PHYSICAL REVIEW D VOLUME 13, NUMBER 10 15 MAY 1976

General-relativistic kinetic theory of waves in a massive particle medium

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(Received 25 September 1974; revised manuscript received 17 September 1975)

In this paper, we give a general-relativistic kinetic theory of waves propagating in a medium filled with massive particles. A major difficulty of this problem is to handle simultaneously dispersive and expansion effects. Matter itself is at the root of both phenomena, and in our treatment they are conveniently separated by using a two-time scale approximation. It turns out that the expansion modifies both the amplitude and the frequency of the waves. Dispersion effects give rise to proper modes, which are shown to be the 0, 1, and 2 helicity components of the total field. The dispersion equations for these different components are obtained in a general form. The propagation of gravitational modes is examined in more detail for the two extreme cases of cold and ultrarelativistic matter. A lower cutoff frequency appears, and no Landau damping is found in the case of a thermalized gas.

I. INTRODUCTION

In this paper, we study the propagation and growth of perturbations in an expanding, homogeneous, and isotropic universe filled with massive particles. A kinetic description of the medium is used: Such an approach is now quite common in the field of relativistic cosmology.[1-4]

Up to now, gravitational-wave propagation has been studied in an empty and usually stationary space-time.[5-8] Matter is generally included through a hydrodynamic approximation.[9,10] Kinetic studies have appeared only rather recently, and have been limited either to zero-mass particle media[11] or to gravitational waves propagating in a nonexpanding universe.[12,13] Other types of waves have been considered only in the hydrodynamic approximation,[14] and that in relation to the stability of Friedmann models.

Our aim is to give a single, unified description for every type of wave propagating in a gas consisting of nonzero-mass particles interacting gravitationally. Now, the very existence of a background requires that the universe expand, and so the waves propagate in a nonstationary medium. Furthermore, matter also gives rise to dispersive effects; the two phenomena are therefore coupled in a rather complicated way.

We have overcome this difficulty by considering only waves of period shorter than the Hubble time. Then, on a short time scale (of the order of a few periods), the amplitude and phase of a wave behave as if the medium were stationary, but undergo a slow variation on the longer Hubble time scale. We may therefore use these two characteristic times as the basis for a two-time scale approximation which allows us to readily separate expansion from dispersion effects. The same idea was first used in this context by McCallum and Taub who introduced it in the form "averaged Lagrangian" method. We finally study in some detail gravitational-wave dispersion in cold as well as in ultrarelativistic media.

II. FUNDAMENTAL EQUATIONS

In the kinetic theory, the structure of space-time is described in a self-consistent manner by coupling the Einstein equations to the Liouville equations. Neglecting correlations between particles (i.e., collisions), the Liouville equation reduces to a Vlasov equation for the one-particle distribution function $\mathfrak{N}(x^\alpha, u^\alpha)$. The coupled system is then written as

$$u^\mu \frac{\partial}{\partial x^\mu}\mathfrak{N}(x^\lambda, u^\lambda) - \Gamma^\alpha_{\rho\sigma} u^\rho u^\sigma \frac{\partial \mathfrak{N}(x^\lambda, u^\lambda)}{\partial u^\alpha} = 0\,, \quad (1)$$

$$R_{\mu\nu} = \chi(T_{\mu\nu} - \tfrac{1}{2} G_{\mu\nu} T)\,, \quad (2)$$

$$T_{\mu\nu} = \int \det(-G_{\mu\nu}) \frac{d^3u}{u_4} u_\mu u_\nu\, \mathfrak{N}(x^\lambda, u^\lambda)\,, \quad (3)$$

where $R_{\mu\nu}$ is the Ricci tensor, $G_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy-momentum tensor, and $\Gamma^\alpha_{\rho\sigma}$ is the Christoffel symbol; χ is the Einstein constant, which is related to the gravitational constant G by

$$\chi = \frac{8\pi G}{c^2}\,.$$

We assume that we already know some particular solution of the system (1), (2), (3), which we refer to as the background solution. This specifies in principle a background metric $g_{\mu\nu}$, together with a background distribution function $N(x^\lambda, u^\lambda)$.[1,2] We want to study small deviations from this background solution as $h_{\mu\nu}$ for the metric perturbation and as Z for the perturbed distribution function:

$$G_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}\,,$$
$$\mathfrak{N}(x^\lambda,u^\lambda)=N(x^\lambda,u^\lambda)+Z(x^\lambda,u^\lambda)\,. \tag{4}$$

Since h and Z are small perturbations, the system (1), (2), (3) (Ref. 15) may be linearized.

The perturbations of the Christoffel symbols may be written as

$$X^{\alpha}_{\rho\sigma}=\tfrac{1}{2}(\nabla_\rho h^\alpha_\sigma+\nabla_\sigma h^\alpha_\rho-\nabla^\alpha h_{\rho\sigma})\,. \tag{5}$$

If the perturbations of the left- and right-hand sides of the Einstein equation (2) are written as $(Lh)_{\mu\nu}$ and $\Sigma_{\mu\nu}$, respectively, then

$$(Lh)_{\mu\nu}=\Delta h_{\mu\nu}+(\nabla_\mu I_\nu+\nabla_\nu I_\mu)-h_{\mu\nu}S_\lambda{}^\lambda-g_{\mu\nu}h_{\alpha\beta}S^{\alpha\beta}+2(h_{\mu\alpha}S_\nu{}^\alpha+h_{\nu\alpha}S_\mu{}^\alpha)\,, \tag{6}$$

$$\Sigma_{\mu\nu}=-2(K_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}K_\lambda{}^\lambda)\,, \tag{7}$$

$$K_{\mu\nu}=\chi\int\frac{d^3u}{u_4}\,u_\mu u_\nu(Z+\tfrac{1}{2}h_\lambda{}^\lambda N)\,. \tag{8}$$

Here $S^{\alpha\beta}$, $R^{\alpha\beta}$, and $R^{\alpha\beta\rho\sigma}$ are the Einstein, Ricci, and curvature tensors of the background metric, respectively, and Δ is the de Rham–Lichnerowicz operator for a symmetric tensor:

$$\Delta h_{\mu\nu}=\nabla_\rho\nabla^\rho h_{\mu\nu}-R_{\mu\rho}h^\rho{}_\nu-R_{\nu\rho}h^\rho{}_\mu+2R_{\mu\rho\nu\sigma}h^{\rho\sigma}\,. \tag{9}$$

We next choose the de Donder gauge by putting I_μ equal to zero:

$$I_\mu=\nabla_\nu(h_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu h_\lambda{}^\lambda)=0 \tag{10}$$

(we note that Weinberg uses a different gauge). The full linearized system then becomes

$$u^\mu\partial_\mu Z-\Gamma^\alpha_{\rho\sigma}u^\rho u^\sigma\frac{\partial Z}{\partial u^\alpha}-X^\alpha_{\rho\sigma}\frac{\partial N}{\partial u^\alpha}u^\rho u^\sigma=0, \tag{11}$$

$$(\mathfrak{L}h)_{\mu\nu}=\Sigma_{\mu\nu}\,. \tag{12}$$

In these equations, $\Gamma^\alpha_{\rho\sigma}$ is taken as the background Christoffel symbol.

We next consider a background Robertson-Walker metric:

$$ds^2=dt^2-S^2(t)\,\frac{dx^2+dy^2+dz^2}{(1+kr^2/4)^2}\,. \tag{13}$$

For the sake of simplicity, k will be put equal to zero. This is rigorously correct in the case of an expanding Minkowski universe, and constitutes a good approximation when the wavelength is much smaller than the radius of the universe. It then follows that

$$\Gamma^i_{4j}=\frac{\dot S}{S}\,\delta^i_j\quad(i,j=1,2,3)\,,$$
$$\Gamma^4_{ij}=S\dot S\delta_{ij}\,, \tag{14}$$

$$X^4_{44}=\frac{1}{2}\,\frac{\partial}{\partial x^4}\,h_{44}\,,$$
$$X^4_{i4}=\frac{1}{2}\left(\frac{\partial}{\partial x^i}\,h_{44}-2\frac{\dot S}{S}\,h_{4i}\right), \tag{15}$$
$$X^4_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial x^i}\,h_{4j}+\frac{\partial}{\partial x^j}\,h_{4i}-\frac{\partial}{\partial x^4}\,h_{ij}-2S\dot S h_{44}\delta_{ij}\right),$$

$$(\mathfrak{L}h)_{44}=\left(\frac{\partial^2}{\partial(x^4)^2}-\frac{1}{S^2}\,\frac{\partial^2}{\partial x^i\partial x^i}\right)h_{44}+3\,\frac{\dot S}{S}\,\frac{\partial h_{44}}{\partial x^4}+4\,\frac{\dot S}{S^3}\,\frac{\partial}{\partial x^l}\,h_{4l}-g\left(\frac{\dot S}{S}\right)^2 h_{44}-3\,\frac{\dot S^2}{S^4}\,h_{ll}\,,$$

$$(\mathfrak{L}h)_{4i}=\left(\frac{\partial^2}{\partial(x^4)^2}-\frac{1}{S^2}\,\frac{\partial^2}{\partial x^j\partial x^j}\right)h_{4i}+\frac{\dot S}{S}\,\frac{\partial}{\partial x^4}\,h_{4i}+2\,\frac{\dot S}{S^3}\,\frac{\partial}{\partial x^l}\,h_{il}+2\,\frac{\dot S}{S}\,\frac{\partial}{\partial x^i}\,h_{44}-\left(\frac{\ddot S}{S}+11\,\frac{\dot S^2}{S^2}\right)h_{4i}\,, \tag{16}$$

$$(\mathfrak{L}h)_{ij}=\left(\frac{\partial^2}{\partial(x^4)^2}-\frac{1}{S^2}\,\frac{\partial^2}{\partial x^i\partial x^i}\right)h_{ij}-\frac{\ddot S}{S}\,\frac{\partial}{\partial x^4}\,h_{ij}+2\,\frac{\dot S}{S}\left(\frac{\partial}{\partial x^i}\,h_{4j}+\frac{\partial}{\partial x^j}\,h_{4i}\right)-\left(6\,\frac{\ddot S}{S}+2\,\frac{\dot S^2}{S^2}\right)h_{ij}+(2S\ddot S-5\dot S^2)h_{44}\delta_{ij}+\left(2\,\frac{\ddot S}{S}-\frac{\dot S^2}{S^2}\right)h_{ll}\delta_{ij}\,,$$

where

$$h_{ll}=h_{11}+h_{22}+h_{33}\,.$$

The basic equations are simply obtained by inserting expressions (14), (15), (16) into the linearized self-consistent system (5), (6), (7).

III. THE TWO-TIME SCALE APPROXIMATION

Let us emphasize that dispersion and expansion effects occur necessarily simultaneously by the characteristic time of expansion, which is the Hubble time

$$\tau_H=\left(\frac{\dot S}{S}\right)^{-1}=\left(\frac{3}{\chi\rho}\right)^{1/2}. \tag{17}$$

It is shown in a later part of this paper that dispersion effects alter the phase of the wave on a time scale comparable to τ_H, and so neither phenomenon can be neglected. If waves whose periods are comparable to τ_H are considered, then full account should be taken of the fact that the medium is not stationary. However, in the other limit $\omega^{-1}\ll\tau_H$, the medium remains essentially stationary for a number of periods, and expansion effects will appear as a slow modulation of the instantaneous dispersion characteristics. This feature gives rise to the two-time scale approximation: It was first introduced for the analysis of slightly nonlinear oscillations by

Poincaré and its application to that case is described rather well in the book by Cole.[16]

We begin by describing this method briefly. It will be seen later that our linearized set of equations can be reduced to a system which resembles, for example,

$$[\partial_4{}^2+\omega_k{}^2(t)]\psi=\frac{\dot S(t)}{S(t)}\,\partial_4\psi\ . \tag{18}$$

$\omega_k(t)$ is an eigenfrequency which varies slowly with time on the same scale as $S(t)$ itself. We now introduce two different time variables, t_c and t_L; the former will apply to phenomena occurring in times of the order of $\omega_k{}^{-1}$ while the latter describes phenomena which occur over time periods of the order of τ_H. As a result, ω_k and S will both be functions of the long-time variable t_L only, and (18) becomes

$$[\partial_4{}^2+\omega_k{}^2(t_L)]\,\psi=\frac{\dot S}{S}\,(t_L)\partial_4\psi\ . \tag{19}$$

A slowly drifting frequency appears in this equation; this snag (see Ref. 16, p. 102) is eliminated immediately by redefining the short-time scale variable. Let us define τ such that

$$d\tau=\omega_k(t)dt\ ,\qquad \tau=\int_0^t\omega_k(t)dt\ . \tag{20}$$

In terms of the variables τ and t_L, the total derivative with respect to t can be written as

$$\frac{d}{dt}=\frac{d\tau}{dt}\,\frac{\partial}{\partial\tau}+\frac{dt_L}{dt}\,\frac{\partial}{\partial t_L}\ . \tag{21}$$

The second term of this derivative involves the long-time scale, so that when it operates on some function ψ it will yield a contribution which is of the order of $(\omega_k\tau_H)^{-1}$ smaller than the first term. If we put

$$\epsilon=\frac{dt_L}{dt} \tag{22}$$

we may the more easily assess the order of magnitude of each term; ϵ will be put equal to 1 at the end of the calculation. Substituting (22) and (20) in (21) and iterating, we obtain the formal expansion of the operators

$$\frac{\partial}{\partial t}=\omega_k(t_L)\,\frac{\partial}{\partial\tau}+\epsilon\,\frac{\partial}{\partial t_L}\ ,$$
$$\frac{\partial^2}{\partial t^2}=\omega_k{}^2(t_L)\,\frac{\partial^2}{\partial\tau^2}+\epsilon\left[2\omega_k(t_L)\,\frac{\partial}{\partial t_L}\,\frac{\partial}{\partial\tau}+\frac{\partial\omega_k}{\partial t_L}\,\frac{\partial}{\partial\tau}\right]+\epsilon^2\,\frac{\partial^2}{\partial t_L{}^2}\ . \tag{23}$$

Substituting now (23) into (19) and considering ψ to be a function of the two variables τ and t_L, we obtain

$$\left[\frac{\partial^2}{\partial\tau^2}+\frac{2\epsilon}{\omega_k}\,\frac{\partial^2}{\partial t_L\partial\tau}+\frac{\epsilon}{\omega_k{}^2}\left(\frac{\partial\omega_k}{\partial t_L}\right)\frac{\partial}{\partial\tau}+\frac{\epsilon^2}{\omega_k{}^2}\,\frac{\partial^2}{\partial t_L{}^2}+1\right]\psi=\epsilon\,\frac{\dot S}{S}\,(t_L)\,\frac{1}{\omega_k{}^2}\left[\omega_k(t_L)\,\frac{\partial}{\partial\tau}+\epsilon\,\frac{\partial}{\partial t_L}\right]\psi\ . \tag{24}$$

Henceforth $\partial\psi/\partial t_L$ will be written as $\dot\psi$. The function ψ is next expanded as a polynomial in ϵ:

$$\psi=\psi^{(0)}+\epsilon\psi^{(1)}+\epsilon^2\psi^{(2)}+\cdots\ . \tag{25}$$

Using (25), (24) can now be split into a hierarchy of coupled equations for $\psi^{(0)},\psi^{(1)},\psi^{(2)},\cdots$,

$$\frac{\partial^2\psi^{(0)}}{\partial\tau^2}+\psi^{(0)}=0\ ,$$
$$\epsilon\left(\frac{\partial^2\psi^{(1)}}{\partial\tau^2}+\psi^{(1)}\right)+\left[\frac{2\epsilon}{\omega_k}\,\frac{\partial^2}{\partial t_L\partial\tau}+\frac{\epsilon}{\omega_k{}^2}\left(\frac{\partial\omega_k}{\partial t_L}\right)\frac{\partial}{\partial\tau}\right]\psi^{(0)}=\epsilon\,\frac{\dot S}{S}\,\frac{1}{\omega_k}\,\frac{\partial}{\partial\tau}\,\psi^{(0)}\ , \tag{26}$$
$$\epsilon^2\left(\frac{\partial^2\psi^{(2)}}{\partial\tau^2}+\psi^{(2)}\right)+\epsilon\left(\frac{2\epsilon}{\omega_k}\,\frac{\partial^2}{\partial t_L\partial\tau}+\epsilon\,\frac{\dot\omega_k}{\omega_k{}^2}\,\frac{\partial}{\partial\tau}\right)\psi^{(1)}+\frac{\epsilon^2}{\omega_k{}^2}\,\frac{\partial^2}{\partial t_L{}^2}\,\psi^{(0)}=\epsilon^2\,\frac{\dot S}{S}\,\frac{1}{\omega_k}\,\frac{\partial}{\partial\tau}\,\psi^{(1)}+\epsilon^2\frac{\dot S}{S}\,\frac{1}{\omega_k{}^2}\,\frac{\partial}{\partial t_L}\,\psi^{(0)}\ ,$$

etc. (the other equations are obtained by replacing $\psi^{(2)}$ by $\psi^{(3)}$, $\psi^{(1)}$ by $\psi^{(2)}$, $\psi^{(0)}$ by $\psi^{(1)}$, etc.). The system can then be solved easily by iteration. Indeed, the first equation gives immediately the solution for $\psi^{(0)}$

$$\psi^{(0)}=\varphi^{(0)}(t_L)e^{i\tau}\ . \tag{27}$$

Here, $\varphi^{(0)}$ is some as yet undetermined function; substituting (27) into the second equation of the system, we obtain

$$\frac{\partial^2\psi^{(1)}}{\partial\tau^2}+\psi^{(1)}=e^{i\tau}\left[i\left(\frac{\dot S}{S}\,\frac{1}{\omega_k}-\frac{\dot\omega_k}{\omega_k}\right)\varphi^{(0)}{}_{(t_L)}-\frac{2i}{\omega_k}\,\frac{\partial\varphi^{(0)}{}_{(t_L)}}{\partial t_L}\right]\ . \tag{28}$$

Now, it is well known that the $\psi^{(1)}$ derived from (28) increases in a linear way: In effect, the eigenfrequencies of the left-hand side of the equation resonate at the frequency of the right-hand-side forcing term. Consequently, the terms in

(25) increase without limit and so invalidate the expansion for times in excess of some rather small limit. This characteristic can be circumvented by choosing $\varphi^{(0)}$ in such a way that the factor in the square brackets in (28) vanishes—we are allowed to do this because $\varphi^{(0)}$ is not so far determined, and this is a consequence of the way in which the physical time was decomposed into two mathematical variables. This procedure is the only one that allows the series (25) to converge over a time interval of the order of τ_H; applying it, (28) becomes

$$\frac{d^2\psi^{(1)}}{d\tau^2}+\psi^{(1)}=0\,,\qquad \frac{\dot S}{S}-\frac{\dot\omega_k}{\omega_k}=2\,\frac{\dot\varphi^{(0)}}{\varphi^{(0)}}\,. \tag{29}$$

Each order may be handled in a similar way.

Let us now return to the system (11), (12). It readily follows from (17) that the factor φ is $O(\epsilon^2)$. The unperturbed distribution function as well as its perturbation Z is proportional to ρ, and so from (8) and (7) we see that $\Sigma_{\mu\nu}$ is also $O(\epsilon^2)$. $\Sigma_{\mu\nu}$ includes the dispersion effect of matter, and so we have to expand the solution up to second order. The zeroth and first order will then describe geometrical effects (trailing effect[17]) from the dispersion effects of matter, we combine the zeroth order of $\mathfrak{L}$ with $\Sigma_{\mu\nu}$. In order to obtain Z in terms of $h_{\mu\nu}$ we solve the linearized Liouville equation (11). The explicit expression (5) for the perturbed Christoffel symbols $X^{\alpha}_{\rho\sigma}$ shows that the gravitational force due to the perturbation [third term in (11)] is of the same order of magnitude as the first free-streaming term. The second term, however, is of higher order because the long-range gravitational force which produces the expansion of the universe is weak; i.e., it is of order $\dot S/S$ as can be seen in the expression of the Christoffel symbol (14). The Liouville equation then reduces to

$$u^\mu\partial_\mu Z=P(t),\qquad P(t)\equiv\tfrac{1}{2}(\partial_\rho h^\alpha_\sigma+\partial_\sigma h^\alpha_\rho-\partial^\alpha h_{\rho\sigma})u^\rho u^\sigma\,\frac{\partial N}{\partial u^\alpha}\,. \tag{30}$$

At this stage it is convenient to introduce a spatial Fourier transform (the definition of this is given by Weinberg[10]). Then

$$Z=\mathfrak{z}\exp(-i\vec q\cdot\vec x)\,,\qquad P=\mathcal{P}\exp(-i\vec q\cdot\vec x)\,,\qquad u^4\,\frac{\partial Z}{\partial t}+iq_iu^iZ=P(t)\,. \tag{31}$$

Equation (31) is formally solved by taking $P_j(t)$ as a second member:

$$Z=\int_0^\infty d\tau\,\frac{P(t-\tau)}{u_4}\exp\left(-i\,\frac{q_ju^j}{u_4}\tau\right)\,. \tag{32}$$

We look for solutions of the form

$$Z=\hat Z\exp\left[-i\int_a^t\omega(\tau')d\tau'\right],\qquad P=\hat P\exp\left[-i\int_a^t\omega(\tau')d\tau'\right], \tag{33}$$

where $\hat Z$ and $\hat P$ are slowly varying functions of time. This allows us to expand the function P in Eq. (32) in the neighborhood of t. Keeping only the lowest-order terms, we obtain

$$\hat Z=\hat P(t)\int_a^\infty\exp\left[i\left(\omega-\frac{q_iu^i}{u_4}\right)\tau\right]\frac{d\tau}{u_4}=i\,\frac{\hat P(t)}{q_\lambda u^\lambda}\,. \tag{34}$$

This solution is substituted into (8) and (7) to give

$$\Sigma_{\mu\nu}=O^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}\,.$$

Equation (12) is then reduced to a linear homogeneous system for $\hat h_{\alpha\beta}$

$$\mathfrak{D}^{\alpha\beta}_{\mu\nu}\hat h_{\alpha\beta}=\mathcal{E}^{\alpha\beta}_{\mu\nu}\hat h_{\alpha\beta}\,. \tag{35}$$

The left-hand side then represents the dispersion characteristics of matter. It is obtained by putting the Laplace operator which is in $(\mathfrak{L}h)$ [Eq. (16)] together with $\Sigma_{\mu\nu}$. The right-hand-side term contains all the trailing effects due to the expansion, which originate in terms proportional to $\dot S$ and $\ddot S$ in Eq. (16). After some algebraic manipulation, the operator $\mathfrak{D}^{\alpha\beta}_{\mu\nu}$ is found to be given by

$$\mathfrak{D}^{\alpha\beta}_{\mu\nu}=\left(-\frac{\omega^2}{c^2}+\frac{q^2}{S^2}+2i\,\frac{\omega}{c}\,\frac{\partial}{\partial t}+\frac{i}{c}\,\frac{\partial\omega}{\partial t}+\frac{\partial^2}{\partial t^2}\right)\eta^\alpha_\mu\eta^\beta_\nu-O^{\alpha\beta}_{\mu\nu}\,. \tag{36}$$

The system has a nontrivial solution at zero order given by

$$\left[\left(-\frac{\omega^2}{c^2}+\frac{q^2}{S^2}\right)\eta^\alpha_\mu\eta^\beta_\nu-O^{\alpha\beta}_{\mu\nu}\right]\hat h_{\alpha\beta}=0 \tag{37}$$

if the determinant of this latter operator vanishes. This defines a set of eigenmodes $\hat h^{(E)}_{\alpha\beta}$, to each of which corresponds a particular instantaneous eigenfrequency $\omega_k(t)$.

Let us consider a particular form of the solution which we shall call a shift-type solution:

$$h^{(E)}(t)=\hat h^{(E)}(t)\exp\left[-i\int_a^t\omega_k(\tau')d\tau'\right]. \tag{38}$$

The operator $\mathfrak{D}^{\alpha\beta}_{\mu\nu}$ is the time Fourier transform of a nonlocal time operator. This arises as a consequence of the dispersion. The interest of shift-type solutions is that $\mathfrak{D}^{\mu\nu}_{\alpha\beta}$ acts on them in exactly the same way as does the simpler, localtime operator:

$$\left[\frac{\partial^2}{\partial t^2}+\omega_k{}^2(t)\right]\eta_\mu^\alpha\eta_\nu^\beta\,. \qquad (39)$$

We limit ourselves to shift-type solutions, because the operator (39) is appropriate to the two-time scale approximation. (35) is then seen to reduce to the following system for the various polarizations (E):

$$\left[\frac{\partial^2}{\partial t^2}+\omega_k{}^2(t)\right]\eta_\mu^\alpha\eta_\nu^\beta h_{\alpha\beta}^{(E)}(t)=\mathcal{E}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta}^{(E)}(t). \qquad (40)$$

This equation is of the form (18). We emphasize that the limitation to a shift-type of solution is not particularly important; in fact, equations of the type (18) have just this form of solution. Coupling between the different eigenpolarizations introduced by the trailing term in (40) does not occur before the third order; we note that this independence is by no means as obvious as is often assumed in the literature.

IV. PROPER MODES OF THE DISPERSION OPERATOR

We shall first study the properties of the left-hand-side dispersion operator $\mathfrak{D}_{\mu\nu}^{\alpha\beta}$; after this, we can come back to the full equation (35). The dispersion operator has a certain number of proper modes whose polarization can immediately be specified as a consequence of the isotropy of space. In the case of a plane wave propagating in an isotropic medium, the only perferred space direction is that of the wave vector $\vec{K}$. We choose the coordinate system in such a way that $\vec{K}$ lies along the x axis,

$$K_\alpha=(q,0,0,\omega/c)\,.$$

One might think that these independent proper polarizations would correspond to the 2, 1, 0 spin components of the field $h_{\mu\nu}$. It is, in effect, possible to define such components by the way they transform under space rotations; this is usual in the theory of the Pauli-Fierz fields. However, this procedure is not applicable to our case because the field of the wave is organized around the privileged direction of the wave vector $\vec{K}$; it is instead more profitable to define a helicity for the wave. Following Weinberg,[18] a given plane wave ψ is said to have helicity h if a space rotation of angle θ around the wave vector transforms it to ψ', given by

$$\psi'=e^{ih\theta}\psi\,. \qquad (41)$$

In our case, the components of the symmetric tensor $h_{\mu\nu}$ (the field of the wave) can be grouped into a number of variables, each of which represents a partial field of given spin and helicity. These are given in Table I. The dispersion operator can be written explicitly in terms of the variables $h_{\alpha\beta}$:

$$\mathfrak{D}_{\mu\nu}^{\alpha\beta}=\tfrac{1}{2}(K_\mu K^\mu)(\eta_\mu^\alpha\eta_\nu^\beta+\eta_\mu^\beta\eta_\nu^\alpha)+2\chi(\eta_\mu^\alpha T_\nu^\beta+\eta_\nu^\alpha T_\mu^\beta+\eta_\mu^\beta T_\nu^\alpha+\eta_\nu^\beta T_\mu^\alpha)+3\chi\eta^{\alpha\beta}(T_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}T)+2\chi(K_\mu K^\mu)(J_{\mu\nu}^{\alpha\beta}-\tfrac{1}{2}\eta_{\mu\nu}J^{\alpha\beta})$$
$$-2\chi(K_\mu I_\nu^{\alpha\beta}+K_\nu I_\mu^{\alpha\beta})-2\chi[K^\alpha(I_{\mu\nu}^\beta-\tfrac{1}{2}\eta_{\mu\nu}I^\beta)+K^\beta(I_{\mu\nu}^\alpha-\tfrac{1}{2}\eta_{\mu\nu}I^\alpha)]\,. \qquad (42)$$

The notation is formally covariant because the time axis $\bar{u}^\alpha=(0,0,0,1)$ and space are well defined by the background solution. The Robertson-Walker metric is flat, over the short-time scale only, because the scale function is a function of t_L only. Then

$$K_i=q_i,\quad K_4=\omega/c,\quad K^i=-\frac{1}{S^2}\,q_i,\quad q^2=\delta^{ij}q_iq_j\,. \qquad (43)$$

TABLE I. Variables H_λ characterized by a definite spin and helicity, given as a function of the components $h_{\mu\nu}$ of the wave's field.

Spin \ Helicity	2	1	0
2	$H^+(2,2)=\frac{1}{2}(h_{22}-h_{33})+ih_{23}$	$H^+(2,1)=h_{12}+ih_{13}$	$H(2,0)=h_{11}-\frac{1}{3}h_{ll}$
	$H^-(2,2)=\frac{1}{2}(h_{22}-h_{33})-ih_{23}$	$H^-(2,1)=h_{12}-ih_{13}$	
1		$H^+(1,1)=h_{42}+ih_{43}$	$H(1,0)=h_{41}$
		$H^-(1,1)=h_{42}-ih_{43}$	
0			$H^T(0,0)=h_{44}$
			$H^S(0,0)=h_{ll}$

$T_{\mu\nu}$ is the energy-momentum tensor and $I_{\mu}^{\alpha\beta}$, $J_{\mu\nu}^{\alpha\beta}, \ldots$ are integrals involving the distribution function:

$$I_{\mu}^{\alpha\beta} = S^3 \int \frac{d^3u}{u_4} \frac{u^\alpha u^\beta u_\mu}{(K_\lambda u^\lambda)} N(x^\alpha, u^\alpha),$$

$$J_{\mu\nu}^{\alpha\beta} = S^3 \int \frac{d^3u}{u_4} \frac{u^\alpha u^\beta u_\mu u_\nu}{(K_\lambda u^\lambda)^2} N(x^\alpha, u^\alpha), \tag{44}$$

$$T_{\mu\nu} = S^3 \int \frac{d^3u}{u_4} u_\mu u_\nu N(x^\alpha, u^\alpha).$$

For simple distribution functions (such as the Jüttner-Synge distribution function, which depends on the single vector parameter $\bar{u}_\alpha$), most of these integrals vanish. This is most easily shown by noting that each is some linear combination of the tensors that can be formed from $\bar{u}^\alpha$ and the four-vector K_α, and have the same order.[19]

We can see that I and J vanish when they are labeled by an odd number of index 2 or 3. This results in a straightforward block diagonalization of $\mathcal{D}_{\mu\nu}^{\alpha\beta}$, as indicated in Fig. 1.

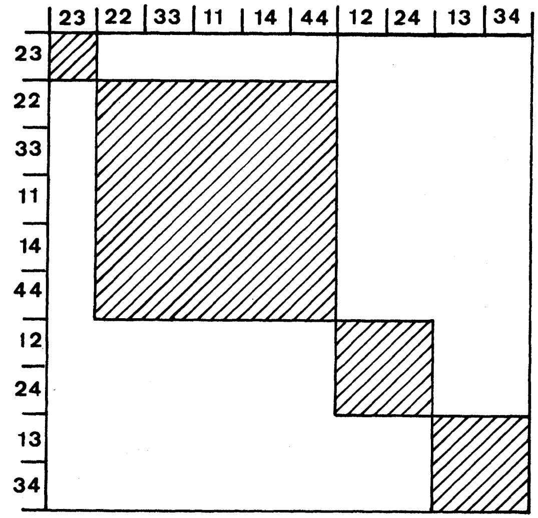

FIG. 1. Block diagonalization of $\mathcal{D}_{\mu\nu}^{\alpha\beta}$ for an isotropic distribution function that depends on one parameter. Shaded blocks contain all nonzero elements.

We now introduce spin-helicity variables H_λ. These are defined in such a way that each of them is an element of given spin and helicity (see Table I):

$$\begin{aligned}
&H_1 = h_{23}, && h_{23} = H_1,\\
&H_2 = \tfrac{1}{2}(h_{22} - h_{23}), && h_{22} = H_2 - \tfrac{1}{6}H_3 + \tfrac{1}{3}H_4,\\
&H_3 = 3h_{11} - h_{ll}, && h_{33} = -H_2 - \tfrac{1}{6}H_3 + \tfrac{1}{3}H_4,\\
&H_4 = h_{ll}, && h_{11} = \tfrac{1}{3}H_3 + \tfrac{1}{3}H_4,\\
&H_5 = h_{41}, && h_{41} = H_5,\\
&H_6 = h_{44}, && h_{44} = H_6,\\
&H_7 = h_{12}, && h_{12} = H_7,\\
&H_8 = h_{24}, && h_{24} = H_8,\\
&H_9 = h_{13}, && h_{13} = H_9,\\
&H_{10} = h_{34}, && h_{34} = H_{10},
\end{aligned} \tag{45}$$

In the H representation, the dispersion operator can be reduced to the block-diagonal form represented in Fig. 2.

The elements are given in terms of $\mathcal{D}_{\mu\nu}^{\alpha\beta}$ in Appendix A.

It can be seen that the proper modes, defined as nontrivial solutions of Eq. (37) can be grouped into five sets defined by:

$$\mathcal{D}_1^1 = 0, \quad H_\lambda = 0, \text{ except } H_1; \tag{46a}$$

$$\mathcal{D}_2^2 = 0, \quad H_\lambda = 0, \text{ except } H_2; \tag{46b}$$

$$\det \mathcal{D}_\lambda^{\lambda'} = 0 \ (3 \le \lambda \le 6, 3 \le \lambda' \le 6),$$
$$H_\lambda = 0 \text{ except } 3 \le \lambda \le 6; \tag{46c}$$

$$\det \mathcal{D}_\lambda^{\lambda'} = 0 \ (7 \le \lambda \le 8, 7 \le \lambda' \le 8),$$
$$H_\lambda = 0 \text{ except } 7 \le \lambda \le 8; \tag{46d}$$

$$\det \mathcal{D}_\lambda^{\lambda'} = 0 \ (9 \le \lambda \le 10, 9 \le \lambda' \le 10),$$
$$H_\lambda = 0 \text{ except } 9 \le \lambda \le 10. \tag{46e}$$

In fact, sets (46d) and (46e) have the same dispersion relation, because the directions perpendicular to $\vec{K}$ are all equivalent and the modes of helicity 1, represented by (46d) and (46e), are identical. Similarly, explicit calculations show

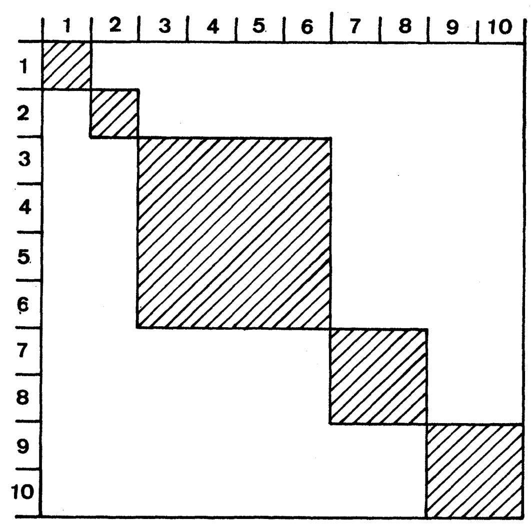

FIG. 2. Block diagonalization of $\mathcal{D}_{\mu\nu}^{\alpha\beta}$ in the representation using variables H_λ.

the $\mathfrak{D}_1^1=\mathfrak{D}_2^2$, so that modes of helicity 2, represented by (46a) and (46b), are also identical. We see that, as expected, the proper modes decompose into independent components of helicity 2, 1, and 0.

Helicity-2 waves represent radiative gravitational waves. Their polarization is determined by the two arbitrary parameters H_1 and H_2. All other variables vanish. Coming back to the usual $h_{\alpha\beta}$ variable, we see that the gravitational wave field is determined by

$$h_{\alpha\beta}=0, \text{ except } h_{23}=h_{32},\ h_{22}=-h_{33}. \tag{47}$$

This result is well known; we have generalized it here to the case where waves propagate in the presence of matter.

Helicity-1 and -0 waves do not exist in a vacuum. Consequently, they do not appear before the second order in ϵ, which is the point at which the $\Sigma_{\mu\nu}$ term can be consistently included in (12).

We still have to prove that our solution does actually satisfy the gauge condition, because at the outset we dropped the gauge terms in (6). The gauge condition is a constraint on the polarization of the proper modes. In order to show directly that it is satisfied, we would first have to find these modes, and this is beyond the scope of the paper, except for gravitational radiating modes. We show in the Appendix that this condition is automatically satisfied for the order in which we are interested.

V. CASE OF GRAVITATIONAL WAVES

Gravitational waves have helicity 2, and their dispersion equation is just

$$\mathfrak{D}_1^1=\mathfrak{D}_2^2=K_\lambda K^\lambda(1+\pi\chi J_{\perp\perp})-8\chi T^{22}, \tag{48}$$

with

$$J_{\perp\perp}=S^3\int du_\perp\int_{-\infty}^{+\infty}du_\parallel\,\frac{u_\perp{}^4}{u_4}\,\frac{N}{(K_\mu u^\mu)^2}\,. \tag{49}$$

The subscripts $\perp$ and $\parallel$ refer to directions perpendicular and parallel to the $\vec{K}$ vector, respectively.

As we explain in Sec. III, we use the two-time scale approximation to separate dispersion from expansion effects. The former determined the eigenfrequency, which in this case is the solution of (46a), and the latter are given by the expansion scheme which we have already described.

A. Expansion effects

Substituting the explicit expressions (42) and (46) for $\mathfrak{D}_{\alpha\beta}^{\mu\nu}$ and $\mathcal{S}_{\alpha\beta}^{\mu\nu}$ we obtain for successive orders

$$\begin{aligned}
h_{ij}^{(0)}&=h_{ij}^{(0)}\exp\left[i\int_0^t\omega_k(t')dt'\right],\\
h_{ij}^{(0)}+h_{ij}^{(1)}&=e_{ij}^{(0)}\frac{S(t)}{S(0)}\exp\left[i\int_0^t\omega_k(t')dt'\right]+h_{ij}^{(1)}\exp\left[i\int_0^t\omega_k(t')dt'\right],\\
h_{ij}^{(0)}+h_{ij}^{(1)}+h_{ij}^{(2)}&=e_{ij}^{(0)}\frac{S(t)}{S(0)}\exp\left\{i\left[\int_0^t dt'\omega_k(t')-\frac{1}{2q}\left(5\ddot{S}+3\frac{\dot{S}^2}{S}\right)\right]\right\}+\left[e_{ij}^{(1)}\frac{S(t)}{S(0)}+e_{ij}^{(2)}\right]\exp\left[i\int_0^t\omega_k(t')dt'\right].
\end{aligned} \tag{50}$$

Of course $h_{ij}=0$ if i or j are neither 2 nor 3, and $h_{23}=h_{32}$, $h_{22}=-h_{33}$ as shown in Sec. IV.

The last equation of the set (50) can also be written correct to the same order in a more compact form:

$$h_{ij}^{(0)+(1)+(2)}=e_{ij}\frac{S(t)}{S(0)}\times\exp\left[i\int_0^t\omega_k(t')dt'-\frac{1}{2q}\int_0^t\left(5\ddot{S}+3\frac{\dot{S}^2}{S}\right)dt'\right].$$

It can be seen readily that, at this order, expansion makes the amplitude vary, and also shifts the phase of the wave slowly. The instantaneous frequency of the wave, which is defined as the time derivative of its phase factor, is

$$\omega(t)=\omega_k(t)-\frac{1}{2q}\left(5\ddot{S}+3\frac{\dot{S}^2}{S}\right). \tag{51}$$

B. Dispersive effects

The dispersive part of the frequency is obtained by solving Eqs. (48) and (49). This can be done for a given background distribution function. It is well known that the equilibrium Jüttner-Synge distribution function is, strictly speaking, not a solution of the self-consistent cosmological equations, since in an expanding universe, neither the energy nor the three-momentum are constants of motion. However, this function is a solution when the background is made up of a gas of massless particles; the Jüttner-Synge distribution is therefore a good approximation in a very hot medium—i.e., when the rest mass is negligible with respect to the kinetic energy. When the medium is quite cold and the universe expands slowly, the Maxwell distribution is a good approximation[20]; however, in the cold case the Maxwell distribution is equivalent to the Jüttner-Synge distribution.

For the sake of simplicity, we shall use the following Jüttner-Synge-type background distribution function[21]:

$$N=\frac{nm\xi}{4\pi K_2(\xi)}\exp(-\xi u^4)\,. \tag{52}$$

As we have already noted, we may for small-time scales take space-time to be Minkowskian, modulo the scale factor $S(t)$ which we leave out of the calculations themselves. Note that the time dependence has nevertheless not been lost, because integrals such as $J_{\perp\perp}$ are functions of the parameters which define the medium:

$$\xi=mc^2/kT(t),\quad \omega_G{}^2=\chi n(t)mc^2\,. \tag{53}$$

The time variation of these quantities is well known for the two cases in which we are interested: for a hot medium,

$$n(t)\simeq n_0S^4(0)/S^4(t),\quad T(t)\simeq T_0S(0)/S(t)\,, \tag{54}$$

and for a cold medium,

$$n(t)\simeq n_0S^3(0)/S^3(t)\,,\quad T(t)\simeq T_0S^2(0)/S^2(t)\,. \tag{55}$$

The kinetic integral $J_{\perp\perp}$ in (49) may become singular due to the factor $(K_\lambda u^\lambda)^{-1}$. This is a well-known difficulty, first studied in plasma physics, when kinetic-theory calculations of the propagation of electrostatic waves were made. Landau[22] noted, however, that the time Fourier transform should be replaced by a time Laplace transform. Indeed, the correct way to attack the problem is to put it as an initial-value problem, and seek for the asymptotic solution, for which all transitory modes are negligible. Therefore, ω should be looked at as a complex variable of the convergence domain of the Laplace transform, which is, with our present sign conventions, the upper half plane. Evaluation of the causal response for t positive is best made by shifting the Laplace inversion contour towards $(-i\infty)$, then enclosing the singularities of the integrand. This integrand is made of some usually regular function of ω divided by the "dispersion function" $D^+(k,\omega)$. In classical problems, the singularities are the zeros of $D^+(k,\omega)$, and the analytic continuation of this function when ω is real is readily obtained by putting

$$\begin{aligned}\frac{1}{K_\lambda u^\lambda}&=\frac{1}{(\omega/c)u^4-qu^1}\\&=\frac{1}{u^4[(\omega/c)+i\nu]-qu^1}\\&=\mathrm{P}\,\frac{1}{(\omega/c)u^4-qu^1}-i\pi\delta\left(\frac{\omega}{c}\,u^4-qu^1\right).\end{aligned} \tag{56}$$

In that case, the asymptotic response varies as $\exp[-i\omega_k(t)]$, where ω_k is the complex zero of $D^+(k,\omega)$ which has the largest imaginary part. However, in the case of relativistic plasmas, the situation has been recognized as more complicated due to the existence of branch-point singularities $\omega=\pm ck$ in the Laplace transform of the electric field (23), so that, to the usual Landau pole contribution, a contribution is added which originates in the cut between $-ck$ and $+ck$ (see Fig. 3).

This cut usually contributes a damped term,[23] which may decay faster than the Landau contribution or not, according to the physical conditions. Strictly speaking, the whole problem of gravitational-wave propagation would be best treated in this initial-value point of view. Here, we shall, however, keep this problem apart because our method is not suited for treating initial values; the cut contribution is presumably damped, as in the case studied in Ref. 23, and we shall show that the Landau pole contribution is actually undamped, so that the latter would asymptotically dominate. Let us switch now to an easier notation:

$$u^1=v_\parallel\,,\quad \gamma=[1-(u^1)^2-(u^2)^2-(u^3)^2]^{-1/2}\,. \tag{57}$$

For an arbitrary isotropic distribution function $f(\gamma)$, $J_{\perp\perp}$ is given unambiguously by (56), and, putting $k=q/S$,

$$\mathrm{Re}\,J_{\perp\perp}=\frac{\partial}{\partial\omega}\frac{c}{k}\int_1^\infty\frac{d\gamma}{\gamma}f(\gamma)\int_{-(\gamma^2-1)^{1/2}}^{(\gamma^2-1)^{1/2}}dv_\parallel\,\mathrm{P}\,\frac{(\gamma^2-1-v_\parallel{}^2)^2}{v_\parallel-\omega\gamma/ck}\,. \tag{58}$$

The imaginary part of $J_{\perp\perp}$ is nonzero if the res-

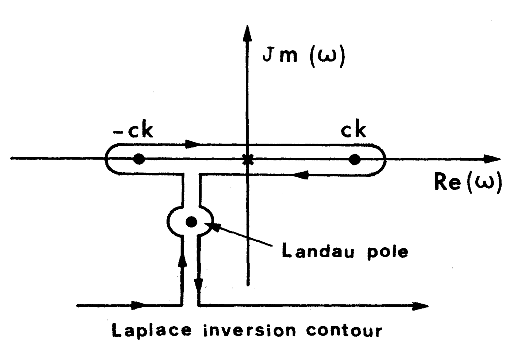

FIG. 3. Example of Laplace inversion contour.

onance condition implied by the δ function in (56) can be satisfied. When the dispersive phase velocity exceeds c, $\mathrm{Im}\, J_{\perp\perp}$ vanishes. In this case, the physically interesting solutions of the dispersion relation are purely real, and there is no resonant "Landau damping." Equation (58) can be explicitly calculated by integrating over $v_{\parallel}$, which is the same as integrating over the pitch angle of the particle. We put $v=\omega/ck$:

$$\mathrm{Re}J_{\perp\perp}=\frac{1}{k^2}\int_1^\infty f(\gamma)d\gamma\left\{(\gamma^2-1)^{1/2}\left[8\gamma^2\nu^2-\tfrac{16}{3}(\gamma^2-1)\right]-4\gamma\nu(\gamma^2-1-\gamma^2\nu^2)\ln\left|\frac{(\gamma^2-1)^{1/2}-\gamma\nu}{(\gamma^2-1)^{1/2}+\gamma\nu}\right|\right\}. \quad (59)$$

It is possible to find approximate expressions for the extreme cases of cold ($kT \ll mc^2$) and ultrarelativistic ($kT \gg mc^2$) matter. In the former case $\gamma=1+t$ is near unity; expanding around this value gives

$$\mathrm{Re}J_{\perp\perp}=\frac{1}{k^2}\int_0^\infty dt f(1+t)\,\frac{64\sqrt{2}}{15\nu^2}\,t^{5/2}, \quad (60a)$$

$$\mathrm{Re}J_{\perp\perp}=\frac{1}{k^2}\frac{nm}{2\pi}\frac{1}{\xi^2}\frac{c^2k^2}{\omega^2}\quad \text{(Jüttner-Synge)}. \quad (60b)$$

In the second case, we can expand (59) in γ^{-1}; this result is the ultrarelativistic approximation:

$$\mathrm{Re}J_{\perp\perp}=\frac{1}{k^2}\int_1^\infty \gamma^3 f(\gamma)d\gamma\left[8\gamma^2-\tfrac{16}{3}-4\nu(1-\nu^2)\ln\left|\frac{1-\nu}{1+\nu}\right|\right], \quad (61a)$$

$$\mathrm{Re}J_{\perp\perp}=\frac{1}{k^2}\frac{3nm}{4\pi}\frac{1}{\xi}\left[8\nu^2-\tfrac{16}{3}-4\nu(1-\nu^2)\ln\left|\frac{1-\nu}{1+\nu}\right|\right]\quad \text{(Jüttner-Synge)}. \quad (61b)$$

In the case of cold matter, the dispersion relation is obtained by substituting (60) in (48). It is seen that $J_{\perp\perp}$ is $O(\xi^{-2})$ and then negligible with respect to the pressure term on the right-hand side of (48), which is $O(\xi^{-1})$. We are left, for a Jüttner-Synge distribution, with

$$\omega_q^2=k^2c^2+8\,\frac{\omega_G^2}{\xi}. \quad (62)$$

This represents gravitational waves propagating in the presence of matter. The eigenmodes have a cutoff at low frequencies: No wave can propagate with a frequency lower than $(8/\xi)^{1/2}\omega_G$. This frequency, however, is so low that our two-time scale approximation breaks down very much earlier. Equation (62) describes essentially a correction to the vacuum propagation in the frequency range $\omega_q \gg \omega_G$ which is of thermal origin and would vanish in a cold universe. Furthermore, the dispersive phase velocity is supraluminous, so that gravitational waves do not suffer Landau damping, at least when the distribution functions are approximately Maxwellian. The complete instantaneous frequency including trailing effects is given by (51):

$$\omega(t)=ck\left(1+\frac{4}{\xi}\frac{\omega_G^2}{c^2k^2}\right)-\frac{1}{2k}\left(5\ddot{S}+3\,\frac{\dot{S}^2}{S}\right). \quad (63)$$

In the case of ultrarelativistic matter, the dispersion equation (48) can be written [using (61) in (48)], for a Jüttner-Synge gas, as

$$(\omega^2-c^2k^2)\left\{1+\frac{3}{4}\frac{\omega_G^2}{\xi c^2k^2}\left[8\frac{\omega^2}{c^2k^2}-\frac{16}{3}+4\frac{\omega}{ck}\left(\frac{\omega^2}{c^2k^2}-1\right)\ln\left|\frac{(\omega/ck)-1}{(\omega/ck)+1}\right|\right]\right\}=8\,\frac{\omega_G^2}{\xi}. \quad (64)$$

This equation bears similarities with the equation for a hot photon gas derived by Chester.[13] Actually Chester's equation (2.3) approaches

$$\omega^2=c^2k^2-\frac{144\pi}{5}\frac{GP}{c^2}.$$

Not only is there a difference in the coefficient, but Chester's result gives a subluminous phase velocity that gives rise to a Landau damping [his equations (2.1), (2.2), (2.3)]. Our result, on the contrary, gives rise to supraluminous phase velocities, and consequently to undamped waves. We think that Chester's result is in error. Indeed, his equation (2.3) gives for the small-wavelength gravitational waves a group velocity

$$v_g=\frac{\partial\omega}{\partial k}=c\left(1+\frac{72\pi P}{5c^2k^2}\right)$$

which is faster than light. In the short-wavelength limit, $k\to\infty$, our Eq. (64) gives essentially the same dispersion relation as in the cold-matter case, namely

$$\omega^2=c^2k^2+8\,\frac{\omega_G{}^2}{\xi}\,. \tag{65}$$

Our result (64) can be solved in principle numerically. It is possible, however, to see that it gives supraluminous phase velocities and subluminous group velocities in the whole wave-vector range. Indeed, Eq. (64) can be arranged to the form

$$x^2=G(\nu)\,,$$

$$G(\nu)=\frac{32}{3}\,\frac{1}{\nu^2-1}-\left[4\nu(\nu^2-1)\ln\left|\frac{\nu-1}{\nu+1}\right|+8\nu^2-\frac{16}{3}\right],$$

$$x^2=\frac{k^2}{k_G{}^2}\,,\quad k_G{}^2=\frac{3}{4\xi}\,\frac{\omega_G{}^2}{c^2}\,,\quad \nu=\frac{\omega}{ck}\,,$$

which relates the wave vector k/k_G to the phase velocity ω/ck. $G(\nu)$ is a universal function which is plotted in Fig. 4. The intersection with the line of ordinate x^2 gives a phase velocity between c and ∞, so that no Landau damping appears. For small wavelengths ($k\to\infty$) we recover the approximate solution (65) and for large wavelengths ($k\to 0$) we get approximately

$$\omega^2=6\,\frac{\omega_G{}^2}{\xi}\,.$$

The dispersion curve is shown in Fig. 5. We note the existence of a cutoff frequency at $\sqrt{6}\,\omega_G\,\xi^{-1/2}$ which is of the order of the Hubble time scale for a hot universe. Our calculations become invalid in that region. Note, however, that the collisionless approximation was used. At very high densities this might not be good enough, and a hydrodynamic viscous fluid model would be more appropriate.

APPENDIX A: EXPRESSION OF THE OPERATOR $\mathfrak{D}_\lambda^{\lambda'}$ IN TERMS OF THE VARIABLES H_λ

In this appendix each element $\mathfrak{D}_\lambda^{\lambda'}$ is given in terms of the components $\mathfrak{D}_{\mu\nu}^{\alpha\beta}$:

$$(\mathfrak{D}_\lambda^{\lambda'})=\begin{bmatrix}\mathfrak{D}_1 & & & \\ & \mathfrak{D}_2 & & \\ & & \mathfrak{D}_3 & \\ & & & \mathfrak{D}_3\end{bmatrix}.$$

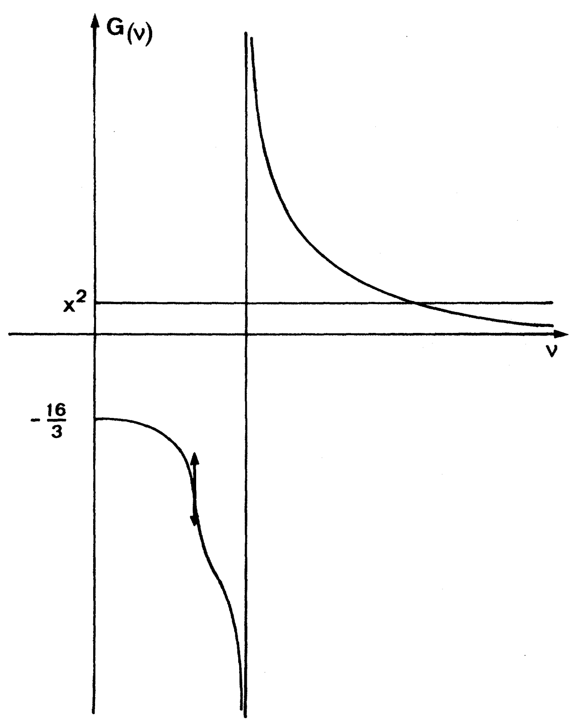

FIG. 4.

$$G(\nu)=\frac{32}{3}\,\frac{1}{\nu^2-1}-\left[4\nu(\nu^2-1)\ln\left|\frac{\nu-1}{\nu+1}\right|+8\nu^2-\frac{16}{3}\right]$$

as a function of ν: $\nu\to 0$: $G(\nu)\simeq-\frac{16}{3}-\frac{80}{3}\nu^2$; $\nu\to 1$: $G(\nu)\simeq 32/3(\nu^2-1)$; $\nu\to\infty$: $G(\nu)\simeq 8/\nu^2$.

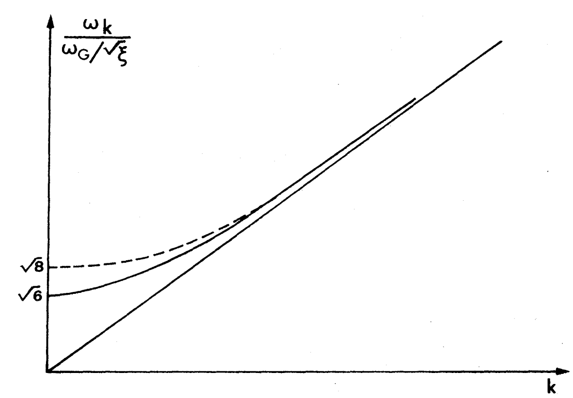

FIG. 5. Dispersion curves for gravitational waves in the hot universe $\xi\ll 1$ (solid line) and cold universe $\xi\gg 1$ (dashed line). The calculation is valid for $\omega\gg\omega_G$.

λ is the column index and λ' is the row index; they run from 1 to 10. $\mathfrak{D}_1$ and $\mathfrak{D}_3$ are 2×2 matrices,

$$\mathfrak{D}_1=\begin{pmatrix} 2\mathfrak{D}_{23}^{23} & 0 \\ 0 & \mathfrak{D}_{22}^{22}-\mathfrak{D}_{33}^{33}\end{pmatrix}, \quad \mathfrak{D}_3=\begin{pmatrix} \mathfrak{D}_{12}^{12} & \mathfrak{D}_{12}^{24} \\ \mathfrak{D}_{24}^{12} & \mathfrak{D}_{24}^{24}\end{pmatrix},$$

and $\mathfrak{D}_2$ is a 4×4 matrix,

$$\mathfrak{D}_2=\begin{bmatrix} \frac{\mathfrak{D}_{22}^{11}}{3}-\frac{\mathfrak{D}_{22}^{22}+\mathfrak{D}_{22}^{33}}{6} & \frac{\mathfrak{D}_{22}^{11}}{3}+\frac{\mathfrak{D}_{22}^{22}+\mathfrak{D}_{22}^{33}}{3} & \mathfrak{D}_{22}^{14} & \mathfrak{D}_{22}^{44} \\ \frac{\mathfrak{D}_{11}^{11}}{3}-\frac{\mathfrak{D}_{11}^{22}+\mathfrak{D}_{11}^{33}}{6} & \frac{\mathfrak{D}_{11}^{11}}{3}+\frac{\mathfrak{D}_{11}^{22}+\mathfrak{D}_{11}^{33}}{3} & \mathfrak{D}_{11}^{14} & \mathfrak{D}_{11}^{44} \\ \frac{\mathfrak{D}_{14}^{11}}{3}+\frac{\mathfrak{D}_{14}^{22}+\mathfrak{D}_{14}^{33}}{6} & \frac{\mathfrak{D}_{14}^{11}}{3}+\frac{\mathfrak{D}_{14}^{22}+\mathfrak{D}_{14}^{33}}{3} & \mathfrak{D}_{14}^{14} & \mathfrak{D}_{14}^{44} \\ \frac{\mathfrak{D}_{14}^{11}}{3}+\frac{\mathfrak{D}_{44}^{22}+\mathfrak{D}_{44}^{33}}{6} & \frac{\mathfrak{D}_{14}^{11}}{3}+\frac{\mathfrak{D}_{44}^{22}+\mathfrak{D}_{44}^{33}}{3} & \mathfrak{D}_{44}^{14} & \mathfrak{D}_{44}^{44} \end{bmatrix}.$$

APPENDIX B

Equation (12) can be written explicitly to any order, using Eq. (16) and the fact that $\Sigma_{\mu\nu}$ is $O(\epsilon^2)$. The zeroth-order approximation yields the vacuum system of equations:

$$\Box^{(0)}h_{\mu\nu}^{(0)}=0,\quad I_{\mu}^{(0)}=0. \tag{B1}$$

Here we introduce the flat-space wave operator:

$$\Box^{(0)}=\partial_4^{(0)^2}-\frac{1}{S^2(t_L)}\Delta.$$

We stress that, to this order of approximation, the scale factor must be regarded as a constant. (B1) gives the usual gravitational-wave solutions:

$$h_{23}^{(0)}\neq 0,\quad h_{22}^{(0)}=-h_{23}^{(0)}\neq 0,\quad h_{\alpha\beta}^{(0)}=0 \text{ for the others.} \tag{B2}$$

Continuing as in Sec. III, the next term in the expansion gives a system that determines $h_{\mu\nu}^{(1)}$ as a function of the short time and $h_{\mu\nu}^{(0)}$ as a function of the long time:

$$\Box^{(0)}h_{\mu\nu}^{(1)}=0, \tag{B3}$$

$$(\text{some operator acting on } t_L)\times h_{\mu\nu}^{(0)}=0. \tag{B4}$$

Using the fact that most of the $h_{\alpha\beta}^{(0)}$ vanish [see (B2)] and depend on x and t only, we get

$$\begin{aligned} I_4^{(1)}&=\partial_4^{(0)}h_4^{4(1)}+\partial_i h_4^{i(1)}-\tfrac{1}{2}\partial_4^{(0)}h_\lambda^{\lambda(1)}=0,\\ I_j^{(1)}&=\partial_4^{(0)}h_j^{4(1)}+\partial_i h_j^{i(1)}-\tfrac{1}{2}\partial_j h_\lambda^{\lambda(1)}=0.\end{aligned} \tag{B5}$$

(B4) describes the trailing effects on gravitational waves, whereas (B3) and (B5) give the usual flat-space wave operator and flat-space gauge condition. The solutions to first order for $h^{(1)}$ are then exactly the same as those for zeroth order for $h^{(0)}$. Let us now go to the second order in ϵ:

$$\Box^{(0)}h_{\mu\nu}^{(2)}=\Sigma_{\mu\nu}, \tag{B6}$$

$$(\text{some operator acting on } t_L)\times h_{\mu\nu}^{(1)}=0. \tag{B7}$$

Proceeding in the same way as for (B5), we obtain

$$\begin{aligned} I_4^{(2)}&=\partial_4^{(0)}h_4^{4(2)}+\partial_i h_4^{i(2)}-\tfrac{1}{2}\partial_4^{(0)}h_\lambda^{\lambda(2)},\\ I_j^{(2)}&=\partial_4^{(0)}h_j^{4(2)}+\partial_i h_j^{i(2)}-\tfrac{1}{2}\partial_j h_\lambda^{\lambda(2)}.\end{aligned} \tag{B8}$$

Equation (B6) can be written as

$$\Box^{(0)}(h_\mu^{\nu(2)}-\tfrac{1}{2}\delta_\mu^\nu h_\lambda^{\lambda(2)})=2\chi\int\frac{d^3u}{u_4}u_\mu u^\nu(Z+\tfrac{1}{2}h_\lambda^\lambda N). \tag{B9}$$

The second member of this equation is easily recognized as twice the perturbation $K_{\mu\nu}$ of the energy-momentum tensor. Now, the total energy-momentum tensor is conservative, a property which is indeed satisfied by (3) provided that N is a solution of the Liouville equation (1) [for a proof see Israel[24]]. This property is true for both the background energy-momentum tensor, with the background covariant derivation, and for the total energy-momentum tensor, with the complete covariant derivation. Let ∇_μ and $T^{\mu\nu}$ be the quantities referring to the background and $X_{\mu\nu}^{\rho}$ the perturbation of the Christoffel symbol; the above properties are written as

$$\begin{aligned} &\nabla_\mu T^{\mu\nu}=0,\\ &\nabla_\mu(T^{\mu\nu}+K^{\mu\nu})+X_{\mu\alpha}^{\mu}(T^{\alpha\nu}+K^{\alpha\nu})+X_{\mu\alpha}^{\nu}(T^{\mu\alpha}+K^{\mu\alpha})=0.\end{aligned} \tag{B10}$$

By subtracting the first from the second, and keeping terms linear in the perturbation, we are left with

$$\nabla_\mu K^{\mu\nu}+X_{\mu\alpha}^{\mu}T^{\alpha\nu}+X_{\mu\alpha}^{\nu}T^{\mu\alpha}=0. \tag{B11}$$

We now calculate this quantity to the second order in the time-scale ratio ϵ, keeping in mind that $K^{\mu\nu}$ and $T^{\mu\nu}$ are of the second order in ϵ. Then, we must retain that part of ∇_μ and $X_{\mu\alpha}^{\nu}$ which is zeroth order in ϵ. This yields

$$\nabla_\mu^{(0)}=\partial_\mu^{(0)}, \tag{B12}$$

and, using expressions (15) and (B2),

$$\begin{aligned} X^{(0)4}{}_{44}&=\tfrac{1}{2}\partial_4^{(0)}h_{44}^{(0)}=0,\\ X^{(0)4}{}_{i4}&=\tfrac{1}{2}\partial_i h_{44}^{(0)}=0,\\ X^{(0)4}{}_{ij}&=\tfrac{1}{2}(\partial_i h_{4j}^{(0)}+\partial_j h_{4i}^{(0)}-\partial_4^{(0)}h_{ij}^{(0)})\\ &=-\tfrac{1}{2}\partial_4^{(0)}h_{ij}^{(0)}.\end{aligned} \tag{B13}$$

So, Eq. (B11) to the second order in ϵ may be finally simplified. We used the fact that $T_{\mu\nu}$ is diagonal and expressions (B2) once more:

$$\begin{aligned} &\partial_\mu^{(0)}K^{\mu i}+X^{(0)4}_{4i}T^{ jj}+X^{(0)i}_{\mu\alpha}T^{\mu\alpha}=0,\\ &\partial_\mu^{(0)}K^{\mu 4}+X^{(0)\mu}_{\mu 4}T^{44}+X^{(0)4}_{\mu\alpha}T^{\mu\alpha}=0.\end{aligned} \tag{B14}$$

But using (B13) and (B2) we are finally left with

$$\partial_\mu^{(0)} K^{\mu i} = 0, \qquad \partial_\mu^{(0)} K^{\mu 4} = 0. \tag{B15}$$

Operating with $\partial_\mu^{(0)}$ on Eq. (B9) gives

$$\partial_\nu^{(0)} \Box^{(0)} (h_\mu^{\nu(2)} - \tfrac{1}{2}\delta_\mu^\nu h_\lambda^{\lambda(2)}) = 0. \tag{B16}$$

$\partial_\mu^{(0)}$ commutes with $\Box^{(0)}$ and (B16) is reduced to

$$\Box^{(0)} I_\mu^{(2)} = 0. \tag{B17}$$

We can choose the conditions on some timelike initial hypersurface such that $I_\mu^{(2)} = 0$ on this surface. The hyperbolic character of the operator $\Box^{(0)}$ ensures that $I_\mu^{(2)}$ vanishes everywhere.

This proves that our solution is a solution of the full system of linearized Einstein equations (6), (12).

This would be due to the fact that up to the second order the universe can be considered to be locally flat, and curvature is locally gauge invariant in the weak-field linear theory. This has already been stressed by Isaacson.[5]

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