Class of scalar-field soliton solutions in three space dimensions*

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A class of three-space-dimensional soliton solutions is given; these solitons are made of scalar fields and are of a nontopological nature. The necessary conditions for having such soliton solutions are (i) the conservation of an additive quantum number, say Q, and (ii) the presence of a neutral ($Q = 0$) scalar field. It is shown that there exist two critical values of the additive quantum number, Q_C and Q_S, with Q_C smaller than Q_S. Soliton solutions exist for $Q > Q_C$. When $Q > Q_S$, the lowest soliton mass is $< Qm$, where m is the mass of the free charged meson field; therefore, there are solitons that are stable quantum mechanically as well as classically. When Q is between Q_C and Q_S, the soliton mass is $> Qm$; nevertheless, the lowest-energy soliton solution can be shown to be always classically stable, though quantum-mechanically metastable. The canonical quantization procedures are carried out. General theorems on stability are established, and specific numerical results of the soliton solutions are given.

I. INTRODUCTION

In this paper, we shall present a class of soliton solutions in three space dimensions; these solitons are made of scalar fields and are of a nontopological nature[1] (to be distinguished from the monopole-type solutions given by 't Hooft[2] and by Nielsen and Olesen[3]).[4] A brief description of such nontopological solitons has been given in Ref. 1. As we shall see, they serve as prototypes of a rather general class of soliton solutions, whose realization hinges on the existence of some integral constraints on the fields, which are, in turn, the consequences of the appropriate physical conservation laws in the theory, such as charge, isospin, etc. Generalization of such soliton solutions to include fields of nonzero spins will be given in a subsequent paper. Throughout our discussion, we consider only relativistic local fields with nonlinear couplings that are renormalizable in the usual sense (i.e., in terms of the usual perturbation series expansion around the plane-wave solutions of the free-field equations).

To begin with, it may be useful to give the definition of a soliton solution that is appropriate to particle physics. Following Ref. 1, we define a classical soliton solution to be one that (i) has a finite and nonzero rest mass and (ii) is confined in a finite region in space at *all* times (i.e., nondispersive). It can then be shown[5-7] that for every such classical soliton solution there exists a corresponding quantum soliton solution. The quantum soliton solution (i) also has a finite and nonzero mass, expressed in terms of the usual renormalized quantities which are defined by the usual perturbation series around the free plane-wave solutions, and (ii) has a spatial extension which gives rise to "soft" form factors[8] that go to zero at large momentum transfer. Because of the uncertainty principle, it is clearly not possible to construct a nondispersive wave packet of the quantum soliton solution. However, it can be readily shown that when the appropriately defined nonlinear coupling constant g becomes sufficiently small, both the mass and the form factor reduce asymptotically to their respective classical expressions; the mass is $O(g^{-2})$ and the form factor $O(g^0)$. When g decreases, the spatial extension of a quantum soliton, as determined by its form factor, remains *finite*, in accordance with its classical limit. This remarkable feature distinguishes a quantum soliton from either an atom or a molecule, whose size approaches infinity as the fine-structure constant $\alpha \to 0$. This difference underscores, once again, that in the context of a relativistic field theory the soliton solution already exists on a classical level, while atoms and molecules exist only in the quantum theory. (Our definition of soliton differs from a more narrow one, used in some mathematical and engineering literature.[9] In this narrow definition, the term soliton is confined only to some extremely specialized nonlinear equations that have solitary wave solutions whose shape and velocity remain unchanged even after a head-on collision. Such a highly restrictive definition would automatically

exclude all the four-dimensional local field theories that are of interest to particle physicists.)

For clarity, we consider first the simplest example of such a soliton solution. We assume the system to consist of only two spin-0 fields: a complex field ϕ and a Hermitian field χ. The Lagrangian density $\mathcal{L}$ is assumed to be (generalization will be given later in Sec. VI)

$$\mathcal{L} = -\frac{\partial\phi^\dagger}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}\left(\frac{\partial\chi}{\partial x_\mu}\right)^2 - f^2\chi^2\phi^\dagger\phi - \tfrac{1}{8}g^2(\chi^2-\chi_{\rm vac}{}^2)^2 , \tag{1.1}$$

where $x_\mu=(\vec{r},it)$, $\phi^\dagger$ is the Hermitian conjugate of ϕ, and f, g, and $\chi_{\rm vac}$ are constants. The theory possesses a discrete symmetry

$$\chi \to -\chi , \tag{1.2}$$

besides the U(1) symmetry

$$\phi \to e^{i\theta}\phi ; \tag{1.3}$$

because of this U(1) symmetry, there is the current conservation $\partial j_\mu/\partial x_\mu = 0$ where $j_\mu = -i[\phi^\dagger(\partial\phi/\partial x_\mu) - \phi(\partial\phi^\dagger/\partial x_\mu)]$. Consequently, the charge

$$Q = -i\int j_4\, d^3r \tag{1.4}$$

is a constant of motion.

We shall consider first the classical solution, and leave the discussion of the quantum solution to a later section, Sec. IV. Since Q depends linearly on $\dot{\phi}$, the classical solution for $Q\neq 0$ must be time dependent; for the lowest energy state, $\phi \propto \exp(-i\omega t)$. It is convenient to scale away both the physical dimension and the nonlinear coupling g. We introduce

$$\begin{aligned}\chi(\vec{r},t) &\equiv (\mu/g)A(\vec{\rho}) ,\\ \phi(\vec{r},t) &\equiv 2^{-1/2}(\mu/g)\,B(\vec{\rho})\,e^{-i\omega t} ,\end{aligned} \tag{1.5}$$

where A and B are both dimensionless and real,

$$\vec{\rho} \equiv \mu\vec{r} , \tag{1.6}$$

and

$$\mu = g\chi_{\rm vac} \tag{1.7}$$

is the mass of the neutral χ meson as can be readily seen from the Lagrangian (1.1). The corresponding mass of the charged ϕ meson is

$$m = f\chi_{\rm vac} . \tag{1.8}$$

From (1.1) and (1.5) it follows that the functions $A(\vec{\rho})$ and $B(\vec{\rho})$ satisfy

$$\vec{\nabla}^2 A - \kappa^2 B^2 A - \tfrac{1}{2}(A^2-1)A = 0 \tag{1.9}$$

and

$$\vec{\nabla}^2 B - \kappa^2 A^2 B + \nu^2 B = 0 , \tag{1.10}$$

where $\vec{\nabla}$ is the gradient operator with respect to the dimensionless parameter $\vec{\rho}$,

$$\nu \equiv \omega/\mu, \text{ and } \kappa \equiv m/\mu . \tag{1.11}$$

The charge Q is related to the frequency ω, or ν, by

$$Q = (\nu/g^2)\int B^2 d^3\rho . \tag{1.12}$$

The energy of the system is given by

$$E = (\mu/g^2)\int \mathcal{E}\, d^3\rho , \tag{1.13}$$

where

$$\mathcal{E} = \tfrac{1}{2}(\vec{\nabla}A)^2 + \tfrac{1}{2}(\vec{\nabla}B)^2 + \tfrac{1}{2}(\nu^2+\kappa^2A^2)B^2 + \tfrac{1}{8}(A^2-1)^2 \tag{1.14}$$

Through (1.12), ν may be regarded as a function of Q and a functional of $B(\vec{\rho})$. Upon substituting $\nu = \nu(Q,B)$ into (1.14), we may express E as a function of Q and a functional of $A(\vec{\rho})$ and $B(\vec{\rho})$. Equations (1.9) and (1.10) can also be derived by keeping Q fixed and setting the functional derivatives

$$\delta E/\delta A(\vec{\rho}) = \delta E/\delta B(\vec{\rho}) = 0 . \tag{1.15}$$

As we shall see, there exist *two* critical values of charge: Q_S and Q_C with $Q_S > Q_C$. Soliton solutions exist when the total charge Q is greater than the lower critical value, $Q > Q_C$. In general, at any given $Q > Q_C$, there is more than one soliton solution. A stability theorem will be established, which states that among these soliton solutions the one with the lowest energy is classically always stable against arbitrary small functional variations, while the others are not. When Q is greater than the upper critical value Q_S, the rest mass of the soliton solution (with the lowest energy) is less than Qm, the corresponding energy of the free ϕ-meson solution. Thus, when $Q > Q_S$, the soliton solution (with the lowest energy) is absolutely stable against complete dissociation into free mesons. (In fact, it is absolutely stable against any decay.) When $Q_S > Q > Q_C$, the soliton mass is $> Qm$. Nevertheless, classically one still has stable soliton solutions. The corresponding quantum soliton solution is, of course, metastable; however, its lifetime can be quite long in the weak-coupling limit, the lifetime $\to\infty$ when the nonlinear coupling $\to 0$.

The existence of the soliton together with some general properties of the solution are discussed in Sec. II. The question of stability is examined in Sec. III, and also partly in Sec. IV. The quantization is carried out in Sec. IV. The result of a numerical calculation of the soliton solution is given in Sec. V. Most of the methods developed for the simple Lagrangian (1.1) are applicable to a much wider class of problems. In particular,

if (1.1) is generalized to

$$\mathcal{L} = -\frac{\partial\phi^\dagger}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}\left(\frac{\partial\chi}{\partial x_\mu}\right)^2 - f^2\chi^2\phi^\dagger\phi - u(\chi)\ , \tag{1.16}$$

where $u(\chi)$ is an arbitrary function of χ, then except for some obvious changes, our entire discussions of the classical soliton solution (existence, stability, etc.) for the simple Lagrangian can be directly applied. For quantum soliton solutions, because of renormalization, $u(\chi)$ has to be a fourth-order polynomial of χ. The details are given in Sec. VI.

Throughout the paper we use the natural units $\hbar = c = 1$.

II. SOME GENERAL PROPERTIES

In this section we discuss some simple properties of the classical solutions of Eq. (1.9) and (1.10). Most of the properties derived are of a rather general character which can be applied to a much wider class of equations (see Sec. VI).

A. Free-meson solution

In order to derive the classical free-meson solution, it is convenient to enclose the system within a finite but extremely large volume Ω/μ^3 where Ω is dimensionless,

$$\Omega = \int d^3\rho\ . \tag{2.1}$$

We may assume Ω to be of cubic shape, and impose the familiar periodic boundary condition on the fields. The lowest-energy free-meson solution for a fixed charge Q can then be readily derived by using (1.9) and (1.10), and by assuming $A(\vec{\rho})$ and $B(\vec{\rho})$ to be constants. We introduce a small constant angle ϵ, defined by

$$(\sin^2\epsilon)\cos\epsilon = 2g^2 Q\kappa/\Omega\ ; \tag{2.2}$$

for Ω large, $\sin^2\epsilon = 2g^2 Q\kappa\Omega^{-1} + O(\Omega^{-2})$. The fields A and B are given by

$$A = \cos\epsilon \tag{2.3}$$

and

$$B = (Qg^2/\Omega\nu)^{1/2} = (2\kappa^2)^{-1/2}\sin\epsilon\ . \tag{2.4}$$

The corresponding energy of the system is

$$E = Qm(\sec\epsilon)(1 - \tfrac{3}{4}\sin^2\epsilon)\ , \tag{2.5}$$

and the frequency is given by $\nu = \kappa\cos\epsilon$, or

$$\omega = m\cos\epsilon\ . \tag{2.6}$$

At a fixed Q, in the limit $\Omega\to\infty$, we have $\epsilon = 0$, and therefore, as expected,

$$\omega = m \text{ and } E = Qm\ . \tag{2.7}$$

The corresponding infinite-volume limits for the fields are $A = 1$ and $B = 0$. Had we started with an infinite volume directly, the usual plane-wave solution for B would not be square-integrable, rendering the relation (1.12) between Q and B ambiguous.

B. Existence of solitons

In this section, we shall show that when the charge Q is larger than a critical value Q_S, there exists a soliton solution which is absolutely stable.[10] A simple way to show the existence of the soliton solution is to follow the variational approach. For the soliton solution, the volume of the system can be safely set to be infinite. We assume the trial functions[11]

$$A = \begin{cases} 0 & \text{for } r \le R\ , \\ 1 - \exp[-(r-R)/l] & \text{for } r \ge R \end{cases}$$

and (2.8)

$$B = \begin{cases} \dfrac{B_0}{r}\sin\omega r & \text{for } r \le R\ , \\ 0 & \text{for } r \ge R\ , \end{cases}$$

where $r = |\vec{r}|$, R and l are two length parameters,

$$\omega R = \pi \tag{2.9}$$

and because of (1.12), $B_0 = (\pi\mu)^{-1} g(\frac{1}{2}Q)^{1/2}$. By using (1.13), (1.14), and (2.8) we derive an upper bound for the lowest-energy value $E_{\min}$ at a given Q:

$$E_{\min} \le \frac{\pi Q}{R} + \frac{\pi\mu^2}{6g^2}\left[R^3 + \tfrac{11}{4}R^2 l + \tfrac{89}{24}Rl^2 + \tfrac{635}{288}l^3 + \frac{6}{\mu^2 l}(R^2 + Rl + \tfrac{1}{2}l^2)\right]. \tag{2.10}$$

This inequality holds for arbitrary lengths R and l. As Q increases, the optimal value of R increases with Q, while that of l remains $O(\mu^{-1})$. Thus, for Q large, the right-hand side of (2.10) becomes $R^{-1}\pi Q + \frac{1}{6}(\pi\mu^4 R^3/g^2) + O(R^2\mu^3/g^2)$. Taking its minimum, which occurs at

$$R \cong (2Qg^2)^{1/4}/\mu\ , \tag{2.11}$$

we find

$$E_{\min} \le \tfrac{4}{3}\pi\mu(2g^2)^{-1/4}Q^{3/4} + O(Q^{1/2}\mu/g)\ . \tag{2.12}$$

By comparing this value with Qm, the energy of the plane-wave solution, we see that

$$E_{\min} < Qm \text{ when } Q > Q_S\ , \tag{2.13}$$

where

$$Q_S \sim \left(\frac{4\pi\mu}{3m}\right)^4 \frac{1}{2g^2} . \tag{2.14}$$

Therefore, when $Q > Q_S$ the soliton solution exists and is absolutely stable.

It can be readily seen that the trial function (2.8) actually satisfies the differential equations (1.9) and (1.10) when $r \leq R$, and it approaches the correct boundary condition at ∞. Because R increases with Q, for Q sufficiently large the upper bound (2.12) is the correct asymptotic expression for $E_{\min}$. Thus, (2.14) also gives the correct limiting value of Q_S when $\kappa = m/\mu \to 0$. As will be shown in Appendix A, when $\kappa \to \infty$ on upper bound on Q_S can be obtained:

$$Q_S < 75\,\pi^4(\pi^4 - 36)^{1/2}/(512\,g^2\kappa^3) \cong 111.8\,(g^2\kappa^3)^{-1} . \tag{2.15}$$

Therefore, when $\kappa \to \infty$, $Q_S \to 0$. As we shall see, in the quantum theory Q must be an integer. According to (2.15), for κ sufficiently large, it is possible to stay in the weak-coupling region (g and $f = g\kappa$ both small) and yet have $Q_S < 1$; the quantum solitons would then be *stable for all* $Q \neq 0$.

C. Variational principles and virial theorem

As already noted in the Introduction, the differential equations (1.9) and (1.10) can be derived from the variational principle (1.15), keeping Q fixed, i.e.,

$$(\delta E)_Q = 0 . \tag{2.16}$$

The resulting stationary value of E is a function of Q. Its derivative is given by

$$\frac{d}{dQ} E(Q) = \omega . \tag{2.17}$$

The Legendre transformation

$$F \equiv E - \omega Q \tag{2.18}$$

defines a functional F of A and B. We may write

$$F = \frac{\mu}{g^2} \int \mathfrak{F}\, d^3\rho , \tag{2.19}$$

where the function $\mathfrak{F}$ is related to $\mathcal{E}$ of (1.14) by

$$\mathfrak{F} = \mathcal{E} - \nu^2 B^2 . \tag{2.20}$$

The variational principle (1.15), or (2.16), is equivalent to requiring F stationary against arbitrary functional variations in A and B but keeping ω fixed; i.e.,

$$(\delta F)_\omega = 0 . \tag{2.21}$$

We may regard the resulting F as a function of ω; it follows then

$$\frac{d}{d\omega} F(\omega) = -Q . \tag{2.22}$$

There exists still another variational formalism that is particularly useful in our later study of the stability problem. We define a functional G of A and B:

$$G \equiv \frac{\mu}{g^2} \int \mathcal{G}\, d^3\rho , \tag{2.23}$$

where

$$\mathcal{G} = \tfrac{1}{2}(\vec{\nabla}A)^2 + \tfrac{1}{2}(\vec{\nabla}B)^2 + \tfrac{1}{2}\kappa^2 A^2 B^2 + \tfrac{1}{8}(A^2 - 1)^2 . \tag{2.24}$$

The stationary condition (2.16), or (2.21), can also be expressed in terms of G. We require the functional G to be stationary, keeping

$$I \equiv (\mu g^2)^{-1} \int B^2\, d^3\rho \tag{2.25}$$

fixed. The condition

$$(\delta G)_I = 0 \tag{2.26}$$

implies

$$\frac{\delta G}{\delta A(\vec{\rho})} = 0 \tag{2.27}$$

and

$$\frac{\delta G}{\delta B(\vec{\rho})} = (\mu g^2)^{-1} \omega^2 B , \tag{2.28}$$

which are identical to (1.9) and (1.10), with ω^2 now appearing as the Lagrange multiplier. The resulting stationary value of G is a function of the constraint I. From (2.28), it follows that

$$\frac{d}{dI} G(I) = \tfrac{1}{2}\omega^2 . \tag{2.29}$$

The functional G is related to E and F by

$$\begin{aligned} G &= E - \tfrac{1}{2}\omega Q \\ &= F + \tfrac{1}{2}\omega Q , \end{aligned} \tag{2.30}$$

and I is related to Q and ω by

$$Q = I\omega . \tag{2.31}$$

The above three variational formulations, (2.16), (2.21), and (2.26), are applicable to *both* the soliton solution and the plane-wave solution, provided that we assume a finite (but large) volume and impose the periodic boundary condition. Of course, for the soliton solution, we may directly assume an infinite volume with the boundary condition: At ∞, $A \to 1$, and $B \to 0$; in addition, B is square-integrable. We note that both $\mathcal{E}$ and $\mathcal{G}$ are positive, but $\mathfrak{F}$ is not.

The virial theorem for the soliton solution can be most easily derived by using the variational

formalism (2.21). We consider the variation

$$A(\vec{r}) \to A(\lambda\vec{r}) \text{ and } B(\vec{r}) \to B(\lambda\vec{r})$$

for a soliton solution in an ∞ volume, where $\lambda = 1+\epsilon$ and $\epsilon = 0+$. By setting

$$\left(\frac{\partial F}{\partial \lambda}\right)_\omega = 0 \text{ at } \lambda = 1 ,$$

we find

$$\int U\, d^3\rho = \tfrac{1}{3}\int T\, d^3\rho , \tag{2.32}$$

where

$$T = \tfrac{1}{2}(\vec{\nabla}A)^2 + \tfrac{1}{2}(\vec{\nabla}B)^2$$

and (2.33)

$$U = \tfrac{1}{2}(\nu^2 - \kappa^2 A^2)B^2 - \tfrac{1}{8}(A^2-1)^2 .$$

At $\omega = 0$, since $\nu = 0$, U becomes ≤ 0 but T remains ≥ 0; the soliton solution disappears, in agreement with a well-known result derived by Derrick,[12] and by Goldstone and Jackiw.[7]

D. Soliton solution when ω is near m

From (2.9) and (2.11) we see that as ω increases from 0, the radius R of the soliton varies as π/ω and its charge Q decreases from ∞ in proportion to ω^{-4}. On the other hand, for the plane-wave solution, as the volume $\to \infty$, the lowest-energy state at a given Q is one with $\omega = m$. Now, both the plane-wave solution and the soliton solution satisfy the same set of differential equations (1.9) and (1.10). As we shall see, assuming that the volume of the system is sufficiently large, when ω increases from 0 to m, the soliton solution would evolve continuously and finally join onto the plane-wave solution. To find the connection between these two types of solutions, we shall investigate the soliton solution when ω is near m (i.e., ν near κ).

We define

$$\xi \equiv (\kappa^2 - \nu^2)^{1/2} . \tag{2.34}$$

When $\xi \to 0+$, it is convenient to introduce

$$A \equiv 1 - \tfrac{1}{2}(\xi/\kappa)^2 x ,$$
$$B \equiv 2^{-1/2}(\xi/\kappa^2)\, y , \tag{2.35}$$

and

$$|\vec{\rho}| \equiv \tau/\xi .$$

For the spherically symmetric solution, x and y are functions of τ. By substituting (2.35) for (1.9), one finds that to the lowest order in ξ^2

$$x = y^2 . \tag{2.36}$$

Likewise, from (1.10) one derives

$$\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2 \frac{dy}{d\tau}\right) - y + y^3 = 0 . \tag{2.37}$$

Let

$$K \equiv \frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 - \tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 . \tag{2.38}$$

Equation (2.37) may be written as

$$\frac{dK}{d\tau} = -\frac{2}{\tau}\left(\frac{dy}{d\tau}\right)^2 < 0 . \tag{2.39}$$

The solution of (2.37) has a simple mechanical analog. We may consider a point particle at "position" y and "time" τ, moving in a potential

$$W = -\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 \tag{2.40}$$

and under a frictional force $= -(2/\tau)(dy/d\tau)$; K denotes the energy of the point particle, and (2.39) gives the rate of energy dissipation by the frictional force.

The soliton solution is one that satisfies the boundary conditions

$$\frac{dy}{d\tau} = 0 \text{ at } \tau = 0$$

and (2.41)

$$y = 0 \text{ at } \tau = \infty .$$

The former is necessary so that the term $(2/\tau)\times(dy/d\tau)$ in (2.37) does not become singular at $\tau = 0$, and the latter is because of the square-integrability of B.

From the shape of W, given in Fig. 1(a), one sees that there exists a family of soliton solutions; each solution begins at a different initial position $y(\tau)$ at $\tau = 0$. We may arrange the initial positions of these different solutions in an ascending order:

$$y(0) = y_0(0), y_1(0), \ldots, y_n(0), \ldots . \tag{2.42}$$

Because of the gauge symmetry (1.3), $y \to -y$ is equivalent to a gauge rotation $\theta = \pi$. Thus, all these $y_n(0)$ can be taken to be positive.

For $n = 0$, the particle begins at $y_0(0)$ at $\tau = 0$ and ends at $y = 0$ at $\tau = \infty$. Its entire path is on the side $y \geq 0$, as shown in Fig. 1(b). For $n = 1$, $y(\tau)$ begins at $y_1(0)$ at $\tau = 0$. After a finite τ, it passes $y = 0$ to the negative-y region; later, it returns to $y = 0$ at $\tau = \infty$, as shown in Fig. 1(c). In the soliton language, these solutions are all radially symmetric. The $n = 0$ solution has no radial node, the $n = 1$ solution has one radial node, etc. (Numerical calculation of these solutions will be discussed in Sec. V.)

The charge Q is, according to (1.12) and (2.35), given by

$$Q = 2\pi(\xi g^2 \kappa^3)^{-1}\int_0^\infty y^2 \tau^2\, d\tau + O(\xi) . \tag{2.43}$$

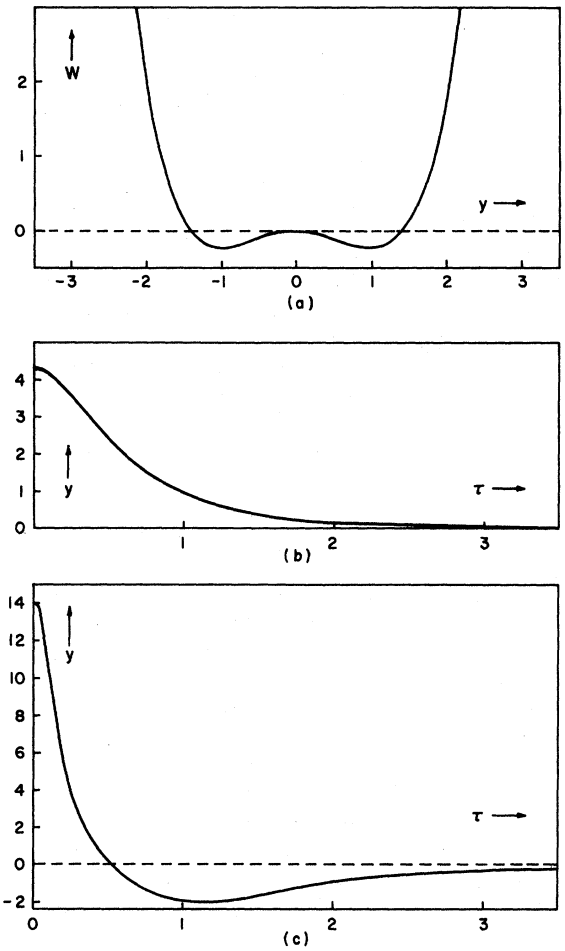

FIG. 1. (a) Potential $W=-\frac{1}{2}y^2+\frac{1}{4}y^4$, (b) ground state $y=y_0(\tau)$, and (c) first excited state $y=y_1(\tau)$ of Eq. (2.37).

Since by using (2.37) one sees that as $\tau\to\infty$, y is $\sim\tau^{-1}e^{-\tau}$, the integral (2.43) is clearly convergent. Thus, in the limit $\xi\to 0+$, $\omega\to m$ and $Q\to\infty$.

There also exists a variational formalism for (2.37). We define a functional L of $y(\tau)$:

$$L\equiv\int_0^\infty\left[\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2+\tfrac{1}{2}y^2-\tfrac{1}{4}y^4\right]\tau^2 d\tau\ . \tag{2.44}$$

The condition

$$\frac{\delta L}{\delta y(\tau)}=0 \tag{2.45}$$

gives (2.37). By following an argument similar to the one used in the derivation of (2.32), we obtain a virial relation

$$\tfrac{1}{3}N+M_2-M_4=0\ , \tag{2.46}$$

where

$$N=\tfrac{1}{2}\int_0^\infty\left(\frac{dy}{d\tau}\right)^2\tau^2 d\tau,\quad M_2=\tfrac{1}{2}\int_0^\infty y^2\tau^2\,d\tau\ ,$$

and

$$M_4=\tfrac{1}{4}\int_0^\infty y^4\tau^2\,d\tau\ . \tag{2.47}$$

Another relation between these moments can be derived by multiplying (2.37) by $\tau^2 y$ and integrating over $d\tau$. We find

$$N+M_2-2M_4=0\ . \tag{2.48}$$

Together, (2.46) and (2.48) yield

$$M_4=2M_2 \text{ and } N=3M_2\ . \tag{2.49}$$

By using (1.13), (2.35), (2.43), and (2.49), we derive for the soliton mass

$$E=Qm[1+\tfrac{1}{2}(\xi/\kappa)^2+O(\xi^4)]\ . \tag{2.50}$$

Thus, when ω is near m, the soliton solution has a *higher* mass than Qm; it approaches the plane-wave solution as $\omega\to m-$.

From (2.43), one sees that the product $Q\xi=4\pi(g^2\kappa^3)^{-1}M_2$ is independent of Q. Equation (2.50) may also be written as

$$E=Qm+\text{const}\times Q^{-1}+O(Q^{-3})\ ,$$

where the constant $=8m(\pi M_2/g^2\kappa^4)^2$.

Next, we would like to examine how these soliton solutions in the region ω near m are related to the ones in ω near 0. Let us consider the functions A and B given by (2.8). We note that (2.9) may be changed into

$$\omega R=(n+1)\pi\ , \tag{2.51}$$

where $n=0,1,2,\ldots$. It can be readily verified that, as before, the field equations (1.9) and (1.10) are satisfied when $r\leq R$. Different n denotes the different numbers of radial nodes in the solution. It is reasonable to expect that when ω varies from 0 to m, each of these solutions should change continuously into the corresponding solution of (2.37) with the same number of radial nodes. The lowest-energy soliton solution is one with no node.

E. Critical points

In Fig. 2(a), we plot the energy E vs the charge Q for the lowest-energy soliton solution. Since E is less than Qm when ω is near 0, but greater than Qm when ω is near m, the curve $E(Q)$ must intersect the straight line $E=Qm$ at least once (actually, only once); the point of intersection defines the critical point S. Since $Q\to\infty$ in both limits $\omega\to 0+$ and $\omega\to m-$, the soliton solution exists only if Q is above a critical value Q_C. Furthermore, because $dE/dQ=\omega$ which is always positive, when $Q=Q_C$ the curve $E(Q)$ must develop a spike; this determines the other critical point

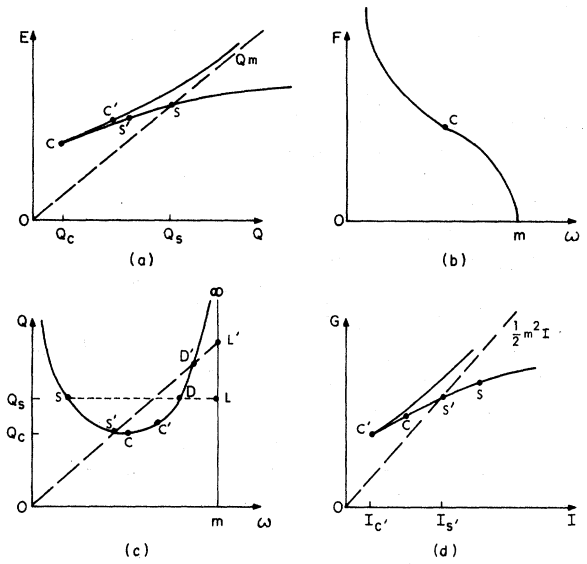

FIG. 2. Schematic drawings of (a) E vs Q, (b) F vs ω, (c) Q vs ω, and (d) G vs I. [See Eqs. (2.18), (2.23), and (2.25).]

C. In Fig. 2(a) the solid curve is $E(Q)$ for the soliton solution and the dashed line $E = Qm$ denotes the plane-wave solution. Both are only schematic drawings.

In Fig. 2(b), we plot the corresponding function F vs ω, where F is defined by (2.18). One sees that the critical point C is simply the point of inflexion:

$$\left(\frac{d^2F}{d\omega^2}\right)_C = -\left(\frac{dQ}{d\omega}\right)_C = 0 \ . \tag{2.52}$$

The charge Q of the same soliton solution is plotted vs ω in Fig. 2(c). The critical point C is the minimum of $Q(\omega)$. In the same figure, the plane-wave solution is represented by the straight line $\omega = m$. At the other critical point S, the soliton mass $E(Q) = Qm$. Since $dE = \omega dQ$, the critical point S may be determined graphically by the familiar rule of equal area: the area of the wedge $DL\infty$ = the area between the dashed horizontal line SD and the solid curve SCD in Fig. 2(c).

In Fig. 2(d), we plot the function G vs I for the same soliton solution, where G and I are respectively defined by (2.23) and (2.25). The dashed straight line $G = \frac{1}{2}m^2I$ denotes the plane-wave solution. There exist now two *new* critical points S' and C'. At S', G of the soliton solution $= \frac{1}{2}m^2I$. At C', the curve $G(I)$ develops a spike; since $dG/dI = \frac{1}{2}\omega^2$, this implies $(dI/d\omega)_{C'} = 0$. By using (2.31), $Q = I\omega$, we find

$$\frac{dQ}{d\omega} = I \quad \text{at } C' \ . \tag{2.53}$$

Thus, the point C' can be easily determined graphically from the curve $Q(\omega)$ in Fig. 2(c) by drawing from the origin a straight line $0C'$ tangent to the curve $Q(\omega)$. The point S' can also be determined graphically. We note that under the transformation $(Q, \omega) \to (I, \frac{1}{2}\omega^2)$ the area is invariant:

$$dQ\, d\omega = dI\, d(\tfrac{1}{2}\omega^2) \ . \tag{2.54}$$

Since $dG = (\frac{1}{2}\omega^2)\, dI$ and $Q = I\omega$, in order that at S' the soliton solution has the same G as that of the plane-wave solution, we may again apply the rule of equal area: In Fig. 2(c) the area of the wedge $D'L'\infty$ = the area between the straight line $0S'D'$ and the curve $S'C'D'$. The relative locations of these two pairs of critical points C, S and C', S' are also given (schematically) in the other two figures, 2(a) and 2(d).

III. STABILITY

The segment CS in Fig. 2(a) covers the region in which the lowest soliton mass is $\geq Qm$. In this section we want to prove that throughout the entire region CS the soliton solution is classically stable, though quantum mechanically only metastable.

A. Second-order variation

Let us keep Q fixed and, for clarity, consider first only an arbitrary real variation δA and δB from a solution A and B of Eqs. (1.9) and (1.10). (The case of a complex δB will be discussed in Sec. IV.) The first-order variation in E is zero, in accordance with (2.16). The second-order variation is given by

$$(\delta^2 E)_Q = \tfrac{1}{2}(\mu/g^2) \int \tilde{\psi} H \psi d^3\rho + 2\mu\nu^3 (Qg^4)^{-1} \left(\int \tilde{\psi} b d^3\rho \right)^2 , \tag{3.1}$$

where $\tilde{\psi}$ is the transpose of ψ,

$$\psi = \begin{pmatrix} \delta A \\ \delta B \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ B \end{pmatrix}, \tag{3.2}$$

and

$$H = -\vec{\nabla}^2 + \begin{pmatrix} \kappa^2 B^2 + \frac{1}{2}(3A^2 - 1) & 2\kappa^2 AB \\ 2\kappa^2 AB & \kappa^2 A^2 - \nu^2 \end{pmatrix}. \tag{3.3}$$

In deriving (3.1), we start from (1.14) in which ν is regarded as a functional of B through (1.12). Both δA and δB are completely arbitrary; the constraint Q being fixed simply induces the appropriate variation in ν, which in turn gives rise to both the $-\nu^2$ term in (3.3) and the second term on the right-hand side of (3.1).

Similarly, we may keep ω fixed and evaluate the

variation in F. According to (2.21), the first-order variation $(\delta F)_\omega$ is 0. The second-order variation can be easily seen to be

$$(\delta^2 F)_\omega = \tfrac{1}{2}(\mu/g^2)\int \vec{\psi} H \psi d^3\rho, \tag{3.4}$$

in which δA and δB are again *completely* arbitrary.

We may also keep I fixed and evaluate the variation in G. The first-order variation $(\delta G)_I$ is 0, on account of (2.26). As we shall see, the second-order variation is

$$(\delta^2 G)_I = \tfrac{1}{2}(\mu/g^2)\int \vec{\psi} H \psi d^3\rho, \tag{3.5}$$

in which because of the constraint I being fixed, δB satisfies

$$\int [2B\delta B + (\delta B)^2] d^3\rho = 0. \tag{3.6}$$

To derive (3.5), we may first consider an arbitrary variation δA and δB. By using (2.23)–(2.28), we find that, to second-order in δA and δB, the change $\Delta G \equiv G(A+\delta A, B+\delta B) - G(A,B)$ is given by

$$\Delta G = \frac{\mu}{g^2}\int \{\nu^2 B\delta B + \tfrac{1}{2}[\vec{\psi} H\psi + \nu^2(\delta B)^2]\} d^3\rho, \tag{3.7}$$

which leads to (3.5) because of (3.6).

B. Eigenvalue equation

The stability problem is closely connected with the eigenvalue equation

$$H\psi_i = \lambda_i \psi_i, \tag{3.8}$$

where ψ_i satisfies the usual boundary conditions of a Schrödinger wave function. There are a few simple but quite general properties of this eigenvalue equation. For clarity, these properties are stated in the form of mathematical theorems. Throughout our discussions, we assume that $A(\vec{\rho})$ and $B(\vec{\rho})$ satisfy (1.9) and (1.10), and they are both radially symmetric.

Theorem 1. H has at least one negative eigenvalue.

Proof. Because of translational invariance, $A(\vec{\rho}+\vec{\epsilon})$ and $B(\vec{\rho}+\vec{\epsilon})$ must also satisfy (1.9) and (1.10). For an infinitesimal $\vec{\epsilon}$, their deviations from the original soliton solution $A(\vec{\rho})$ and $B(\vec{\rho})$ are $\vec{\epsilon}\cdot\vec{\nabla}A(\vec{\rho})$ and $\vec{\epsilon}\cdot\vec{\nabla}B(\vec{\rho})$. By using these deviations, we can construct three p-state eigenfunctions of H, all with zero eigenvalues; i.e.,

$$H\psi_k = 0, \tag{3.9}$$

where

$$\psi_k = \vec{\nabla}_k \begin{pmatrix} A \\ B \end{pmatrix} \tag{3.10}$$

and $k = 1$, or 2, or 3. Since the lowest s-state eigenvalue of H must be lower than the lowest p-state eigenvalue, Theorem 1 is proved.

In Sec. II E, we have shown that the soliton solution exists only if $I \geq I_{C'}$. At a given I, the *soliton* solution with the lowest G value lies on the lower branch $C'CS$ of the curve $G(I)$ in Fig. 2(d).

Theorem 2. At any given $I \geq I_{C'}$, the H, evaluated by using the soliton solution with the lowest G value, has only one eigenvalue less than 0.

Proof. We assume that the total volume Ω of the system is sufficiently large. Let us consider a point P on the branch $C'CS$ in Fig. 2(d). For $I > I_{S'}$, P is the absolute minimum of the functional $G(A,B)$. The corresponding H must have at most only one eigenvalue < 0. Otherwise, if there are more than one, say both λ_1 and λ_2 are < 0, we may choose for ψ a suitable linear combination of the two corresponding eigenfunctions. By using (3.2), (3.5), and (3.6), we can easily satisfy the constraint that I be a constant, but we can make $(\delta^2 G)_I < 0$. This leads to a contradiction. The theorem is then established for $I > I_{S'}$.

The difficulty in proving the theorem is in the region $I_{C'} \leq I \leq I_{S'}$, when P is no longer the absolute minimum of the functional $G(A,B)$. To bypass this difficulty, let us consider a related problem.

We introduce two constraints, keeping both

$$I_1 = \int B d^3\rho \text{ and } I_2 = \int B^2 d^3\rho \text{ fixed}, \tag{3.11}$$

where, because of (2.25), I_2 is related to I by

$$I = (\mu g^2)^{-1} I_2. \tag{3.12}$$

Let $G_{\min}(I_1, I_2)$ be the minimum value of the functional $G(A,B)$ under the two constraints (3.11). We then plot $G_{\min}(I_1, I_2)$ against I_1 at a fixed I_2.

As before, let P be a point on the lower branch $C'CS$ in Fig. 2(d). For $I_2 > \mu g^2 I_{S'}$, P must lie on the curve $G_{\min}(I_1, I_2)$ vs I_1 and be its absolute minimum, since P is the absolute minimum of G under only one constraint, keeping only I_2 fixed. By using the results derived in Sec. II A, one can show that on the same curve there is another local minimum, called p, representing the plane-wave solution. The point p is located at an $I_1 \sim O(\Omega^{1/2})$ while P is at $I_1 \sim O(\Omega^0) \sim O(1)$. Thus, these two points, p and P, are always well separated in I_1. When $I_2 = \mu g^2 I_{S'}$, these two points have the same $G = \frac{1}{2}m^2 I_{S'}$; both are the absolute minima of $G_{\min}(I_1, I_2)$ vs I_1. For $I_2 < \mu g^2 I_{S'}$, p becomes the absolute minimum, while P is a local minimum. As we continuously decrease I_2, P will remain a local minimum of the curve $G_{\min}(I_1, I_2)$ vs I_1, until at some critical value, P becomes a point of inflexion. Thus as we approach this critical value, P must approach a local maximum, say P', of the curve $G_{\min}(I_1, I_2)$ vs I_1. By definition, P' satisfies

the stationary condition (2.26), $(\delta G)_I = 0$; therefore, P' also lies on the soliton curve $G(I)$ in Fig. 2(d). A glance at the figure tells us that $P \to P'$ only at the critical point C'. Thus, throughout the segment $C'S'$, the soliton solution is a local minimum of the functional $G(A, B)$ under *one* constraint, keeping only I_2 fixed. The same argument used in the beginning of this proof can now be extended to the region $I_{C'} < I < I_{S'}$. Theorem 2 is then established.

Remarks. The introduction of an additional constraint on I_1 separates in a clear way the plane-wave solution from the soliton solution; thereby it enables us to distinguish these two types of solutions even when they cross in Fig. 2(d). If one wishes, one may introduce, in place of the constraint on I_1, a constraint on $I_n = \int B^n d^3\rho$, where $n > 2$. The plane-wave solution has an $I_n \to 0$ as $\Omega \to \infty$, while that of the soliton solution remains nonzero.

C. Classical stability

Since Q is a constant of motion but I is not, the classical stability condition for the physical soliton solution is that its E should be a local minimum at a fixed Q.

Theorem 3. The necessary and sufficient conditions for $(\delta^2 E)_Q \geq 0$ under arbitrary variations δA and δB are (i) H has only one negative eigenvalue and (ii)

$$\frac{1}{Q}\frac{dQ}{d\omega} < 0, \tag{3.13}$$

where ω is, by choice, positive.

Proof. Condition (i) is necessary because otherwise, if there are two eigenfunctions of H that have negative eigenvalues, by a suitable linear combination of these two eigenfunctions one can easily construct $a\ \psi$ orthogonal to b; because of (3.1), the corresponding $(\delta^2 E)_Q$ is < 0.

To understand condition (ii), we first define

$$a(\vec{\rho}) \equiv \frac{\partial}{\partial \nu}\begin{pmatrix} A(\vec{\rho}) \\ B(\vec{\rho}) \end{pmatrix}, \tag{3.14}$$

in which $A(\vec{\rho})$ and $B(\vec{\rho})$, being solutions of (1.9) and (1.10), are regarded as functions of the parameter ν that appears in (1.10). By differentiating (1.9) and (1.10), we find

$$Ha = 2\nu b, \tag{3.15}$$

where b is given by (3.2). The eigenfunctions of H can be normalized to form a complete orthonormal set of real functions $\{\psi_i\}$. We may expand

$$a(\vec{\rho}) = \sum_i a_i\, \psi_i(\vec{\rho})$$

and (3.16)

$$b(\vec{\rho}) = \sum_i b_i\, \psi_i(\vec{\rho}).$$

From (3.15), it follows that

$$a_i \lambda_i = 2\nu b_i \tag{3.17}$$

thus,

$$b_i = 0 \text{ if } \lambda_i = 0. \tag{3.18}$$

We note that $\int \tilde{a} H a d^3\rho = \sum_i a_i^2 \lambda_i$; furthermore, because of (3.2), (3.14), and (3.15) the same integral is also equal to $2\nu \int \tilde{a} b d^3\rho = 2\nu \int B(\partial B/\partial \nu) d^3\rho$. By using (1.12), we find

$$\sum_i a_i^2 \lambda_i = g^2 \nu \frac{d}{d\nu}\left(\frac{Q}{\nu}\right), \tag{3.19}$$

which together with (3.17) leads to

$$\sum_i{}' \frac{b_i^2}{\lambda_i} = \frac{1}{4}\left(\frac{g^2}{\nu}\right)\frac{d}{d\nu}\left(\frac{Q}{\nu}\right), \tag{3.20}$$

where the sum $\sum_i'$ extends over all $\lambda_i \neq 0$.

Next, we expand an arbitrary ψ:

$$\psi = \sum_i c_i \psi_i. \tag{3.21}$$

Because of (3.1) and (3.18), the corresponding $(\delta^2 E)_Q$ can be written as

$$\left(\frac{2g^2}{\mu}\right)(\delta^2 E)_Q = \sum_i{}' \sum_j{}' (c_i M_{ij} c_j), \tag{3.22}$$

where

$$M_{ij} = \lambda_i \delta_{ij} + 4\nu^3 (Qg^2)^{-1} b_i b_j. \tag{3.23}$$

The eigenvalue z of the matrix (M_{ij}) can be readily determined by examining the equation

$$\sum_j{}' M_{ij} c_j = z c_i, \tag{3.24}$$

which implies that z is the root of

$$\mathcal{J}(z) \equiv \sum_i{}' \frac{4\nu^3 b_i^2}{Qg^2(z - \lambda_i)} - 1 = 0. \tag{3.25}$$

By using (3.20), we find

$$\mathcal{J}(0) = -\left(\frac{\nu}{Q}\right)\frac{dQ}{d\nu}. \tag{3.26}$$

In order for $(\delta^2 E)_Q$ to be always ≥ 0, none of the roots of (3.25) can be negative.

Now, assume that there is *only* one negative eigenvalue of H, say only λ_1 is < 0. From (3.25), one sees that $\mathcal{J}(z)$ is negative when $z < \lambda_1$, it $\to -\infty$ as $z \to \lambda_1 -$, then jumps to $+\infty$ as $z \to \lambda_1 +$. Its derivative $d\mathcal{J}/dz$ is always negative. Consequently, if

$\mathcal{J}(0)$ is >0, $\mathcal{J}(z)=0$ has no negative root. Therefore, conditions (i) and (ii) are sufficient to insure that $(\delta^2 E)_Q \geq 0$.

From Theorem 1, H is known to have at least one negative eigenvalue. Hence, by examining the plot $\mathcal{J}(z)$ vs z, one can easily verify that condition (ii) is also necessary. This completes the proof of Theorem 3.

As shown in Fig. 2(a), the soliton solution exists only if $Q \geq Q_C$.

Theorem 4. At any given $Q \geq Q_C$ the lowest-energy soliton solution is always classically stable.

Proof. In Fig. 2(d), the point C lies to the right of the point C'. By using Theorem 2, one sees that in Fig. 2(a), along the entire lower branch $CS'S$ H has only one negative eigenvalue. Since $Q^{-1}(dQ/d\omega)$ is negative, it follows from Theorem 3 that the soliton solutions on this entire lower branch $CS'S$ are all classically stable. Theorem 4 is then proved.

We note that the soliton solutions along the upper branch $CC'\infty$ in Fig. 2(a) are all classically unstable because $Q^{-1}(dQ/d\omega)$ is positive.

In Sec. IID, it was shown that at a given Q there can be other excited soliton solutions which have radial nodes. As we shall see, those solutions are also classically unstable.

Theorem 5. If in the spherically symmetric soliton solution the function B has one or more nodes, then $(\delta^2 E)_Q$ can be negative.

Proof. Let us consider Eq. (1.10), which is linear in B. The assumption that B has nodes implies that there exists a $\overline{B}$ with no node, or fewer nodes than B, which satisfies

$$(\vec{\nabla}^2 - \kappa^2 A^2)\overline{B} + \overline{\nu}^2 \overline{B} = 0, \tag{3.27}$$

with $\overline{\nu}^2 < \nu^2$. The function $\overline{B}$ is, of course, orthogonal to B. Thus by choosing

$$\psi = \begin{pmatrix} 0 \\ \overline{B} \end{pmatrix} \tag{3.28}$$

we find, through (3.1) and (3.3),

$$(\delta^2 E)_Q = \tfrac{1}{2}(\mu/g^2)\int (\overline{\nu}^2 - \nu^2)\overline{B}^2 d^3\rho < 0, \tag{3.29}$$

which establishes the theorem.

We note that if B has n nodes, then there are at least n such linearly independent $\overline{B}$ functions. (There may be more, since $\overline{B}$ does not have to be spherically symmetric.) Thus, the higher the number of nodes a soliton solution contains, the more unstable it is.

Remarks. The classical solutions A and B are real functions. So far, for clarity of presentation, their variations δA and δB are also assumed to be real. Since in a classical theory, χ is a real field, the variation δA must be real. On the other hand, ϕ is a complex field; the variation δB can be complex. The general case of a complex δB will be examined in Sec. IV. As we shall prove, the classical stability theorem (Theorem 4 given above) remains valid even when δB is complex. [See (4.78) and the remarks made at the end of Sec. IV D.]

IV. QUANTIZATION

To derive the quantum soliton solution, we shall follow the general canonical method developed by Christ and Lee.[5] For clarity of presentation, in this section the quantization procedure is carried out only for the center-of-mass system in which the total momentum $\vec{P}$ is zero. (The details for a moving system, $\vec{P} \neq 0$, are discussed in Appendix B.)

A. Collective coordinates

We introduce four collective coordinates, the three components $R_k(f)$ of the center-of-mass position vector ($k=1,2,3$) and an over-all phase variable $\zeta(t)$ for the charged field $\phi(\vec{r},t)$. Following Ref. 5, the quantum expansion of the operators $\chi(\vec{r},t)$, $\phi(\vec{r},t)$ and its Hermitian conjugate $\phi^\dagger(\vec{r},t)$ is given by

$$\chi(\vec{r},t) = \frac{\mu}{g} A(\vec{\rho}) + \sum_{N=5}^{\infty} q_N(t)\alpha_N(\vec{\rho}),$$

$$\phi(\vec{r},t) = \frac{1}{\sqrt{2}}\left[\frac{\mu}{g} B(\vec{\rho}) + \sum_{N=5}^{\infty} q_N(t)\beta_N(\vec{\rho})\right] e^{-i\zeta(t)}, \tag{4.1}$$

and

$$\phi^\dagger(\vec{r},t) = \frac{1}{\sqrt{2}}\left[\frac{\mu}{g} B(\vec{\rho}) + \sum_{N=5}^{\infty} q_N(t)\beta_N^*(\vec{\rho})\right] e^{i\zeta(t)},$$

where the q_N's are the (Hermitian) coordinates for the vibrational modes, μ is, as before, the mass of the neutral χ meson,

$$\vec{\rho} \equiv [\vec{r} - \vec{R}(t)]\mu, \tag{4.2}$$

$\alpha_N(\rho)$ and $\beta_N(\rho)$ are c-number functions of $\vec{\rho}$, and $A(\vec{\rho})$ and $B(\vec{\rho})$ are the same previously derived radically symmetric classical solutions of (1.9) and (1.10); α_N, A, and B are real functions of $\vec{\rho}$, but β_N is complex. We note that if we could set $q_N=0$, $\vec{R}=0$, and $\zeta=\omega t$, then (4.1) would give the classical soliton solution. The subscript N varies from 5 to ∞, which serves as a reminder that the four collective coordinates $R_1(t)$, $R_2(t)$, $R_3(t)$, and $\zeta(t)$ are excluded from the vibrational coordinates.

In order to separate the vibrational modes from the motion of the collective coordinates, we impose the constraints

$$\int [(\vec{\nabla}A)\alpha_N + \tfrac{1}{2}(\vec{\nabla}B)(\beta_N + \beta_N^*)]d^3\rho = 0 \tag{4.3}$$

and

$$\int (\beta_N - \beta_N^*)B d^3\rho = 0 \tag{4.4}$$

for all $N = 5, 6, \ldots$. These two conditions, (4.3) and (4.4), are introduced to exclude, respectively, the translational mode in space and that in the phase of ϕ from the vibrational modes. [Mathematically, as we shall see in (4.14) and (4.15), these two conditions insure that the coefficients of $\dot{R}_k \dot{q}_N$ and $\dot{\zeta}\dot{q}_N$ in the Lagrangian are $O(g^0)$, not $O(g^{-1})$.] In addition, the α_N's and the β_N's satisfy the orthonormality conditions

$$\int [\alpha_N \alpha_{N'} + \tfrac{1}{2}(\beta_N^* \beta_{N'} + \beta_N \beta_{N'}^*)]d^3r = \delta_{NN'}, \tag{4.5}$$

where $\delta_{NN'}$ is the usual Kronecker symbol. [See (4.35) and (4.56) for an explicit choice of these c-number functions.]

By using (1.1) and (4.1), the Lagrangian $L = \int \mathcal{L} d^3r$ can be readily expressed in terms of the coordinate vector

$$q \equiv \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \zeta \\ q_5 \\ q_6 \\ \vdots \end{bmatrix} \tag{4.6}$$

and its time derivative $\dot{q}$. We write

$$L = \tfrac{1}{2}\tilde{\dot{q}}\, \mathfrak{M} \dot{q} - V(q), \tag{4.7}$$

where

$$V = \int [\tfrac{1}{2}(\vec{\nabla}_r \chi)^2 + (\vec{\nabla}_r \phi^\dagger)\nabla_r \phi + f^2 \chi^2 \phi^\dagger \phi + \tfrac{1}{8} g^2 (\chi^2 - \chi_{\text{vac}}^2)^2] d^3r \tag{4.8}$$

and $\vec{\nabla}_r$ denotes the gradient with respect to $\vec{r}$ (while $\vec{\nabla}$ without any subscript always denotes the gradient with respect to $\vec{\rho}$). The mass matrix $\mathfrak{M}(q)$ is $\infty \times \infty$. We find

$$\mathfrak{M}_{kk'} = M\delta_{kk'} + \sum_N g^{-1} \int d^3\rho \left[2 \frac{\partial A}{\partial \rho_k} \frac{\partial \alpha_N}{\partial \rho_{k'}} + \frac{\partial B}{\partial \rho_k} \frac{\partial}{\partial \rho_{k'}} (\beta_N + \beta_N^*) \right] q_N + \sum_{N,N'} \mu^{-1} \int d^3\rho \left(\frac{\partial \alpha_N}{\partial \rho_k} \frac{\partial \alpha_{N'}}{\partial \rho_{k'}} + \frac{\partial \beta_N^*}{\partial \rho_k} \frac{\partial \beta_{N'}}{\partial \rho_{k'}} \right) q_N q_{N'}, \tag{4.9}$$

where

$$M = \tfrac{1}{3}(\mu/g^2) \int d^3\rho [(\vec{\nabla}A)^2 + (\vec{\nabla}B)^2]. \tag{4.10}$$

Throughout the paper, the subscripts k (or k') are treated differently from N (or N'):

$$k \text{ (or } k') = 1, 2, 3, \tag{4.11}$$

while

$$N \text{ (or } N') = 5, 6, \ldots .$$

The other matrix elements of $\mathfrak{M}(q)$ are

$$\mathfrak{M}_{44} = I + \sum_N (\mu^2 g)^{-1} \int d^3\rho (\beta_N + \beta_N^*) B q_N + \sum_{N,N'} \mu^{-3} \int d^3\rho (\beta_N \beta_{N'}^*) q_N q_{N'}, \tag{4.12}$$

where I is given by (2.25),

$$\mathfrak{M}_{4k} = \mathfrak{M}_{k4} = i \sum_N (\mu g)^{-1} \int d^3\rho \left(\frac{\partial B}{\partial \rho_k} \right) (\beta_N - \beta_N^*) q_N + \frac{1}{2} i \sum_{N,N'} \mu^{-2} \int d^3\rho \left[\beta_N \frac{\partial}{\partial \rho_k} \beta_{N'}^* - \beta_N^* \frac{\partial}{\partial \rho_k} \beta_{N'} \right] q_N q_{N'}, \tag{4.13}$$

$$\mathfrak{M}_{kN} = \mathfrak{M}_{Nk} = -\sum_{N'} \mu^{-2} \int d^3\rho \left[\alpha_N \frac{\partial}{\partial \rho_k} \alpha_{N'} + \frac{1}{2} \left(\beta_N \frac{\partial}{\partial \rho_k} \beta_{N'}^* + \beta_N^* \frac{\partial}{\partial \rho_k} \beta_{N'} \right) \right] q_{N'}, \tag{4.14}$$

$$\mathfrak{M}_{4N} = \mathfrak{M}_{N4} = -\frac{1}{2} i \sum_{N'} \mu^{-3} \int d^3\rho (\beta_N^* \beta_{N'} - \beta_N \beta_{N'}^*) q_{N'}, \tag{4.15}$$

and

$$\mathfrak{M}_{NN'} = \delta_{NN'}. \tag{4.16}$$

B. Canonical formalism

The conjugate momentum operators of $\vec{R}$, ζ, and q_N $(N=5,6,\ldots)$ are respectively

$$\vec{P}=-i\vec{\nabla}_R\,,\quad Q=-i\frac{\partial}{\partial\zeta}$$

and

$$p_N=-i\frac{\partial}{\partial q_N}\,. \tag{4.17}$$

Since the variation $\vec{R}\to\vec{R}+\delta\vec{R}$ is a space-translation, $\vec{P}$ is the total momentum operator. Similarly, since $\zeta\to\zeta+\delta\zeta$ gives an over-all shift in the phase of ϕ, Q is the total charge operator. Because ζ is a cyclic variable with a period $=2\pi$, the eigenvalues of its conjugate momentum Q are all integers: $0,\pm1,\pm2,\ldots$. By following the standard canonical procedure, we find from (4.7)

$$p=\mathfrak{M}\dot{q}\,, \tag{4.18}$$

where

$$p=\begin{bmatrix}P_1\\P_2\\P_3\\Q\\p_5\\p_6\\\cdot\\\cdot\\\cdot\end{bmatrix}\,. \tag{4.19}$$

The Schrödinger equation for the state vector $|\,\rangle$ can be written as

$$\mathcal{H}|\,\rangle=E|\,\rangle\,. \tag{4.20}$$

In the representation where the q's are diagonal, the Hamiltonian operator $\mathcal{H}$ is

$$\mathcal{H}=(2J)^{-1}\tilde{p}\mathfrak{M}^{-1}Jp+V(q)\,, \tag{4.21}$$

where p is the differential operator defined by (4.17) and (4.19), $\mathfrak{M}^{-1}(q)$ is the inverse matrix of $\mathfrak{M}(q)$, and $J(q)$ is the "Jacobian" given by

$$J(q)=[\det\mathfrak{M}(q)]^{1/2}\,.$$

As shown in Ref. 5, although $\mathfrak{M}(q)$ is $(\infty\times\infty)$, $J(q)$ can be evaluated in terms of a 4×4 matrix $\mathfrak{A}(q)$ whose square is given by

$$(\mathfrak{A}^2)_{ab}=\mathfrak{M}_{ab}-\sum_{N=5}^{\infty}\mathfrak{M}_{aN}\mathfrak{M}_{Nb}\,, \tag{4.22}$$

where the subscript a, or b, varies from 1 to 4:

$$J(q)=\det\mathfrak{A}(q)\,. \tag{4.23}$$

Since $J(q)$, $\mathfrak{M}(q)$, and $V(q)$ are all independent of the collective coordinates $\vec{R}$ and ζ, we have, as expected,

$$[\vec{P},\mathcal{H}]=0\quad\text{and}\quad[Q,\mathcal{H}]=0\,. \tag{4.24}$$

For simplicity, let us assume the center-of-mass system; i.e.,

$$\vec{P}|\,\rangle=0\,. \tag{4.25}$$

So far, the state vector $|\,\rangle$ can be an arbitrary eigenstate of Q. However, as we shall see, for g sufficiently small, in order that the solution of the Schrödinger equation (4.20) have a power-series expansion in g, we must have, to $O(g^{-1})$, $Q|\,\rangle=I\omega|\,\rangle$, where $I\omega$ is the charge of the classical soliton solution.

C. Power-series expansion

Let us consider the formal expansion of $\mathcal{H}$ in powers of q_N and p_N $(N=5,6,\ldots)$:

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\cdots\,, \tag{4.26}$$

where $\mathcal{H}_0$ is independent of q_N and p_N, $\mathcal{H}_1$ depends on q_N and p_N linearly, $\mathcal{H}_2$ quadratically, $\mathcal{H}_3$ cubically, etc. In explicit form, $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by

$$\mathcal{H}_0=M+(2I)^{-1}[Q^2+(I\omega)^2] \tag{4.27}$$

and, neglecting the $O(g)$ term due to the commutator between p_N and $q_{N'}$,

$$\mathcal{H}_1=(2\mu^2g)^{-1}[\omega^2-(Q/I)^2]\sum_{N=5}^{\infty}q_N\int d^3\rho(\beta_N+\beta_N^*)B\,, \tag{4.28}$$

where, as before, $Q=-i(\partial/\partial\zeta)$, I and M are given by (2.25) and (4.10), respectively, and $\omega=\nu\mu$ is the parameter that enters in (1.9) and (1.10) of which $A(\tilde{\rho})$ and $B(\tilde{\rho})$ are the solutions. In deriving the above expressions, we have set $\vec{P}=0$ and used the virial expression (2.32).

We note that $\mathcal{H}_0$ is proportional to g^{-2}, and $\mathcal{H}_1$ is proportional to g^{-1}. For g sufficiently small, in order that $\mathcal{H}_0$ does give the correct energy to $O(g^{-2})$, we must have $\mathcal{H}_1|\,\rangle$ equal to 0 (or at least to a higher order in g). Otherwise, by combining $\mathcal{H}_1$ and $\mathcal{H}_2$, we would find $q_N\sim O(g^{-1})$ and consequently an additional $O(g^{-2})$ term in energy from $\mathcal{H}_1+\mathcal{H}_2+\cdots$. Thus, we require

$$Q|\,\rangle=I\omega|\,\rangle\,, \tag{4.29}$$

and therefore $\mathcal{H}_1|\,\rangle=0$. Since the eigenvalues of Q are integers, only those classical solutions with $I\omega=$ integers can be used in the quantum solutions.

We may now expand the energy E in a power series in g^2:

$$E=E_0+E_2+E_4+\cdots\,, \tag{4.30}$$

where $E_0\sim O(g^{-2})$, $E_2\sim O(g^2E_0)\sim O(g^0)$, etc. By

using (4.27) and (4.29) we find

$$E_0 = M + I\omega^2 , \tag{4.31}$$

which is identical to the classical expression for the soliton mass.

To derive the lowest-order radiative correction, we need to diagonalize $\mathcal{H}_2$. The details of the diagonalization into its normal modes and the related characteristic frequencies will be given in Sec. IV D. In terms of these characteristic frequencies Ω_N, $\mathcal{H}_2$ is

$$\mathcal{H}_2 = \sum_N (\tfrac{1}{2} + \mathfrak{N}_N)\Omega_N - E_{\rm vac} , \tag{4.32}$$

where $E_{\rm vac}$ is the vacuum energy (without the soliton), and $\mathfrak{N}_N$ is the occupation number of the Nth normal mode. The vacuum energy is, as usual,

$$E_{\rm vac} = \tfrac{1}{2}\sum_n \Omega_n^0 , \tag{4.33}$$

where the sum n extends over all frequencies Ω_n^0 of the plane-wave solutions of both χ and ϕ fields. By setting the occupation number $\mathfrak{N}_N = 0$ for all N, we find that the lowest-order quantum correction to the soliton mass is $O(g^0)$; apart from an additive counterterm due to the usual renormalization of the zeroth-order energy term, it is given by

$$E_2 = \tfrac{1}{2}\sum_N \Omega_N - \tfrac{1}{2}\sum_n \Omega_n^0 . \tag{4.34}$$

Because the classical solutions $A(\vec{\rho})$ and $B(\vec{\rho})$ are regular everywhere, it is expected that the complete $O(g^0)$ correction to the soliton mass is finite.

It can be shown that in order to evaluate E_2, one needs only the bound-state energy and the S matrix of the vibrational modes, which will be studied in Sec. IV D. However, a complete discussion lies outside the scope of this paper.

D. Normal modes

To find the normal modes, we need the explicit form of $\mathcal{H}_2$. It is convenient to define

$$\psi_R \equiv \sum_N q_N \begin{pmatrix} \alpha_N \\ \tfrac{1}{2}(\beta_N + \beta_N^*) \end{pmatrix}$$

and (4.35)

$$\psi_I \equiv -i\tfrac{1}{2}\sum_N q_N(\beta_N - \beta_N^*) .$$

By using the various expressions derived above and after setting $\vec{P} = 0$ and $Q = I\omega$, we find

$$\mathcal{H}_2 = \tfrac{1}{2}\sum_N (p_N - \omega\mathfrak{M}_{4N})^2 + V_R + V_I , \tag{4.36}$$

with

$$V_R = \tfrac{1}{2}\mu^{-1}\int \psi_R^\dagger H_R \psi_R \, d^3\rho + \frac{2\omega^2}{(\mu^2 g)^2 I}\left(\int \psi_R^\dagger b \, d^3\rho\right)^2 \tag{4.37}$$

and

$$V_I = \tfrac{1}{2}\mu^{-1}\int \psi_I H_I \psi_I \, d^3\rho + \frac{2\omega^2}{(\mu g)^2 M}\left[\int \psi_I(\vec{\nabla}B)\, d^3\rho\right]^2 , \tag{4.38}$$

where, as before, the dagger denotes the Hermitian conjugate, b is given by (3.2),

$$H_R = H , \tag{4.39}$$

which is given by (3.3),

$$H_I = -\nabla^2 + \kappa^2 A^2 - \nu^2 , \tag{4.40}$$

and $\mathfrak{M}_{4N}$ is given by (4.15). In terms of ψ_R and ψ_I, the constraints (4.3) and (4.4) become, respectively,

$$\int \psi_R^\dagger \psi_k \, d^3\rho = 0 , \tag{4.41}$$

where

$$\psi_k = \frac{\partial}{\partial \rho_k}\begin{pmatrix} A \\ B \end{pmatrix} , \tag{4.42}$$

and

$$\int \psi_I B \, d^3\rho = 0 . \tag{4.43}$$

To diagonalize $\mathcal{H}_2$, we shall proceed in several steps:

1. *Normal modes of H_R and H_I*

Let ψ_{Ri} and ψ_{Ij} be the (c-number) eigenstates of H_R and H_I; i.e.,

$$H_R\psi_{Ri} = \lambda_{Ri}\psi_{Ri} \tag{4.44}$$

and

$$H_I\psi_{Ij} = \lambda_{Ij}\psi_{Ij} . \tag{4.45}$$

Without any loss of generality, we may choose all the ψ_{Ri}'s and ψ_{Ij}'s to be real.

Throughout this section, we shall consider only those classical solutions A and B that lie on the branch $CS'S$ in Fig. 2(a) (i.e., they are the lowest-energy classical soliton solutions at any given charge Q). We recall that $H_R = H$. As shown in (3.9), H_R has three p-state eigenfunctions ψ_1, ψ_2, and ψ_3, all with zero eigenvalue:

$$H_R\psi_k = 0 , \tag{4.46}$$

where ψ_k is given by (4.42). Furthermore, as proved in Theorem 2 of Sec. III B, H_R has one and only one negative eigenvalue. By combining (1.10) with (4.40), we see that H_I has one zero s-state

eigenvalue:

$$H_I B = 0 . \tag{4.47}$$

Since B has no node, H_I does not have any negative eigenvalue.

Let us define two sets of functions:

$$\{\psi_{Ri}\} \text{ and } \{\psi_{Ij}\}, \tag{4.48}$$

where the former contains all the eigenstates of H_R *except* the three ψ_k's, and the latter contains all the eigenstates of H_I *except* B. If we expand the operators ψ_R and ψ_I in terms of, respectively, the functions in $\{\psi_{Ri}\}$ and $\{\psi_{Ij}\}$, then the constraints (4.41) and (4.43) are clearly satisfied. The orthonormality relation (4.5) requires that

$$\int \bar{\psi}_{Ri} \psi_{Ri'} d^3\rho = \mu^3 \delta_{ii'}$$

and (4.49)

$$\int \bar{\psi}_{Ij} \psi_{Ij'} d^3\rho = \mu^3 \delta_{jj'} .$$

2. *Normal modes of V_R and V_I*

Next, we consider the eigenvalue equations generated by

$$\frac{\delta V_R}{\delta \psi_R} = \Lambda_R \psi_R \quad \text{and} \quad \frac{\delta V_I}{\delta \psi_I} = \Lambda_I \psi_I .$$

By labeling the respective c-number eigenfunctions Ψ_{Ri} and Ψ_{Ij}, and their associated eigenvalues Λ_{Ri} and Λ_{Ij}, we have

$$H_R \Psi_{Ri} + \frac{4\omega^2}{\mu^3 g^2 I} \left(\int d^3\rho \bar{b} \Psi_{Ri} \right) b = \Lambda_{Ri} \Psi_{Ri} \tag{4.50}$$

and

$$H_I \Psi_{Ij} + \frac{4\omega^2}{\mu g^2 M} \left[\int d^3\rho (\vec{\nabla} B) \Psi_{Ij} \right] \cdot \vec{\nabla} B = \Lambda_{Ij} \Psi_{Ij} . \tag{4.51}$$

Without any loss of generality, we may again choose these eigenfunctions Ψ_{Ri} and Ψ_{Ij} to be real.

Because b is spherically symmetric, *except* for the s-state solutions, (4.50) reduces to (4.44), and therefore there are three zero p-state eigenvalues of (4.50), whose eigenstates are the same ψ_k given by (4.42). For the s state, both the eigenfunction Ψ_{Ri} and the eigenvalue Λ_{Ri} can be readily obtained from those of H_R, as shown explicitly in Sec. III C. By following the proof of Theorem 3, we find that for every zero eigenvalue $\lambda_{Ri} = 0$ of H_R, there is a zero eigenvalue $\Lambda_{Ri} = 0$ of (4.50). All nonzero eigenvalues of (4.50) are the roots of (3.25), which may be written as

$$\mathcal{J}_R(z) \equiv \frac{4\omega^2}{I(\mu^2 g)^2} \sum_i{}' \frac{b_i^2}{z - \lambda_{Ri}} - 1 = 0 \tag{4.52}$$

where $b_i = \int \bar{\psi}_{Ri} b d^3\rho$ and the sum $\sum_i{}'$ extends over all $\lambda_{Ri} \neq 0$. [We note that ψ_{Ri} and λ_{Ri} in (4.44) are the same as ψ_i and λ_i in (3.21)–(3.26).] Since the classical solutions A and B are assumed to lie on the branch $CS'S$ in Fig. 2(a), Theorem 3 states that none of the Λ_{Ri}'s is negative.

Almost identical considerations can be applied to (4.51). *Except* for p states, (4.51) reduces to (4.45), and therefore there is one zero s-state eigenvalue of (4.51) whose eigenstate is B, in accordance with (4.47). For p states, both Ψ_{Ii} and Λ_{Ii} can be obtained from those of H_I in a similar way as in the previous case. As mentioned before, none of the eigenvalues of H_I is negative; the same holds for (4.51) [as can be seen directly from (4.38) by noting that the second term on its right-hand side is positive]. Thus, we establish for all i and j

$$\Lambda_{Ri} \geq 0 \text{ and } \Lambda_{Ij} \geq 0 . \tag{4.53}$$

Similar to (4.52), all nonzero eigenvalues of (4.51) are the roots of

$$\mathcal{J}_I(z) \equiv \frac{4\omega^2}{M(\mu g)^2} \sum_j{}' \frac{(\vec{b}_j)^2}{z - \lambda_{Ij}} - 1 = 0 , \tag{4.54}$$

where $\vec{b}_j = \int (\vec{\nabla} B) \psi_{Ij} d^3\rho$ and the sum $\sum_j{}'$ extends over all $\lambda_{Ij} \neq 0$.

Just as in (4.48), we construct two sets of functions

$$\{\Psi_{Ri}\} \text{ and } \{\Psi_{Ij}\}, \tag{4.55}$$

where the former contains all the eigenstates of (4.50) *except* the three ψ_k's, and the latter contains all the eigenstates of (4.51) *except* B. The expansion of the operators

$$\psi_R = \sum_i q_{Ri} \Psi_{Ri} \quad \text{and} \quad \psi_I = \sum_j q_{Ij} \Psi_{Ij} \tag{4.56}$$

in terms of these two sets of functions clearly satisfies the constraints (4.41) and (4.43), where ψ_R and ψ_I are defined by (4.35). In order to retain the orthonormal relation (4.5), we require

$$\int \bar{\Psi}_{Ri} \Psi_{Ri'} d^3\rho = \mu^3 \delta_{ii'}$$

and (4.57)

$$\int \Psi_{Ij} \Psi_{Ij'} d^3\rho = \mu^3 \delta_{jj'} .$$

Equations (4.37) and (4.38) can then be written in terms of their respective normal modes:

$$V_R = \tfrac{1}{2}\sum_{i=1}^{\infty} \bar{\Lambda}_{Ri}\, q_{Ri}^{\,2}$$

and (4.58)

$$V_I = \tfrac{1}{2}\sum_{j=1}^{\infty} \bar{\Lambda}_{Ij}\, q_{Ij}^{\,2},$$

where $\bar{\Lambda}_{Ri} \equiv \mu^2 \Lambda_{Ri}$, $\bar{\Lambda}_{Ij} = \mu^2 \Lambda_{Ij}$, and the sums i and j extend respectively over the two sets of functions in (4.55). (The order in i, or j, is arbitrary.) The same summation convention with respect to the indices i and j will be adopted throughout Sec. IV D 3.

3. *Normal modes of* $\mathcal{K}_2$

We now turn to the problem of finding the normal modes of $\mathcal{K}_2$. In the expression (4.36) for $\mathcal{K}_2$, the subscript N now goes over all the normal modes of V_R and V_I: $N = Ri$ and $N = Ij$. It is convenient to write the respective p_N as p_{Ri} or p_{Ij}. We define

$$p_R \equiv \begin{bmatrix} p_{R1} \\ p_{R2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad p_I \equiv \begin{bmatrix} p_{I1} \\ p_{I2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad (4.59)$$

$$q_R \equiv \begin{bmatrix} q_{R1} \\ q_{R2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad q_I \equiv \begin{bmatrix} q_{I1} \\ q_{I2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

On account of (4.58) and (4.59), (4.36) becomes

$$\mathcal{K}_2 = \tfrac{1}{2}(\tilde{p}_R p_R + \tilde{p}_I p_I) + \tfrac{1}{2}\tilde{q}_R(\bar{\Lambda}_R + \Gamma\tilde{\Gamma})q_R + \tfrac{1}{2}\tilde{q}_I(\bar{\Lambda}_I + \tilde{\Gamma}\Gamma)q_I + \tilde{p}_R\Gamma q_I - \tilde{p}_I\tilde{\Gamma}q_R, \qquad (4.60)$$

where $\bar{\Lambda}_R$ and $\bar{\Lambda}_I$ are diagonal matrices whose diagonal matrix elements are $\bar{\Lambda}_{Ri}$ and $\bar{\Lambda}_{Ij}$, respectively, and $\Gamma = (\Gamma_{ij})$ is a real $\infty \times \infty$ matrix given by

$$\Gamma_{ij} = -(\omega/\mu^3)\int d^3\rho\, \tilde{\Psi}_{Ri} \cdot \begin{pmatrix} 0 \\ \Psi_{Ij} \end{pmatrix}. \qquad (4.61)$$

We may cast the above expression (4.60) into a more compact form by introducing

$$\eta \equiv \begin{pmatrix} \mathcal{P} \\ \mathcal{Q} \end{pmatrix}, \qquad (4.62)$$

where

$$\mathcal{P} \equiv \begin{pmatrix} p_R \\ p_I \end{pmatrix} \text{and } \mathcal{Q} \equiv \begin{pmatrix} q_R \\ q_I \end{pmatrix}. \qquad (4.63)$$

[If in (4.6) and (4.19), one deletes the first four components of q and p, then these two column vectors reduce respectively to the above $\mathcal{Q}$ and $\mathcal{P}$.] $\mathcal{K}_2$ can then be written

$$\mathcal{K}_2 = \tfrac{1}{2}\tilde{\eta}\begin{pmatrix} 1 & \Xi \\ \tilde{\Xi} & \Delta \end{pmatrix}\eta, \qquad (4.64)$$

where

$$\Xi = -\tilde{\Xi} = \begin{pmatrix} 0 & \Gamma \\ -\tilde{\Gamma} & 0 \end{pmatrix} \qquad (4.65)$$

and

$$\Delta = \begin{pmatrix} \bar{\Lambda}_R + \Gamma\tilde{\Gamma} & 0 \\ 0 & \bar{\Lambda}_I + \Gamma\tilde{\Gamma} \end{pmatrix}. \qquad (4.66)$$

To diagonalize $\mathcal{K}_2$, we consider transformations of the form

$$\eta \to \eta' \equiv \begin{pmatrix} \mathcal{P}' \\ \mathcal{Q}' \end{pmatrix} = T \cdot \eta, \qquad (4.67)$$

where in order to retain the Hermiticity of the new coordinates q'_{Ri}, q'_{Ij} and their conjugate momenta p'_{Ri}, p'_{Ij}, we assume T to be real, and in order to preserve the canonical commutation relations between these coordinates and their conjugate momenta, we require

$$\tilde{T}\rho_2 T = \rho_2, \qquad (4.68)$$

where

$$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \qquad (4.69)$$

The transformation matrix T can be most easily derived by considering the Heisenberg equation of motion

$$i\dot{\eta} = [\eta, \mathcal{K}_2] \qquad (4.70)$$

and examining its normal mode solution

$$\eta(t) \propto e^{-i\Omega_N t}. \qquad (4.71)$$

By using the various expressions given above, we find that the coordinate vector $\mathcal{Q} \equiv \mathcal{Q}_N$ of the normal mode (4.71) satisfies

$$\begin{pmatrix} \Omega_N^2 - \bar{\Lambda}_R & -2i\Omega_N\Gamma \\ 2i\Omega_N\tilde{\Gamma} & \Omega_N^2 - \bar{\Lambda}_I \end{pmatrix} \cdot \mathcal{Q}_N = 0. \qquad (4.72)$$

Thus, Ω_N is the root of the quadratic equation

$$-c_1 + c_2\Omega_N + \Omega_N^2 = 0, \qquad (4.73)$$

where

$$c_1 = \mathcal{Q}_N^{\dagger}\begin{pmatrix} \bar{\Lambda}_R & 0 \\ 0 & \bar{\Lambda}_I \end{pmatrix}\mathcal{Q}_N \Big/ (\mathcal{Q}_N^{\dagger}\mathcal{Q}_N), \qquad (4.74)$$

$$c_2 = \mathcal{Q}_N^{\dagger}\begin{pmatrix} 0 & -2i\Gamma \\ 2i\tilde{\Gamma} & 0 \end{pmatrix}\mathcal{Q}_N \Big/ (\mathcal{Q}_N^{\dagger}\mathcal{Q}_N). \qquad (4.75)$$

More explicitly,

$$\Omega_N = \tfrac{1}{2}[-c_2 \pm (c_2^2 + 4c_1)^{1/2}]. \tag{4.76}$$

Since

$$\bar{\Lambda}_{Ri} = \mu^2 \Lambda_{Ri} \geq 0 \text{ and } \bar{\Lambda}_{Ij} = \mu^2 \Lambda_{Ij} \geq 0, \tag{4.77}$$

c_1 is real and positive. Because Γ is a real matrix, c_2 is real. Hence, the frequencies

$$\Omega_N\text{'s are all real.} \tag{4.78}$$

The transformation matrix T can be obtained explicitly in terms of these normal-mode solutions in a standard way. Through T, $\mathcal{H}_2$ takes on the diagonal form (4.32), which gives the radiative correction (4.34) to the soliton mass. From (4.78), it also follows that the classical soliton solution is stable along the entire branch $CS'S$ in Fig. 2(a). (Therefore, Theorem 4 holds even if the variation δB can be complex.) Quantum mechanically, the soliton is stable only if $Q > Q_S$. For $Q_S > Q > Q_C$, the quantum soliton is metastable; however, since the barrier penetration factor depends exponentially on $(-g^{-2})$, its lifetime can be quite long if g^2 is $\ll 1$.

V. NUMERICAL CALCULATION

In this section we discuss briefly our numerical study of the soliton solution of Eqs. (1.9) and (1.10). The aim of this analysis is two fold: (i) to confirm some of the general properties discussed on theoretical grounds in Sec. II and (ii) to give a more precise determination of the critical points Q_S and Q_C.

In order to describe the strategy of the numerical calculation, it is convenient to discuss first the solutions of Eqs. (2.37) and (2.41) which, in conjunction with Eq. (2.35), govern the behavior of the soliton solution in the limit $\nu \to \kappa$. For the ground state, we search for a solution of Eqs. (2.37) and (2.41) with no radial nodes. The simple mechanical analog discussed after Eq. (2.39) suggests then the following approach: We select a tentative positive initial value y_0 at $\tau = 0$ and integrate numerically the differential equation up to the lowest value τ at which either $y(\tau) < 0$ or its derivative $y'(\tau) > 0$. When this occurs the solution is rejected, a new initial value is selected, and the integration repeated. The mechanical analog indicates the direction in which y_0 must be changed: If $y(\tau) < 0$, the initial "potential energy" was too large and y_0 must be decreased; if $y'(\tau) > 0$, y_0 must be increased. In this way we are led to the correct initial value y_0. A similar strategy was used to find the initial value y_1 of the first excited solution, with one radial node. The corresponding solutions are depicted in Figs. 1(b) and 1(c).

Numerical integrations of the differential equation (2.37) were carried out with a fourth-order Runge-Kutta method at intervals of 10^{-3}. As a check, the validity of the equalities $\frac{1}{3}N + M_2 = M_4$ and $N + M_2 = 2M_4$ [see Eqs. (2.46) and (2.48)] was verified with a relative error $\epsilon < 10^{-8}$ for the ground-state solution and $\epsilon \cong 10^{-8}$ for the first excited solution.

For the two-component system of Eqs. (1.9) and (1.10) we restrict ourselves to studying the lowest-energy soliton solution. Thus we search for spherically symmetric solutions of Eqs. (1.9) and (1.10) with no radial nodes, subject to the boundary conditions:

$$\frac{dA}{d\rho} = \frac{dB}{d\rho} = 0 \text{ at } \rho = 0$$

and (5.1)

$$A = 1, \; B = 0 \text{ at } \rho = \infty.$$

The former is necessary so that the terms $(2/\rho)(dA/d\rho)$ and $(2/\rho)(dB/d\rho)$ do not become singular at $\rho = 0$, and the latter is necessary because of the requirement that $\mathcal{E}$ and B^2 be integrable over the infinite volume. The differential equations tell us that A and B approach their asymptotic limits at infinity exponentially, modulo inverse powers of ρ. From the symmetry of the problem, it is clear that we can restrict our considerations to the quadrant $A \geq 0$, $B \geq 0$.

It is convenient to introduce the quantities

$$\hat{E} \equiv \frac{Eg^2}{8\pi m} = \frac{1}{2\kappa} \int_0^\infty \mathcal{E} \rho^2 d\rho$$

and (5.2)

$$\hat{Q} \equiv \frac{Qg^2}{8\pi} = \frac{\nu}{2} \int_0^\infty B^2 \rho^2 d\rho,$$

which can be calculated directly once the appropriate solutions of the differential equations are found. In terms of $\hat{E}$ and $\hat{Q}$ the condition for the critical point S becomes $\hat{E}_S = \hat{Q}_S$.

From Eqs. (1.9) and (1.10) we readily derive

$$\frac{d}{d\rho}\left[\frac{1}{2}\left(\frac{dA}{d\rho}\right)^2 + \frac{1}{2}\left(\frac{dB}{d\rho}\right)^2 + U\right] = -\frac{2}{\rho}\left[\left(\frac{dA}{d\rho}\right)^2 + \left(\frac{dB}{d\rho}\right)^2\right] < 0, \tag{5.3}$$

where U is defined in (2.33). As $\frac{1}{2}(dA/d\rho)^2 + \frac{1}{2}(dB/d\rho)^2 + U$ is a monotonically decreasing function of ρ and vanishes at $\rho = \infty$, it follows that $U > 0$ at $\rho = 0$. Thus we see that the initial point must lie in the region

$$A(0) \leq \frac{\nu}{\kappa}, \qquad B(0) \geq \frac{1}{2} \frac{1 - A^2(0)}{[\nu^2 - \kappa^2 A^2(0)]^{1/2}}. \tag{5.4}$$

The strategy we follow for the numerical integration is a generalization of the method used for the one-dimensional problem of Eqs. (2.37) and (2.41): for a given value of κ and $\nu < \kappa$, we select a tentative initial value $[A(0), B(0)]$ satisfying Eq. (5.4) and integrate numerically the differential equations up to the lowest value ρ at which any of the following possibilities occurs: (i) $A(\rho) > 1$, (ii) $B(\rho) < 0$, (iii) the derivative $A'(\rho) < 0$, or (iv) $B'(\rho) > 0$. When any of these four possibilities occurs, the solution is rejected, a new initial value is selected, and the integration repeated. The sets of initial values leading to the four possibilities described above define four regions in the strip (5.4) which we denote by I, II, III, and IV, respectively. The four regions intersect at a point which defines the correct initial value for the ground-state solutions. The numerical integrations of Eqs. (1.9) and (1.10) were carried out with a fourth-order Runge-Kutta method at intervals of 10^{-2}. Using the solutions $A(\rho), B(\rho)$ corresponding to the "correct initial value," $\hat{E}$ and $\hat{Q}$ are calculated on the basis of Eq. (5.2). Changing ν at a fixed κ, we can obtain the various curves of interest: $\hat{E}$ vs $\hat{Q}$, $\hat{Q}$ vs ν, etc. Among other checks, the validity of the virial theorem of Eq. (2.32) was verified with relative errors $\epsilon \sim 10^{-5}$ to 10^{-4} for the solutions corresponding to the majority of the ν values in the cases $\kappa = 1$ and 5. Figures 3(a) and 3(b) give the $\hat{E}$ vs $\hat{Q}$ and $\hat{Q}$ vs ν curves, for $\kappa = 1$, while Figs. 4(a) and 4(b) illustrate the solutions A and B for a particular case, $\kappa = 1$ and $\nu = 0.96$. The results depicted in Figs. 3(a) and 3(b) confirm the qualitative picture that emerges from the theoretical considerations of Sec. II. From our numerical results we can also determine the critical points $\hat{Q}_S$ and $\hat{Q}_C$. Thus, for $\kappa = 1$ we obtain $\hat{Q}_S = 3.47$ and $\hat{Q}_C = 3.03$. For $\kappa = 5$, we find $\hat{Q}_S = 0.0213$ and $\hat{Q}_C = 0.0198$. Finally, in the case of $\kappa = \frac{1}{2}$, which is calculated with less accuracy, we obtain $\hat{Q}_S = 39$ and $\hat{Q}_C = 32$.

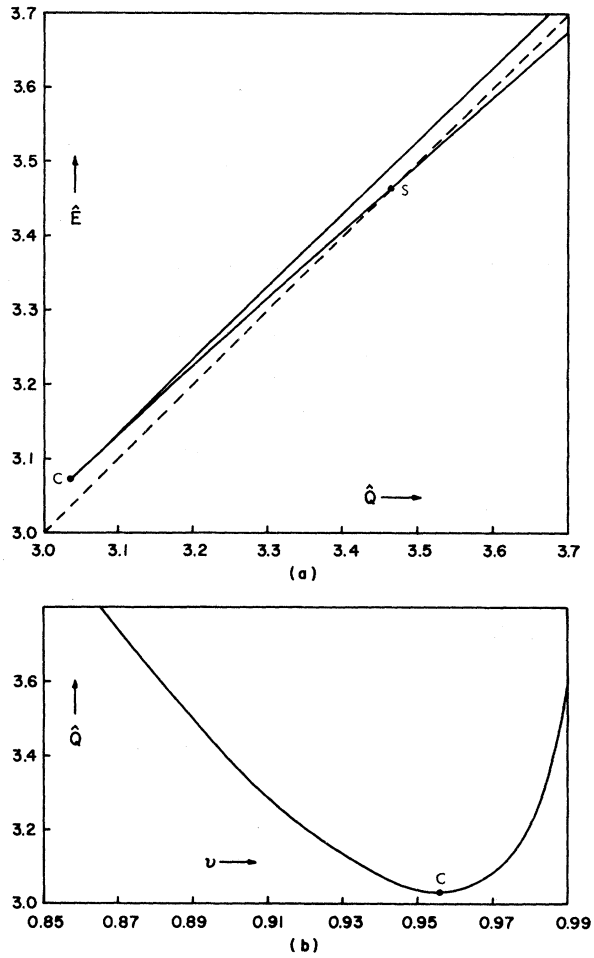

FIG. 3. $\hat{E}$ vs $\hat{Q}$ and $\hat{Q}$ vs ν curves for $\kappa = m/\mu = 1$ in the neighborhood of the critical points, where $\hat{E} = (8\pi m)^{-1} E g^2$, $\hat{Q} = (8\pi)^{-1} Q g^2$, and $\nu = \omega/\mu$. The dashed line in (a) is $\hat{E} = \hat{Q}$, or $E = Qm$.

VI. GENERALIZATION

So far, for clarity of presentation, we assume the Lagrangian to be given by (1.1). As may have already been obvious, throughout our previous discussions, except for some of the numerical results, very little actually depends on that specific Lagrangian. In this section, we will examine a particular class of generalization. We assume the Lagrangian to be of the general form

$$\mathcal{L} = -\frac{\partial \phi^\dagger}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} - \frac{1}{2}\left(\frac{\partial \chi}{\partial x_\mu}\right)^2 - f^2 \chi^2 \phi^\dagger \phi - u(\chi), \tag{6.1}$$

where $u(\chi)$ is an arbitrary fourth-order polynomial of χ, so that the theory remains renormalizable. The function $u(\chi)$ is assumed to have an absolute minimum at $\chi = \chi_{\text{vac}}$, its vacuum expectation value. For convenience, we choose $u(\chi_{\text{vac}}) = 0$, and therefore

$$u(\chi) \geq u(\chi_{\text{vac}}) = 0. \tag{6.2}$$

The mass of the charged ϕ meson is

$$m = f \chi_{\text{vac}}, \tag{6.3}$$

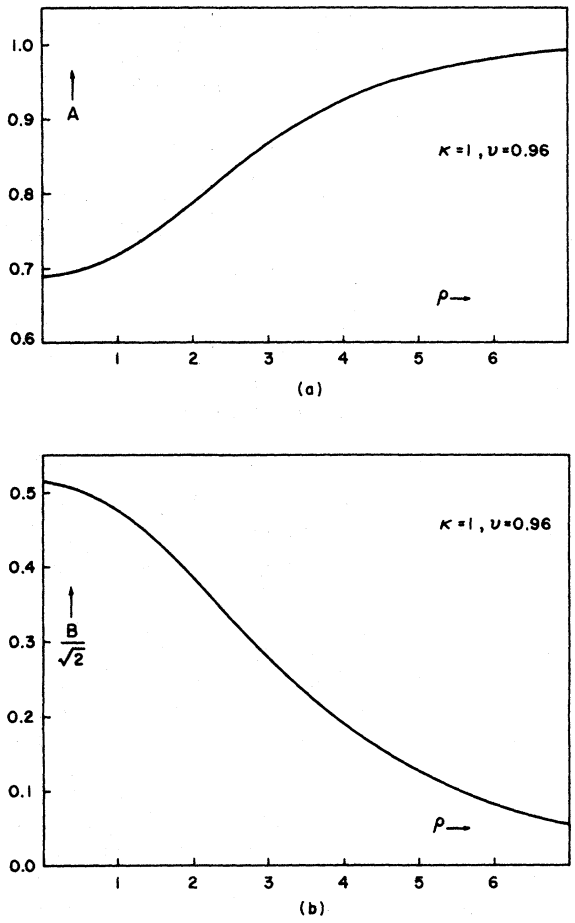

FIG. 4. Ground-state solutions of Eqs. (1.9) and (1.10) for $\kappa=1$ and $\nu=0.96$. As can be seen from Fig. 3(b), this value of ν lies very close to the critical point C.

which will be assumed to be nonzero; consequently,

$$\chi_{\text{vac}} \neq 0. \tag{6.4}$$

We consider first the classical theory. As in (1.5) and (1.6), we define

$$\chi(\vec{r}, t) \equiv \chi_{\text{vac}} A(\vec{\rho})$$

and (6.5)

$$\phi(\vec{r}, t) \equiv 2^{-1/2}\chi_{\text{vac}} B(\vec{\rho}) e^{-i\omega t},$$

where $\vec{\rho} \equiv \mu \vec{r}$, and μ is the mass of the neutral χ meson. The charge and the energy of the system remain given respectively by (1.12) and (1.13), except that $\mathcal{E}$ is now, instead of (1.14),

$$\mathcal{E} = \tfrac{1}{2}(\vec{\nabla}A)^2 + \tfrac{1}{2}(\vec{\nabla}B)^2 + \tfrac{1}{2}(\nu^2 + \kappa^2 A^2)B^2 + w(A), \tag{6.6}$$

where

$$w(A) \equiv (\mu\chi_{\text{vac}})^{-2} u(\chi). \tag{6.7}$$

As before, $\nu \equiv \omega/\mu$ and $\kappa \equiv m/\mu$.

To show the existence of the soliton, we may assume the same trial function (2.8). This leads to an upper bound for the lowest energy E_{min} at a given Q. For Q large,

$$E_{\text{min}} \leq \frac{\pi Q}{R} + \frac{4\pi}{3} u(0) R^3 + O(R^2\mu^3/g^2), \tag{6.8}$$

where $g \equiv \mu/\chi_{\text{vac}}$, as in (1.7). We separate our discussion into two cases:

1. $u(0) \neq 0$. Taking the minimum of (6.8), which occurs at $R \cong [\frac{1}{4}Q/u(0)]^{1/4}$, we find for Q sufficiently large

$$E_{\text{min}} \leq \tfrac{4}{3}\pi\sqrt{2}\,[u(0)]^{1/4} Q^{3/4} + O(R^2\mu^3/g^2). \tag{6.9}$$

2. $u(0) = 0$. In this case, because u is a fourth-order polynomial,

$$u(\chi) = \tfrac{1}{2} g^2 \chi^2 (\chi - \chi_{\text{vac}})^2. \tag{6.10}$$

[To avoid ambiguity in the definition of the vacuum, the factor $\chi^2(\chi - \chi_{\text{vac}})^2$ may be replaced by $(\chi^2 + \epsilon)(\chi - \chi_{\text{vac}})^2$, where $\epsilon = 0+$.] We assume the trial function to be of the form

$$A = [1 + e^{-\mu(r-R)}]^{-1}, \tag{6.11}$$

but B remains given by (2.8). This leads to, in place of (6.8),

$$E_{\text{min}} \leq \pi Q R^{-1} + \tfrac{2}{3}\pi R^2 \mu^3 g^{-2} + O(R\mu^2/g^2). \tag{6.12}$$

Taking the minimum of this upper bound, which occurs at $R \cong (\frac{3}{4} g^2 Q)^{1/3}/\mu$, we find for Q sufficiently large

$$E_{\text{min}} \leq \tfrac{1}{2}\pi (6Q/g)^{2/3}\mu + O(R\mu^2/g^2). \tag{6.13}$$

In either case, (6.9) or (6.13), there exists a finite critical value Q_S: For $Q > Q_S$, $E_{\text{min}} < Qm$. Thus, absolutely stable soliton solutions must exist.

Except for the replacement of $\frac{1}{8}(A^2-1)^2$ by $w(A)$ in (2.24) and (2.33), the entire Sec. II C on the variational principles and virial theorem is applicable to the general case.

To study the soliton solution when ω is near m, we make the same substitution (2.35) for A and B. As $\xi \equiv (\kappa^2 - \nu^2)^{1/2} \to 0$, $A \to 1$, and therefore

$$w(A) \to \tfrac{1}{2}\mu^2 (1-A)^2, \tag{6.14}$$

which is independent of the detailed form of $w(A)$. The discussions given in Sec. II D can be applied to the general Lagrangian (6.1) without any change. Therefore, by following the discussions given in Sec. II E, one sees that in general there exists another critical point $Q_C < Q_S$. For $Q_C < Q < Q_S$, soliton solutions still exist, though their energy is $> Qm$. Furthermore, as $\kappa \to \infty$, the inequality (2.15) holds for the general case as well.

Except for the replacement of $\frac{1}{2}(3A^2-1)$ by $d^2w(A)/dA^2$ in (3.3), the entire discussions given in Secs. III and IV on the stability and quantization are valid in the general case. The numerical integration of the differential equation (2.37) discussed in Sec. V is also applicable to the general case; of course, the specific solutions of (1.9) and (1.10) given there hold only for $w(A)=\frac{1}{8}(1-A^2)^2$.

The generalization to include spin-$\frac{1}{2}$ fermion fields and spin-1 boson fields will be discussed in a subsequent paper.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix, we derive the upper bound (2.15) for Q_S when $\kappa=m/\mu\to\infty$, where μ and m are, respectively, the masses of the neutral χ and the charged ϕ fields.

(1) We first show that when $\kappa\to\infty$, $Q_S\to 0$ faster than κ^{-2}. It is convenient to define

$$z\equiv\rho\kappa=mr \quad \text{and} \quad n\equiv\nu/\kappa=\omega/m. \tag{A1}$$

For a spherically symmetric solution, (1.9) and (1.10) may be written as

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dA}{dz}\right)-B^2A=\frac{1}{2\kappa^2}(A^2-1)A \tag{A2}$$

and

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dB}{dz}\right)-A^2B+n^2B=0. \tag{A3}$$

Correspondingly, (1.12) and (1.13) become

$$Q=4\pi n(g\kappa)^{-2}\int_0^\infty z^2B^2dz \tag{A4}$$

and

$$E=4\pi m(g\kappa)^{-2}\int_0^\infty z^2\bar{E}dz, \tag{A5}$$

where

$$\bar{E}=\frac{1}{2}\left(\frac{dA}{dz}\right)^2+\frac{1}{2}\left(\frac{dB}{dz}\right)^2+\frac{1}{2}(n^2+A^2)B^2+(8\kappa^2)^{-1}(A^2-1)^2. \tag{A6}$$

When $\kappa\to\infty$ and for $Q\sim O(\kappa^{-2})$, the right-hand side of (A2) and the last term in (A6) may be neglected, except at large z. This suggests a trial function of the form

$$A(z)=\begin{cases} a & \text{for } z\le Z \\ 1-(1-a)(Z/z)e^{-(z-Z)/\kappa} & \text{for } z\ge Z \end{cases}$$

and (A7)

$$B(z)=\begin{cases} (b/z)\sin(\pi z/Z) & \text{for } z\le Z, \\ 0 & \text{for } z\ge Z, \end{cases}$$

where a, b, and Z are parameters to be determined by minimizing the energy E. The charge Q of this trial function is given by

$$Q=2\pi n(g\kappa)^{-2}b^2Z. \tag{A8}$$

By substituting (A7) into (A5), we find E is equal to a b-independent term plus

$$\pi m(g\kappa)^{-2}b^2Z[n^2+a^2+(\pi/Z)^2], \tag{A9}$$

where n^2 is, according to (A8), proportional to $(Q/b^2)^2$. The minimization of E with respect to b can be carried out by differentiating (A9) with respect to b, but keeping a, Z, and Q fixed; this leads to

$$n^2=a^2+(\pi/Z)^2. \tag{A10}$$

The same result can also be obtained by requiring the trial function (A7) to satisfy the field equation (A3) for $z<Z$.

It is convenient to define

$$q\equiv(4\pi)^{-1}(g\kappa)^2Q. \tag{A11}$$

By using the trial function (A7), we derive an upper bound on the energy

$$E\le 4\pi m(g\kappa)^{-2}e, \tag{A12}$$

where

$$e=qn+\tfrac{1}{2}(1-a)^2Z^2(Z^{-1}+\kappa^{-1})+(24\kappa^2)^{-1}(1-a^2)^2Z^3+\epsilon, \tag{A13}$$

$$\epsilon=(2\kappa^2)^{-1}\int_Z^\infty z^2(\delta A)^3[1+\tfrac{1}{4}(\delta A)]dz, \tag{A14}$$

and

$$\delta A\equiv-(1-a)(Z/z)\exp[-(z-Z)/\kappa].$$

Let us consider the minimum of e when $\kappa\to\infty$, but assuming $Q\sim O(\kappa^{-2})$, i.e.,

$$q\sim O(1). \tag{A15}$$

As we shall see from (A17) and (A18) below, in this case the optimal values of the parameters a and Z are both $O(1)$. From (A13), it follows that when $\kappa\to\infty$

$$e=qn+\tfrac{1}{2}(1-a)^2Z+O(\kappa^{-1}) \tag{A16}$$

where n is given by (A10). Neglecting $O(\kappa^{-1})$, the minimum of (A16) occurs at

$$a=[1+(q/\pi)^2]^{-1/2} \tag{A17}$$

and

$$Z^{-1}=[\tfrac{1}{2}a(1-a)]^{1/2}/\pi; \tag{A18}$$

the corresponding minimum value of e is

$$\pi\sqrt{2}\{[1+(q/\pi)^2]^{1/2}-1\}^{1/2}. \tag{A19}$$

Since (A19) is always $<q$, we find that, in the limit $\kappa\to\infty$, the soliton mass E is always $<Qm$ provided (A15) holds; i.e., on account of (A11), when $\kappa\to\infty$, Q_S must $\to 0$ *faster* than κ^{-2}.

(2) To derive a better upper bound for Q_S in the limit $\kappa\to\infty$, we assume, instead of (A15),

$$q\sim O(\kappa^{-1}). \tag{A20}$$

As $q\to 0$, because of (A17) and (A18),

$$(1-a)\to(2\pi^2)^{-1}q^2 \text{ and } Z\to 2\pi^2/q. \tag{A21}$$

In the range (A20) we have then, when $\kappa\to\infty$, $(1-a)\sim O(\kappa^{-2})$, $Z\sim O(\kappa)$, and, on account of (A14), $\epsilon\sim O(\kappa^{-5})$.

This suggests that we retain the variable ρ and introduce scaled functions $\hat{x},\hat{y}$ satisfying

$$\begin{aligned} A&=1-(2\kappa^2)^{-1}\hat{x},\\ B&=2^{-1/2}\kappa^{-2}\hat{y}\end{aligned} \tag{A22}$$

so that (1.9), (1.10) become

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\hat{x}}{d\rho}\right)=\hat{x}-\hat{y}^2+O(\kappa^{-2}), \tag{A23}$$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\hat{y}}{d\rho}\right)=(\xi^2-\hat{x})\hat{y}+O(\kappa^{-2}), \tag{A24}$$

where $\xi^2=\kappa^2-\nu^2$. We consider the limit $\kappa\to\infty$ for fixed ξ. Then defining $\hat{q}=q/\kappa$ we have

$$Q=4\pi(g^2\kappa^3)^{-1}\hat{q}, \tag{A25}$$

$$E=4\pi\mu(g^2\kappa^2)^{-1}[\hat{q}+(8\kappa^2)^{-1}\hat{\epsilon}+O(\kappa^{-4})], \tag{A26}$$

where

$$\hat{q}=[1-(\xi/\kappa)^2]^{1/2}\int_0^\infty \tfrac{1}{2}\hat{y}^2\rho^2 d\rho, \tag{A27}$$

$$\hat{\epsilon}=\int_0^\infty\left[\left(\frac{d\hat{x}}{d\rho}\right)^2+2\left(\frac{d\hat{y}}{d\rho}\right)^2-2\hat{x}\hat{y}^2+\hat{x}^2\right]\rho^2 d\rho. \tag{A28}$$

In the limit $\kappa\to\infty$, (A27) becomes

$$\hat{q}=\int_0^\infty \tfrac{1}{2}\hat{y}^2\rho^2 d\rho. \tag{A29}$$

Since $\hat{x}$ and $\hat{y}$ are determined by ξ through (A23) and (A24), it follows that $\hat{q}$ and $\hat{\epsilon}$ are well-defined functions of ξ through (A28) and (A29). Now, when $\xi\ll 1$ the equations (A23) and (A24) reduce (setting $\hat{x}=\xi^2 x$, $\hat{y}=\xi y$) to (2.36) and (2.37), and then from (2.43) we see that $\hat{q}$ increases as ξ^{-1}. On the other hand, when $\xi\gg 1$ the choice of variables $\tau=\rho\xi$, $\bar{x}=x=\hat{x}/\xi^2$, $\bar{y}=\hat{y}/\xi^2$ leads to

$$\begin{aligned}&\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2\frac{d\bar{x}}{d\tau}\right)=\bar{y}-\xi^{-2}\bar{x}\\ &\text{and}\\ &\frac{1}{\tau^2}\frac{d}{d\tau}\left(\tau^2\frac{d\bar{y}}{d\tau}\right)=\bar{y}(1-\bar{x}),\end{aligned} \tag{A30}$$

in which the term $\xi^{-2}\bar{x}$ is unimportant except at large τ, and $\bar{y}(\tau)$ is essentially independent of ξ. But from (A29) we have

$$\hat{q}=\xi\int_0^\infty \tfrac{1}{2}\bar{y}^2\tau^2 d\tau \tag{A31}$$

so that $\hat{q}$ increases as ξ.

Since $\hat{q}$ becomes large for both small and large ξ, it must have a minimum, $\hat{q}_C=\hat{q}(\xi_C)$, where both $\hat{q}_C$ and ξ_C are $O(1)$. Then from (A25),

$$Q_C=4\pi(g^2\kappa^3)^{-1}\hat{q}_C. \tag{A32}$$

The other critical point is given by $\hat{q}_S=\hat{q}(\xi_S)$, where ξ_S is defined by $\hat{\epsilon}(\xi_S)=0$. To obtain an upper bound for $\hat{q}_S$ we introduce the trial functions

$$\hat{x}=\begin{cases}\hat{x}_0\cos(\pi\rho/2\rho_0) & \text{for } \rho\le\rho_0,\\ 0 & \text{for } \rho\ge\rho_0\end{cases} \tag{A33}$$

and

$$\hat{y}=\begin{cases}(\hat{y}_0/\rho)\sin(\pi\rho/\rho_0) & \text{for } \rho\le\rho_0,\\ 0 & \text{for } \rho\ge\rho_0.\end{cases} \tag{A34}$$

Applying (A28) and (A29), we obtain

$$\hat{q}=\rho_0\hat{y}_0{}^2/4, \tag{A35}$$

$$\begin{aligned}\hat{\epsilon}&=\hat{x}_0{}^2\left[\frac{\rho_0}{4}\left(\frac{\pi^2}{6}+1\right)+\frac{\rho_0{}^3}{\pi^2}\left(\frac{\pi^2}{6}-1\right)\right]\\ &\quad+\hat{y}_0{}^2\left(\frac{\pi^2}{\rho_0}-\frac{32}{15\pi}x_0\rho_0\right)\\ &=\hat{x}_0{}^2\left[\frac{\rho_0}{4}\left(\frac{\pi^2}{6}+1\right)+\frac{\rho_0{}^3}{\pi^2}\left(\frac{\pi^2}{6}-1\right)\right]\\ &\quad+4\hat{q}\left(\frac{\pi^2}{\rho_0{}^2}-\frac{32}{15\pi}\hat{x}_0\right).\end{aligned} \tag{A36}$$

We wish to make $\hat{\epsilon}$ vanish for the smallest possible $\hat{q}$. This is achieved by setting

$$\hat{x}_0=\tfrac{1}{4}(15\pi)(\pi^2-6)/(\pi^2+6), \tag{A37}$$

$$\rho_0=\tfrac{1}{2}\pi(\pi^2+6)^{1/2}/(\pi^2-6)^{1/2} \tag{A38}$$

so that $\hat{\epsilon}$ is negative whenever $\hat{q}$ exceeds the value

$$\hat{q}_0=75\pi^3(\pi^4-36)^{1/2}/2048\simeq 8.898. \tag{A39}$$

Since, for given $\hat{q}$, the true $\hat{\epsilon}$ is less than that given by our trial function, it follows that the true $\hat{\epsilon}$ must be <0 when $\hat{q}=\hat{q}_0$. Hence the crossing oc-

curs at $\hat{q}=\hat{q}_S<\hat{q}_0$. [This argument does not give a bound on ξ_S, but from the previous reasoning ξ_S must be $O(1)$ since $\hat{q}_S$ is.] Comparing (A39) with (A25), we obtain (2.15); i.e., as $\kappa\to\infty$,

$$Q_S < 75\pi^4(\pi^4-36)^{1/2}/512g^2\kappa^3 \cong 111.8(g^2\kappa^3)^{-1}. \tag{A40}$$

APPENDIX B

We shall discuss in this appendix how to derive the quantum soliton solution in a moving system, in which the total momentum $\vec{P}\neq 0$.

Let $A(\vec{\rho})$ and $B(\vec{\rho})$ be the *same* classical solutions of (1.9) and (1.10), as in the previous sections. But instead of (4.2), the components of $\vec{\rho}$ are now related to those of the space coordinate $\vec{r}$ and the collective coordinate $\vec{R}(t)$ by

$$\rho_1=\mu[r_1-R_1(t)],\quad \rho_2=\mu[r_2-R_2(t)],$$
and
$$\rho_3=\mu\gamma[r_3-R_3(t)], \tag{B1}$$

where

$$\gamma=(1-v^2)^{-1/2}, \tag{B2}$$

v is a fixed parameter (not a variable), and μ is, as before, the mass of the neutral χ meson. Just as in Sec. IV, in addition to the three components of $\vec{R}(t)$, we shall introduce a fourth collective coordinate $\zeta(t)$, which is the over-all phase variable of the charged ϕ field. The quantum expansion of the operators $\chi(\vec{r},t)$, $\phi(\vec{r},t)$ and $\phi^\dagger(\vec{r},t)$ is now given by the following modification of (4.1):

$$\chi(\vec{r},t)=\frac{\mu}{g}A(\vec{\rho})+\sum_{N=5}^{\infty}q_N(t)\alpha_N(\vec{\rho}),$$
$$\phi(\vec{r},t)=\frac{1}{\sqrt{2}}\left[\frac{\mu}{g}B(\vec{\rho})+\sum_{N=5}^{\infty}q_N(t)\beta_N(\vec{\rho})\right]e^{-i\theta}, \tag{B3}$$

and

$$\phi^\dagger(\vec{r},t)=\frac{1}{\sqrt{2}}\left[\frac{\mu}{g}B(\vec{\rho})+\sum_{N=5}^{\infty}q_N(t)\beta_N^*(\vec{\rho})\right]e^{i\theta},$$

where $\vec{\rho}$ is given by (B1), θ is defined to be

$$\theta\equiv\zeta(t)-\omega v\gamma[r_3-R_3(t)], \tag{B4}$$

and $\alpha_N(\vec{\rho})$ and $\beta_N(\vec{\rho})$ are c-number functions of $\vec{\rho}$; as before, α_N's are real and β_N's are complex. The $q_N(t)$'s are the(Hermitian) coordinates for the vibrational modes. [Note that if $q_N=0$, $R_1=R_2=0$, $R_3=vt$, and $\zeta=\omega t/\gamma$, then $\theta=\omega\gamma(t-vr_3)$, and (B3) would describe a classical soliton moving with velocity v along the r_3 direction.]

Just as in (4.3) and (4.4), in order to maintain the independence of the vibrational modes from the motion of the collective coordinates, $\vec{R}(t)$ and $\zeta(t)$, we impose the following constraints for all N ($N=5,6,\ldots$):

$$\int\left[\frac{\partial A}{\partial\rho_l}\alpha_N+\frac{1}{2}\frac{\partial B}{\partial\rho_l}(\beta_N+\beta_N^*)\right]d^3\rho=0 \tag{B5}$$

and

$$\int(\beta_N-\beta_N^*)Bd^3\rho=0, \tag{B6}$$

where the subscript $l=1$ or 2 or 3. As in (1.11), we define $\nu=\omega/\mu$ and $\kappa=m/\mu$. It is convenient to combine the collective coordinates and the coordinates of the vibrational modes into a single column vector:

$$q\equiv\begin{bmatrix}R_1\\R_2\\R_3\\\zeta\\q_5\\q_6\\\vdots\end{bmatrix}. \tag{B7}$$

The Lagrangian then takes on the standard form, as in (4.7),

$$L=\tfrac{1}{2}\dot{\tilde{q}}\,\mathfrak{M}\,\dot{q}-V(q), \tag{B8}$$

where the matrix $\mathfrak{M}(q)$ and the function $V(q)$ can be obtained in a straightforward way by substituting (B3) into (1.1). [If $v=0$, $\mathfrak{M}(q)$ reduces to that given by (4.9)–(4.16).]

The conjugate momentum operators of $\vec{R}$, ζ, and q_N ($N=5,6,\ldots$) are respectively

$$\vec{P}=-i\vec{\nabla}_R,\quad Q=-i\frac{\partial}{\partial\zeta} \tag{B9}$$

and

$$p_N=-i\frac{\partial}{\partial q_N}, \tag{B10}$$

which may also be combined into a single column vector

$$p\equiv\begin{bmatrix}P_1\\P_2\\P_3\\Q\\p_5\\p_6\\\vdots\end{bmatrix}. \tag{B11}$$

By following the standard canonical procedure we find that, similar to (4.18), (4.20), and (4.21).

$$p = \mathfrak{M}\dot{q}\,, \tag{B12}$$

and the Schrödinger equation for the state vector $|\,\rangle$ is given by

$$\mathcal{H}\,|\,\rangle = E\,|\,\rangle\,, \tag{B13}$$

where

$$\mathcal{H} = (2J)^{-1}\vec{p}\,\mathfrak{M}^{-1}Jp + V(q) \tag{B14}$$

and

$$J(q) = [\det \mathfrak{M}(q)]^{1/2}\,. \tag{B15}$$

It can be readily seen that $J(q)$, $\mathfrak{M}(q)$, and $V(q)$ are all independent of the collective coordinates $\vec{R}$ and ζ, and therefore

$$[\vec{P},\mathcal{H}] = 0 \text{ and } [Q,\mathcal{H}] = 0\,. \tag{B16}$$

This is, of course, to be expected. Since the transformations $\vec{R}\to\vec{R}+\delta\vec{R}$ and $\zeta\to\zeta+\delta\zeta$ are respectively a space translation and an over-all shift in the phase of ϕ, their generators must be constants of motion; these generators are, respectively, $\vec{P}$ = the total momentum and Q = the total charge. Because ζ is a cyclic variable with a period $=2\pi$, its conjugate momentum $Q = -i\partial/\partial\zeta$ has only integer eigenvalues: $0,\pm1,\pm2,\ldots$. On account of (B16), we may choose the state vector $|\,\rangle$ to be also an eigenstate of $\vec{P}$, and Q. (So far, $|\,\rangle$ can be an arbitrary eigenstate of $\vec{P}$ and Q.) As we shall see, in order that the solution of the Schrödinger equation (B13) has a power-series expansion in g, the total momentum and the total charge of $|\,\rangle$ must be the same as those given by the corresponding classical soliton solution, at least when g is sufficiently small.

Let us consider the formal expansion of $\mathcal{H}$ in powers of q_N and p_N $(N=5,6,\ldots)$:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots\,, \tag{B17}$$

where, just as in (4.26), $\mathcal{H}_0$ is independent of q_N and p_N, $\mathcal{H}_1$ depends on q_N and p_N linearly, $\mathcal{H}_2$ quadratically, etc. By using (1.1), (B1)–(B4), and (B14), we find

$$\mathcal{H}_0 = (2M)^{-1}\gamma(P_1^2+P_2^2) + (2M\gamma)^{-1}P_3^2 + \tfrac{1}{2}\gamma(I^{-1}+M^{-1}\omega^2 v^2)Q^2 - M^{-1}\omega v Q P_3 + V_0\,, \tag{B18}$$

where I and M are given by (2.25) and (4.10), respectively,

$$V_0 = (g^2\gamma)^{-1}\mu\int d^3\rho\,\overline{V}\,, \tag{B19}$$

and

$$\overline{V} = \frac{1}{2}\left[\left(\frac{\partial A}{\partial\rho_1}\right)^2 + \left(\frac{\partial A}{\partial\rho_2}\right)^2 + \gamma^2\left(\frac{\partial A}{\partial\rho_3}\right)^2 + \left(\frac{\partial B}{\partial\rho_1}\right)^2 + \left(\frac{\partial B}{\partial\rho_2}\right)^2 + \gamma^2\left(\frac{\partial B}{\partial\rho_3}\right)^2\right] + \tfrac{1}{2}(v^2\gamma^2\nu^2 + \kappa^2A^2)B^2 + \tfrac{1}{8}(A^2-1)^2\,. \tag{B20}$$

By using $\int(\partial A/\partial\rho_k)^2\,d^3\rho = \frac{1}{3}\int(\vec{\nabla}A)^2 d^3\rho$, $\int(dB/\partial\rho_k)^2 d^3\rho = \frac{1}{3}\int(\vec{\nabla}B)^2 d^3\rho$, where $k=1$, or 2, or 3, and the virial expression (2.23), we may reduce (B19) to

$$V_0 = \tfrac{1}{2}[\gamma I\omega^2 + \gamma^{-1}(1+\gamma^2)M]\,. \tag{B21}$$

Formally, $\mathcal{H}_0$ is proportional to g^{-2} and $\mathcal{H}_1$ is proportional to g^{-1}. As explained in Sec. IV C, in order that, when $g\to 0$, $\mathcal{H}_0$ does give the correct energy to $O(g^{-2})$, we must require $\mathcal{H}_1\,|\,\rangle = 0$, at least to $O(g^{-1})$. It can be verified that, in analogy with (4.29), $\mathcal{H}_1\,|\,\rangle = 0$ implies now

$$P_1|\,\rangle = 0,\quad P_2|\,\rangle = 0\,,$$
$$P_3|\,\rangle = \gamma(M+I\omega^2)v|\,\rangle\,, \tag{B22}$$

and

$$Q|\,\rangle = I\omega|\,\rangle\,.$$

To $O(g^{-2})$, (B13) may then be written as

$$\mathcal{H}_0|\,\rangle = E_0|\,\rangle\,, \tag{B23}$$

where because of (B18), (B21), and (B22),

$$E_0 = \gamma(M+I\omega^2)\,. \tag{B24}$$

This gives, to the order calculated, the relativistic energy of a quantum soliton moving with velocity v. When $v=0$, (B24) reduces to (4.31).

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