Energy-momentum tensor near an evaporating black hole

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We calculate the vacuum expectation value, $T_{\mu\nu}$, of the energy-momentum tensor of a massless scalar field in a general two-dimensional spacetime and evaluate it in a two-dimensional model of gravitational collapse. In two dimensions, quantum radiation production is incompatible with a conserved and traceless $T_{\mu\nu}$. We therefore resolve an ambiguity in our expression for $T_{\mu\nu}$, regularized by a geodesic point-separation procedure, by demanding conservation but allowing a trace. In the collapse model, the results support that picture of black-hole evaporation in which pairs of particles are created outside the horizon (and not entirely in the collapsing matter), one of which carries negative energy into the future horizon of the black hole, while the other contributes to the thermal flux at infinity.

It is now generally believed, on the basis of an argument due to Hawking,[1] that the gravitational field of a collapsing object will induce the quantum creation of particles, so that the object radiates with a thermal spectrum at a temperature inversely proportional to the mass of the object.

Most calculations of this effect have examined the behavior of the quantum fields only near infinity. Consequently, it has never been clear precisely where the radiation is being created, and what is happening near the horizon of the "black hole." A knowledge of the energy-momentum tensor of the quantum field in the vicinity of the object would help in clarifying the details of the creation process. Unfortunately, this quantity is always formally divergent, and the meaningful physical component must be extracted by a regularization procedure. Such procedures always contain ambiguities which must be resolved by the application of additional criteria, such as physical reasonableness.

In addition to the problems of regularization, mathematical complexities have prevented detailed discussion of quantum field theory near the surface of a black hole. In what follows, the latter problem is circumvented by studying a simple two-dimensional model[2] of the black-hole-formation process. This model has the advantage of possessing a conformally flat metric, so that the mode functions for the quantum field can be explicitly evaluated everywhere, while retaining the essential features of the Hawking evaporation process. The highly plausible character of the "renormalized" energy-momentum tensor which emerges for this simple model encourages the hope that the qualitative features of the full four-dimensional collapse are contained in this treatment.

The regularization procedure which we employ has already been applied to an evaluation of the energy-momentum tensor near moving mirrors in two-dimensional flat spacetimes by Fulling and Davies,[3,4] a study inspired by the striking formal resemblance between the black-hole-formation process and the accelerating mirror system.[5] The two field operators occuring in a term of the energy-momentum tensor $T_{\mu\nu}(x)$ are evaluated at points separated along a geodesic through x, and derivatives of the field are parallel propagated along the geodesic to the two points. Here we generalize the technique to an arbitrary two-dimensional spacetime and apply it to the simplified collapse model.

The metric for any two-dimensional spacetime is conformally flat and may be written as

$$ds^2 = C(u,v)du\,dv\ , \tag{1}$$

where u, v are null coordinates. [Any other null coordinates $\bar{u}, \bar{v}$ are related to these by $\bar{u} = \bar{u}(u)$, $\bar{v} = \bar{v}(v)$.] We examine the massless scalar field ϕ, which for this metric obeys the simple equation

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = 0\ . \tag{2}$$

The solutions of this equation are

$$\phi = f(u) + g(v)\ , \tag{3}$$

where $f(u)$ and $g(v)$ are, in general, arbitrary functions, restricted only by the spatial boundary conditions.

We wish to calculate the expectation value of the operator

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} \tag{4}$$

in some quantum state. In expanding the operator ϕ in normal modes, we assume (as has always been found to be the case) that there exist null coordinates $\bar{u},\bar{v}$ such that the ingoing and outgoing parts of a normal mode go, respectively, as

$$e^{-i\omega\bar{v}}/(4\pi|\omega|)^{1/2}\,,\quad e^{-i\omega\bar{u}}/(4\pi|\omega|)^{1/2}\,. \tag{5}$$

The state which we have examined is the one annihilated by the operators associated with the modes $\omega>0$ in the field expansion.

If the geometry is initially static or has an asymptotically flat region at infinity, this state is made unique by the requirement that the modes reduce to ordinary plane waves in that region. This state is then that in which no particles are present initially (before the collapse begins, in the problem at hand), and is conventionally called the "vacuum" or "in-vacuum" state, $|0\rangle$.

The expectation value of $T_{\mu\nu}$ in this state (also designated by $T_{\mu\nu}$) is calculated by an expanison of the expression

$$\langle 0|\phi_{,\alpha}(x^+)\phi_{,\beta}(x^-)|0\rangle(e_\mu^{+\alpha}e_\nu^{-\beta}-\tfrac{1}{2}g_{\mu\nu}e_\sigma^{+\alpha}e^{-\sigma\beta}) \tag{6}$$

in powers of ϵ. Here the points x^+,x^- lie on a geodesic through the point x of interest, each at a proper distance of ϵ, but in opposite directions from x, and $e_\mu^{\pm\alpha}$ are the matrices of parallel transport along the geodesic from x to $x^\pm$. The expectation value of $T_{\mu\nu}$ is then defined as the limit as $\epsilon\to 0$ after the pole terms have been eliminated. The calculation, which proceeds along the lines of Ref. 3, yields

$$T_{\mu\nu}=-\left(\frac{\epsilon^{-2}}{4\pi t_\alpha t^\alpha}+\frac{R}{24\pi}\right)\left(\frac{t_\mu t_\nu}{t_\alpha t^\alpha}-\tfrac{1}{2}g_{\mu\nu}\right)+\theta_{\mu\nu}+O(\epsilon)\,, \tag{7}$$

where t_μ is a tangent vector to the geodesic at the point x, R is the curvature scalar, and the tensor $\theta_{\mu\nu}$ as evaluated in the special $\bar{u},\bar{v}$ coordinates has the components

$$\begin{aligned}\theta_{\bar{u}\bar{u}}&=-(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{u}\bar{u}}\,,\\ \theta_{\bar{v}\bar{v}}&=-(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{v}\bar{v}}\,,\\ \theta_{\bar{u}\bar{v}}&=\theta_{\bar{v}\bar{u}}=0\,.\end{aligned} \tag{8}$$

In systems where the spectrum of ω is discrete (e.g., a closed universe), $\theta_{\mu\nu}$ contains an additional "Casimir effect" term, whose evaluation is straightforward.

Equation (7) as it stands is unsatisfactory. In particular, it contains terms which depend on the direction of point separation. It is hard to understand how a physical result could depend on such an arbitrary additional vector field. It appears that such terms, which evidently would arise in any point-separation procedure, must be discarded. The expression as it stands is traceless, and will remain so if the entire first term is discarded. However, in neither case does it obey the conservation law

$$\nabla_\nu T^{\mu\nu}=0\,. \tag{9}$$

On the other hand, by discarding only the terms proportional to $t_\mu t_\nu$ one does obtain a conserved tensor, but one which is no longer traceless.

Although conformal invariance of a theory formally implies tracelessness of $T_{\mu\nu}$, the appearance of a nonvanishing trace is not entirely unexpected. Already in Ref. 3 one naive consequence of conformal invariance, namely that the vacuum expectation value of $T_{\mu\nu}$ be the same in all conformally related spacetimes, was observed to be lost. Other regularization techniques (in different contexts) have also led to a breakdown of conformal invariance in the quantum theory.[6,7]

In two dimensions, quite general arguments imply that conservation, zero trace, and particle production are incompatible. A traceless conserved energy-momentum tensor obeys the equations

$$T_{uu,v}=0\,,\quad T_{vv,u}=0\,, \tag{10}$$

which imply that the energy flux along any null ray must be constant; i.e., radiation cannot be created or destroyed.

In what follows, we therefore adopt the following expression for $T_{\mu\nu}$:

$$T_{\mu\nu}=\frac{\pm g_{\mu\nu}}{8\pi\epsilon^2}+\theta_{\mu\nu}+\frac{R}{48\pi}g_{\mu\nu}\,. \tag{11}$$

(We have normalized t^μ to $t_\alpha t^\alpha=\pm 1$.) The leading divergent term is present even in flat spacetime and may be regarded as a renormalization of the cosmological constant. The final "renormalized" expression is thus

$$T_{\mu\nu}=\theta_{\mu\nu}+\frac{R}{48\pi}g_{\mu\nu}\,. \tag{12}$$

The term $\theta_{\mu\nu}$, evaluated in the special coordinate system in Eq. (8), is not expressible in terms of local geometrical quantities. This is to be expected, as the definition of the state of the system is a global one. In this soluble model, the effect on $T_{\mu\nu}(x)$ of the geometry elsewhere has been encoded in the special choice of coordinates $\bar{u},\bar{v}$.

When $C(\bar{u},\bar{v})$ is a function of $\bar{u}$ alone, the space is flat and the ambiguous finite terms in Eq. (7) vanish identically. $T_{\mu\nu}$ is then in general both traceless and conserved. This situation reduces to that treated in Ref. 3, in which radiation can only be produced at the surface of the moving mirror, where the conservation equations (9) break down.

The expression in Eq. (12) will now be evaluated in the model of collapse described in Ref. 2. In this model, the metric is obtained by eliminating the angular coordinates from that of a four-dimensional shell of matter which collapses at high velocity. Inside the shell spacetime is flat, whereas outside the shell the metric takes the Schwarzschild form

$$ds^2 = (1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2 . \qquad (13)$$

There exist three useful sets of null coordinates for this problem. In the first, given outside the shell by

$$u = t - r^* ,$$
$$v = t + r^* , \qquad (14)$$
$$r^* = r + 2M \ln(r/2M - 1) ,$$

the external metric takes the simple form

$$ds^2 = (1-2M/r)du\,dv , \qquad (15)$$

with r an implicit function of u, v by Eqs: (14).

The second set, U, V, is defined so that the interior metric takes the simple form

$$ds^2 = dU dV . \qquad (16)$$

The relation between the u, v and the U, V coordinates is obtained in Ref. 2 by demanding continuity of the metric across the shell in either coordinate system.

Finally, the coordinates $\bar{u}, \bar{v}$ which are to appear in the mode solutions (5) and in the determination [Eq. (8)] of the energy-momentum tensor are obtained as follows. To ensure that the state considered is the usual initial vacuum, the modes are defined to have the form $e^{-i\omega v}$ near $\mathscr{I}^-$. ($\mathscr{I}^-$ and $\mathscr{I}^+$ are past and future null infinity, respectively. This gives the relation $\bar{v} = v$ everywhere. Using the relation between V and v given in Ref. 2, one obtains

$$\bar{v} = (1-2M/R)^{-1/2} V , \qquad (17)$$

where R is the radius of the static shell before it begins to collapse. This relation holds only for advanced times preceding the beginning of collapse, but the assumption that the shell collapses quickly makes this sufficient. The reflection boundary condition at $r \equiv (V-U)/2 = 0$ implies that the modes take the form

$$\exp[-i\omega(1-2M/R)^{-1/2}V] - \exp[-i\omega(1-2M/R)^{-1/2}U] . \qquad (18)$$

It follows that $\bar{u} = (1-2M/R)^{-1/2}\, U$ for all retarded times which will be of relevance for the calculation of $T_{\mu\nu}$ outside the collapsing shell.

The relation between u and U derived in Ref. 2 implies the following relations between $\bar{u}$ and u. For retarded times before the collapse has begun, one has $\bar{u} = u$. For retarded times u long after the collapse has begun, one obtains

$$u = -4M \ln(A - \bar{u}) + B(\bar{u}) , \qquad (19)$$

where $B(\bar{u})$ is a slowly varying function of $\bar{u}$ whose exact form depends on the nature of the collapse process and does not affect the final result, while $\bar{u} = A$ is the equation for the future horizon.

The above relations lead to an expression for the external metric in $\bar{u}, \bar{v}$ coordinates and to values for a renormalized $T_{\mu\nu}$. For retarded times $\bar{u}$ before the onset of collapse, one obtains

$$ds^2 = (1-2M/r)d\bar{u}\,d\bar{v} , \qquad (20)$$

that is, the conformal factor, $C(\bar{u}, \bar{v})$, to be used in Eq. (8) is

$$C = 1 - 2M/r \qquad (21)$$

in the external region of spacetime. The values of $T_{\mu\nu}$ in this region expressed in u, v and in t, r coordinates are

$$T_{uu} = (24\pi)^{-1}\left(\frac{3M^2}{2r^4} - \frac{M}{r^3}\right),$$
$$T_{uv} = T_{vu} = (24\pi)^{-1}\left(\frac{2M^2}{r^4} - \frac{M}{r^3}\right),$$
$$T_{vv} = (24\pi)^{-1}\left(\frac{3M^2}{2r^4} - \frac{M}{r^3}\right), \qquad (22)$$
$$T_{tt} = (24\pi)^{-1}\left(\frac{7M^2}{r^4} - \frac{4M}{r^3}\right),$$
$$T_{tr} = T_{rt} = 0 ,$$
$$T_{rr} = -(24\pi)^{-1}(1-2M/r)^{-2}\frac{M^2}{r^4} .$$

For retarded times, u, long after the collapse, the external conformal factor in $\bar{u}, \bar{v}$ coordinates takes the form

$$C(\bar{u}, \bar{v}) = \left(1 - \frac{2M}{r}\right)\left[\frac{4M}{A - \bar{u}} + O(1)\right], \qquad (23)$$

where $O(1)$ are terms of order unity in $\bar{u}$. Evaluating $T_{\mu\nu}$ outside the shell, transforming to u, v coordinates, and neglecting terms which die off for large values of u or t, one obtains

$$T_{uu} = (24\pi)^{-1}\left(\frac{3M^2}{2r^4} - \frac{M}{r^3} + \frac{1}{32M^2}\right)$$
$$= (768\pi M^2)^{-1}\left(1 - \frac{2M}{r}\right)^2\left(1 + \frac{4M}{r} + \frac{12M^2}{r^2}\right), \qquad (24)$$

with T_{vv} and T_{uv} remaining as in Eq. (22).

Comparison with Eq. (22) reveals that the effect of collapse is to add a constant term to T_{uu}, which appears at large r as a flux of energy (defined as in Ref. 2 by using the timelike Killing vector field outside the shell) of magnitude $(768\pi M^2)^{-1}$. This is just the energy flux one would expect on the basis of Hawking's arguments (Ref. 1) as applied to this model. Note that we have obtained it without appealing to Bogolubov transformations or backward ray tracing.

From Eqs. (24) and (22), one finds that the flux of energy is given by two components. Near infinity it is dominated by an outward null flux of energy (given by T_{uu}). Near the horizon, however, it is a flux of negative energy going into the horizon of the black hole (represented by T_{vv} for r near $2M$). This negative energy flux would presumably cause the area of the horizon to shrink at a rate consistent with the energy flux observed at infinity. [Note that negative energy densities are possible in quantum field theories even in flat spacetimes (see Ref. 3).]

The energy-momentum tensor obtained here is finite everywhere outside the collapsing shell when expressed in coordinates which are regular across the horizon. Therefore, the constant radiation flux at infinity does not imply [even given Eq. (9)] infinite energy densities or fluxes near the horizon, as one might fear. Therefore our results are entirely consistent with Hawking's speculation (Ref. 1) that the process can be described as the creation of "particles" in pairs near the horizon, with one carrying positive energy to infinity and the other carrying negative energy into the black hole. It also supports the result in Refs. 2 and 8 obtained in analyzing the behavior of freely falling detectors near the horizon.

An alternative view of the Hawking process is that the particles which reach $\mathscr{I}^+$ must have been created entirely inside the collapsing matter. Although the decomposition of $T_{\mu\nu}$ into a static vacuum polarization and a constant outward flux can be construed to support that picture, we feel that the form of the total energy-momentum tensor near the horizon casts doubt on the physical relevance of that description. Of course both descriptions are only figures of speech since particles are not well defined in the regions of high curvature near the horizon.

In conclusion, we emphasize that the energy-momentum tensor has been *calculated* by means of a particular method of regularization for space-time about a model collapsing star, including the regions of high curvature. Althoug certain ambiguities in the point-separation technique of regularization remain, and although this calculation has been for a very simple two-dimensional model of collapse, we believe that the result greatly increases one's understanding of, and confidence in, the surprising effect discovered by Hawking.

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