R{}^2]^{1/2}}\,\frac{1}{\exp[\beta(\vec{K}_1'^2+m_R{}^2)^{1/2}]-1}\,\frac{1}{K_2{}^2-m_R{}^2}\,\frac{1}{(K_1'+K_2-P)^2-m_R{}^2}\Bigg|_{K_1^{0\prime}=\pm[(\vec{K}_1')^2+m_R{}^2]^{1/2}}\\
&\qquad+\frac{1}{2\pi}\int_{-i\infty}^{+i\infty}dK_1^0\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{K_1{}^2-m_R{}^2}\,\frac{1}{K_2{}^2-m_R{}^2}\,\frac{1}{(K_2-K_1-P)^2-m_R{}^2}\Bigg\}\,. \qquad (10)
\end{aligned}
$$

We use Eq. (2) to transform

$$\frac{i}{\beta}\sum_{\nu_2}\int\frac{d^3K_2}{(2\pi)^3}$$

into contour integrals. We see that there are temperature-dependent infinities coming from each term on the right-hand side of Eq. (10). The first two terms have temperature-dependent infinities arising from the $\int_{-i\infty}^{+i\infty}dK_2^0$ integration and contribute

$$\frac{-ig_R{}^2}{12\pi^2(N-4)}\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{2(\vec{K}_1{}^2+m_R{}^2)^{1/2}}\,\frac{1}{\exp[\beta(\vec{K}_1{}^2+m_R{}^2)^{1/2}]-1}\,.$$

The third and last term in Eq. (10) has an infinity when we take the TDK_2 contour integrals and close them over the pole from $(K_2{}^2-m_R{}^2)^{-1}$; then we evaluate the $\int_{-i\infty}^{+i\infty}dK_1^0$ integral for the resulting temperature-dependent infinity

$$\frac{-ig_R{}^2}{24\pi^2(N-4)}\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{2(\vec{K}_1{}^2+m_R{}^2)^{1/2}}\,\frac{1}{\exp[\beta(\vec{K}_1{}^2+m_R{}^2)^{1/2}]-1}\,.$$

Adding these two infinities together we find that Fig. 6 contains the temperature-dependent infinity

$$\frac{-ig_R{}^2}{16\pi^2(N-4)}\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{(\vec{K}_1{}^2+m_R{}^2)^{1/2}}\,\frac{1}{\exp[\beta(\vec{K}_1{}^2+m_R{}^2)^{1/2}]-1}\,. \qquad (11)$$

Adding the temperature-dependent infinities in Eqs. (7), (8), (9), and (11) we see that they cancel. What remains is the usual $T\mu$I arising at this order from

$$(-ig)^2\,\frac{(i)^3}{6}\int_{-i\infty}^{+i\infty}dK_2^0\int\frac{d^3K_2}{(2\pi)^3}\int_{-i\infty}^{+i\infty}dK_1^0\int\frac{d^3K_1}{(2\pi)^3}\,\frac{1}{(K_2{}^2-m_R{}^2)}\,\frac{1}{(K_1{}^2-m_R{}^2)}\,\frac{1}{(K_2-K_1-P)^2-m_R{}^2}\,.$$

This completes the example.

IV. RENORMALIZATION-GROUP EQUATIONS FOR A MEDIUM

In our ϕ^4 example we used dimensional regularization for the temperature-independent infinities. This conveniently allows for mass-independent renormalization procedures such as described in Ref. 6. Coupled with our renormalization of FTF we have then a temperature-, chemical-potential-, and mass-independent renormalization program. Thus the renormalization-group equation for an FTF Green's function has the form

$$\left[\mu \frac{\partial}{\partial\mu} + T \frac{\partial}{\partial T} + K \frac{\partial}{\partial K} - \beta(g_R) \frac{\partial}{\partial g_R} + (1+\gamma_m(g_R)) m_R \frac{\partial}{\partial m_R} + \gamma_\Gamma - D_\Gamma\right] \Gamma_R(KP_0, \mu, T, g_R, m_R, S) = 0. \tag{12}$$

Subscript R refers to renormalized quantities. S is the renormalization point parameter and K is a scale factor for the momentum with P_0 a fixed momentum. γ_Γ and D_Γ are, respectively, the anomalous and canonical dimensions of Γ_R. Using standard methods[6] the solution to Eq. (12) is obtained by defining a K-dependent effective coupling, mass, temperature, and chemical potential through the differential equations

$$K \frac{d}{dK} g(K) = \beta(g(K)),$$

$$K \frac{d}{dK} m(K) = -[1+\gamma_m(g(K))] m(K),$$

$$K \frac{d}{dK} \mu(K) = -\mu(K),$$

$$K \frac{d}{dK} T(K) = -T(K),$$

and the initial conditions

$$g(1) = g_R, \quad m(1) = m_R, \quad T(1) = T, \quad \mu(1) = \mu.$$

Then Eq. (12) has the solution

$$\Gamma_R(KP_0, \mu, T, g_R, m_R, S) = K^{D_\Gamma} \Gamma_R(P_0, \mu(K), T(K), g(K), m(K), S) \exp\left[-\int_1^K \gamma_\Gamma(g(K')) \frac{dK'}{K'}\right]. \tag{13}$$

Note that $\mu(K)$ and $T(K)$ are the same for all theories:

$$\mu(K) = \mu/K, \quad T(K) = T/K.$$

Instead of scaling momentum we could have scaled μ or T and obtained a solution analogous to Eq. (13).

One sees that the same behavior of the Green's function occurs for any two variables held fixed and the remaining one scaled. Thus, for example, asymptotic freedom[7] in non-Abelian gauge theories occurs for large momenta (small distances), large temperature (early hot universe), and large chemical potential (high density of the medium). See Ref. 8 for a calculation of the asymptotic behavior of the equation of state in an asymptotically free theory.

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