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Gravitational mass defect in general relativity

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A fraction of mass is theoretically predicted to be radiated out when particles assemble together from a region of no gravitation to form a gravitating mass. In this paper the mass defect has been calculated for a spherically symmetric shell and solid. It has been shown that there is a maximum upper limit to the decrease in mass.

INTRODUCTION

The gravitational mass and the inertial mass of a particle are equal. Kinetic energy and potential energy can make contributions to the inertial mass and hence to the gravitational mass. So the M which occurs in the Schwarzschild line element is the effective mass and includes the self-energy of the mass due to the gravitational field itself. It is interesting to compare it with the inertial rest mass of the constituent particles when they are separate and outside any field. We have derived by a simple method the relation between the mass of a test particle outside any gravitational effect and in a gravitational field at a certain distance from the gravitating body. Hence we can obtain the self-energy and reduction for effective mass for a spherical shell. This mass defect can also be obtained in the case of an actual solid sphere. The effect of contraction (reduction in radius) on mass defect is found to be similar for spherical shells and solid spheres.

I. MASS OF A TEST PARTICLE

The squared line element in the spherically symmetric Schwarzschild space, using obvious notation, is given by[1]

$$ds^2 = (1 - 2kM/c^2r)c^2dt^2 - (1 - 2kM/c^2r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2. \tag{1}$$

Integrating the $\sigma = 4$ component of the equation of geodesic,

$$\frac{d^2x^\sigma}{ds^2} = -\Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{2}$$

we get

$$(1 - 2kM/c^2r)\frac{dt}{ds} = A, \tag{3}$$

where A is a constant. As $r \to \infty$, the space becomes Minkowskian and $dt/ds \to A$. In the theory of special relativity, for a particle of rest mass m_0, the total relativistic energy is given by[2]

$$m_0c^3\frac{dt}{ds} = mc^2. \tag{4}$$

With this analogy, we can interpret[3] the constant A in Eq. (3). m_0c^3A is the total energy associated with a free particle in a gravitational field and includes rest energy and kinetic energy as well as gravitational potential energy. With the help of Schwarzschild's exterior solution (1), the energy of a particle of inertial rest mass m_0 in the gravitational field of mass M can be written as

$$\frac{(1 - 2kM/c^2r)m_0c^2}{\{(1 - 2kM/c^2r) - (1/c^2)[(1 - 2kM/c^2r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2]\}^{1/2}}. \tag{5}$$

The above equation is identical with the results derived by Jaen[4] by solving the Hamilton-Jacobi equation for the motion of a test particle. Total energy of a particle in any geodesic path is always constant. When the particle is brought into the influence of a gravitational field, it is attracted towards the central force and so gains kinetic energy, which is radiated out in the form of heat and other forms of energies when the body is brought to rest at some position in the field. This loss of energy results in the reduction of rest mass of the body in the field. It will be seen later that the maximum mass which can be radiated out in the formation of a shell is $0.5000\, m_0$ and in a solid sphere it is $0.5756\, m_0$. The rest mass m_0' in the gravitational field is obtained by setting $\dot{r} = \dot{\theta} = \dot{\phi} = 0$,

$$m_0' = (1 - 2kM/c^2r)^{1/2}m_0. \tag{6}$$

The energy of a particle at rest, $m_0'c^2$, in the gravitational field is thus made up of inertial rest energy m_0c^2 and gravitational potential energy

$$m_0c^2[(1-2kM/c^2r)^{1/2}-1]\,.$$

II. SPHERICAL SHELL

Consider a hollow spherical shell of radius a and negligible thickness, having mass m as it exists in the field created by itself. The internal stress required to maintain equilibrium of the shell is irrelevant for calculations involving an external gravitational field. If an infinitesimal mass element dm on its surface has a mass dm_0 in the absence of any gravitational field, then by Eq. (6) we have

$$dm_0=(1-2km/c^2a)^{-1/2}dm\,. \quad (7)$$

Suppose this mass dm is uniformly distributed over the whole surface of the shell; then integration of Eq. (7) between the limits 0 to m_0 and 0 to m reveals the relation of mass m of the shell to the inertial rest mass m_0 of the constituent particles:

$$m=m_0-km_0^2/2c^2a\,. \quad (8)$$

The plot of this relation is shown in Fig. 1 for a constant original mass m_0. When the radius of the shell is decreased the effective gravitational mass also decreases reaching a minimum value equal to half of the original mass when $a=km_0/c^2$. The plot for the Schwarzschild radius $R=2km/c^2$ also resembles this curve if the scale of m is multiplied by a factor $2k/c^2$. Figure 2 shows the relation between m and m_0 for various constant radii. It will

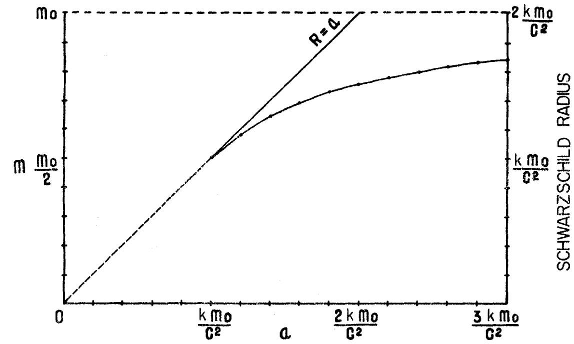

FIG. 1. The plot between m and a for a constant value of m_0. The dashed portion lies in the Schwarzschild singularity region.

be observed that the physical portions of all curves which are parabolic in nature lie within the lines $m=m_0$ and $m=0.5000\,m_0$. The minimum mass that can be assembled on a shell of radius a is obviously zero, and the maximum is limited by

$$m_{\max}=c^2a/2k=\tfrac{1}{2}m_{0\,\max}\,.$$

After attaining so much mass the Schwarzschild radius reaches the surface of the shell, and the time required to bring any additional mass will be infinite. Thus the maximum mass that can be radiated out is $\frac{1}{2}m_0$.

III. SOLID SPHERE OF CONSTANT DENSITY

Consider a solid sphere of constant proper density μ_0 defined by the formula

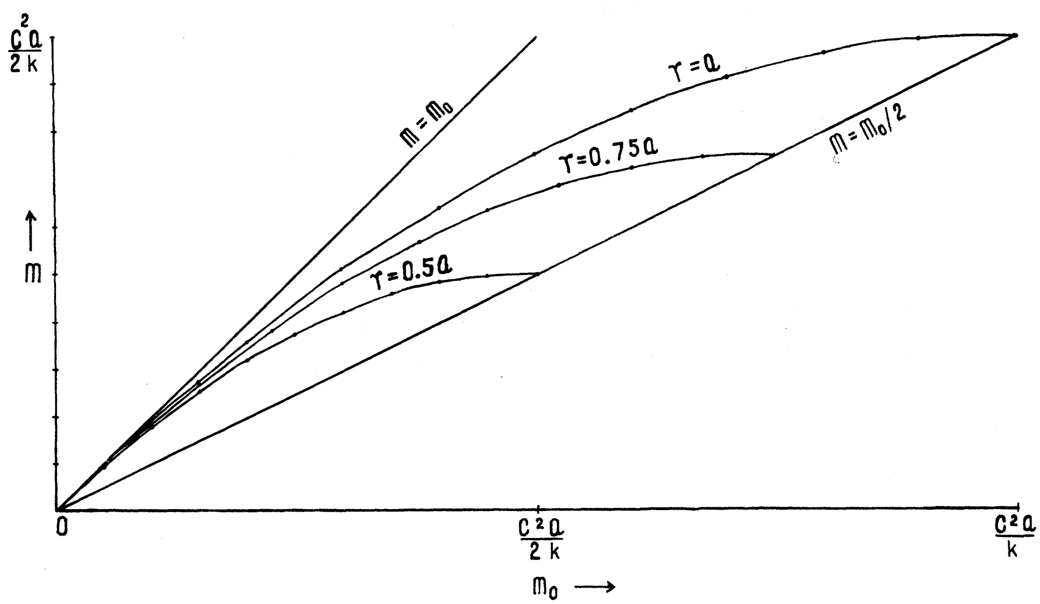

FIG. 2. The plot between m and m_0 for a hollow spherical shell of a constant radius a.

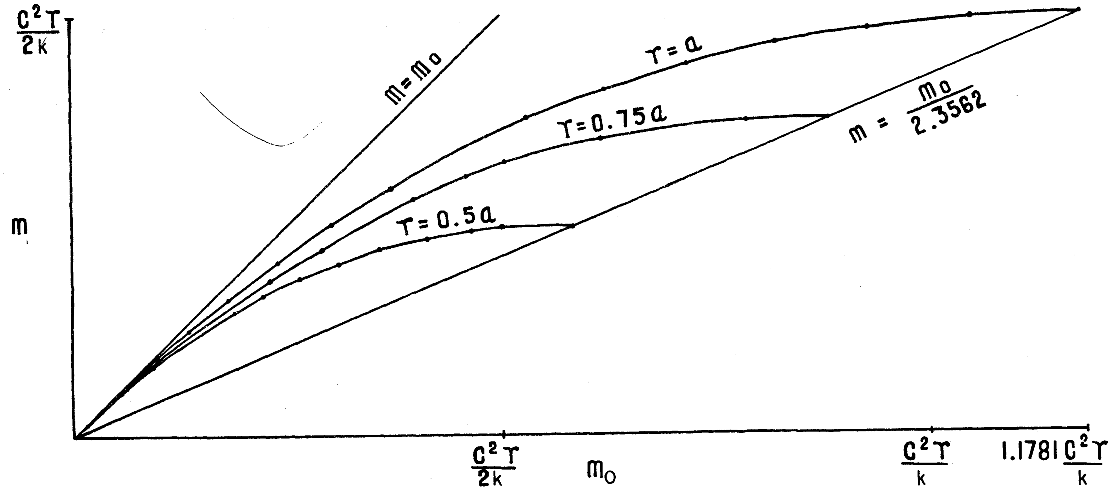

FIG. 3. The plot between m and m_0 for a solid sphere of constant radius a.

$$\tfrac{4}{3}\pi a^3 \mu_0 = m \ . \tag{9}$$

An infinitesimal mass dm on its surface would have mass dm_0 outside the influence of any gravitational field. Suppose this mass forms a shell of radius r and thickness dr, so that $dm = 4\pi r^2 \mu_0 dr$. Therefore, from Eq. (7)

$$dm_0 = 4\pi r^2 \mu_0 (1 - 8\pi k \mu_0 r^2/3c^2)^{-1/2} dr \ .$$

Integration of this, between the limits 0 and a or over the volume occupied by the whole sphere, gives

$$m_0 = (4\pi/2)\mu_0 R^3[\sin^{-1}(a/R) - (a/R)(1 - a^2/R^2)^{1/2}] , \tag{10}$$

where a is the radius of the solid sphere and R is a function of density,

$$R^2 = 3c^2/8\pi k \mu_0 \ .$$

The above result can also be directly written from the Schwarzschild interior solution.[5] By rearranging Eq. (10) with the help of (9), we get the following form in which the masses can be recognized:

$$2km_0/c^2 a = \tfrac{3}{2}[(c^2 a/2km)^{1/2} \sin^{-1}(2km/c^2 a)^{1/2} - (1 - 2km/c^2 a)^{1/2}] \ . \tag{11}$$

When m of a solid sphere is plotted against a from Eq. (11), the curve obtained almost, but not exactly, coincides with the curve in Fig. 1, except that the lowest possible value of a is $0.8488\, km_0/c^2$ (instead of km_0/c^2) and then $m = 0.4244\, m_0$. Variations of m with m_0 for various constant radii are shown in Fig. 3. It has the same general features as in Fig. 2 for a spherical shell except for small quantitative differences. The maximum mass defect or binding energy is thus $0.5756\, m_0$ for a solid sphere of uniform proper density.

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