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Unified theory of gravitation, electromagnetism, and the Yang-Mills field

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The recent modification and extension of Einstein's nonsymmetric unified field theory for gravitation and electromagnetism is generalized to include the Yang-Mills field theory. The generalization consists in assuming that the components of the linear connection and of the fundamental tensor are not ordinary c numbers but are matrices related to some unitary symmetry. As an example we consider the SU(2) case. The theory is applied to the gauge-covariant formulation of electrically and isotopically charged spin-1/2 field theories.

I. INTRODUCTION

The extended nonsymmetric unified field theory is characterized by a tensor field g_{ik} and a linear connection Γ^i_{kj},[1,2] which can be decomposed according to[3]

$$g_{ik}=g_{(ik)}+pF_{ik}, \tag{1.1}$$

$$\Gamma^i_{kj}=\Gamma^{*i}_{kj}-\frac{2}{p}\delta^i_k A_j, \tag{1.2}$$

where A_k is defined by

$$A_k=-\tfrac{1}{3}p\Gamma^a_{[ak]}. \tag{1.3}$$

p is a universal constant, the vanishing of which leads to the Einstein-Maxwell theory with $g_{(ik)}$ and Γ^{*i}_{kj} as the metric and connection of general relativity, and F_{ik} and A_k as the field tensor and potential of Maxwell's electrodynamics, respectively. The field equations for g_{ik} and Γ^i_{kj} are derived from a variational principle with the Lagrangian

$$\mathcal{L}=g'^{ab}R_{ab}+\frac{1}{p^2}g'^{[ab]}g_{[ab]}+16\pi J^a A_a+8\pi L, \tag{1.4}$$

where R_{ik} is the contracted curvature tensor

$$R_{ik}=\Gamma^s_{ik,s}-\Gamma^s_{is,k}-\Gamma^t_{sk}\Gamma^s_{it}+\Gamma^t_{st}\Gamma^s_{ik}, \tag{1.5}$$

and g'^{ik} is the tensor density

$$g'^{ki}=wg^{ki}, \tag{1.6a}$$

$$w=(-\det g_{ik})^{1/2}, \tag{1.6b}$$

$$g^{ia}g_{ja}=\delta^i_j. \tag{1.6c}$$

J^i is the electric current density and L is a density containing matter fields and the metric tensor $g_{(ik)}$. The theory is invariant under the ordinary electromagnetic Abelian gauge transformation

$$\overline{A}_i\to A_i-\tfrac{1}{2}p\lambda_{,i}, \tag{1.7}$$

where λ is an arbitrary function. When applied to the decomposition (1.2), the gauge transformation (1.7) is equivalent to Einstein's λ transformation of the linear connection

$$\overline{\Gamma}^i_{kj}=\Gamma^i_{kj}+\delta^i_k\lambda_{,j}. \tag{1.8}$$

The application of the unified field to the gauge-covariant formulation of Dirac's equation[4] suggests fixing the universal constant p as

$$p=-2i\hbar/e \quad (c=G=1),\qquad |p|=3.8\times10^{-32}\text{ cm}; \tag{1.9}$$

with this value, the deviations from the Einstein-Maxwell theory must be expected to become significant only for phenomena dominated by quantum field effects. The only "classically" detectable deviation found so far is the nonexistence of magnetic monopoles proved by Boal and Moffat,[5] a fact in good agreement with the asymmetry between electricity and magnetism found in nature but not reflected in Maxwell's theory.

The close similarity between Einstein's gravitational field equations and the Yang-Mills field equations is well known, a correspondence arising from the fact that both are self-coupled gauge theories.[6] It is therefore natural to investigate whether it is possible to construct a unified field theory containing both. In the following we are going to show that this is indeed possible, the unified field theory so obtained having a principle of correspondence to the Einstein-Maxwell theory and the Yang-Mills theory.

II. THE YANG-MILLS FIELD EQUATIONS

For the sake of completeness and in order to fix the notation we shall give a brief review of the Yang-Mills field equations.[7] Let χ be a two-component wave function describing a field with isotopic spin $\frac{1}{2}$. Under an isotopic gauge transformation it transforms as

$$\psi=S\psi', \tag{2.1}$$

where S is a 2×2 unitary matrix. The isotopic gauge invariance of the field equations for ψ is secured by four Hermitian traceless matrices B_i, $i=0,1,\ldots,3$ entering the gauge-covariant derivative

$$\partial_i - i\kappa B_i \tag{2.2}$$

and transforming under the isotopic gauge transformation (2.1) as

$$B_i' = S^{-1}B_i S + \frac{i}{\kappa}S^{-1}\frac{\partial S}{\partial x^i}\,. \tag{2.3}$$

From the field B_i we define the field

$$E_{ij} = B_{i,j} - B_{j,i} + i\kappa(B_iB_j - B_jB_i), \tag{2.4}$$

which transforms as

$$E_{ij}' = S^{-1}E_{ij}S, \tag{2.5}$$

with the three Pauli matrices and the unit matrix given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},$$

$$\tau_3 \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \quad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{2.6}$$

$$\vec{\tau} = (\tau_1, \tau_2, \tau_3).$$

The four traceless matrices B_i can be uniquely written as a linear combination of τ_1, τ_2, τ_3,

$$B_i = \vec{\tau}\cdot\vec{b}_i\,, \tag{2.7}$$

where arrows denote three-vectors in isotopic space. Similar to (2.7), we have for E_{ik}

$$E_{ik} = \vec{\tau}\cdot\vec{f}_{ik}\,. \tag{2.8}$$

By use of the commutation relations

$$[\vec{\tau}\cdot\vec{a}, \vec{\tau}\cdot\vec{b}] = 2i\vec{\tau}\cdot(\vec{a}\times\vec{b}) \tag{2.9}$$

we obtain the relationship between $\vec{f}_{ik}$ and $\vec{b}_k$:

$$\vec{f}_{ik} = \vec{b}_{i,k} - \vec{b}_{k,i} - 2\kappa\vec{b}_i\times\vec{b}_k\,. \tag{2.10}$$

The field $\vec{b}_i$ satisfies the gauge-invariant Yang-Mills field equations

$$f_{ia,a} + 2\kappa\vec{b}_a\times\vec{f}_{ia} + \vec{j}_i = 0, \tag{2.11}$$

where j_i is the isotopic current of the source fields. From (2.11), we derive the equation of continuity

$$(2\kappa\vec{b}_a\times\vec{f}_{ia} + \vec{j}_i)_{,i} = 0, \tag{2.12}$$

which leads to the interpretation of $2\kappa\vec{b}_a\times \mathrm{f}_{ia}$ as the isotopic current of the Yang-Mills field itself. The whole theory is trivially transferred from the case of isotopic spin to any unitary symmetry.

III. THE UNIFIED THEORY: FREE-FIELD EQUATIONS AND GAUGE SYMMETRIES

The generalization of the nonsymmetric unified field theory presented in Sec. I, which we are now going to investigate, consists in assuming that the components of the linear connection Γ^i_{kj} and of the fundamental tensor g_{ik} are matrices instead of ordinary c numbers. For the sake of clarity, we are going to consider a definite case, namely that of 2×2 matrices, corresponding to isotopic spin [SU(2)], but the whole analysis can immediately be carried over to $n\times n$ matrices corresponding to SU(n). As matrices do not in general commute, the order of the fields in the nonlinear terms now becomes important.

Under the isotopic gauge transformation (2.1), we assume that the fundamental tensor g_{ik} and the linear connection Γ^i_{kj} transform as

$$\bar{g}_{ik} = S^{-1}g_{ik}S\,, \tag{3.1}$$

$$\bar{\Gamma}^i_{kj} = S^{-1}\Gamma^i_{kj}S + \delta^i_k S^{-1}\frac{\partial S}{\partial x^j}\,. \tag{3.2}$$

Corresponding to (1.8), Einstein's λ transformation becomes

$$\bar{\Gamma}^i_{kj} = \Gamma^i_{kj} + \delta^i_k\lambda_{,j}\tau_0. \tag{3.3}$$

To the fundamental tensor g_{ik} we attach the tensor density g'^{ik},

$$g'^{ki} = wg^{ki}\,, \tag{3.4a}$$

$$w = [-\tfrac{1}{2}\mathrm{Tr}(\det g_{ik})]^{1/2}\,, \tag{3.4b}$$

$$g^{ia}g_{ka} = \tau_0\delta^i_k\,, \tag{3.4c}$$

where Tr means that the trace of the matrix should be taken. From (3.1) and (3.4), it follows that under isotopic gauge transformations, g'^{ik} transforms as

$$\bar{g}'^{ik} = S^{-1}g'^{ik}S. \tag{3.5}$$

As a Lagrangian for the free-field equations we shall use

$$\mathfrak{L} = \mathrm{Tr}\left(g'^{ab}R_{ab} + \frac{1}{p^2}g'^{[ab]}g_{[ab]}\right), \tag{3.6}$$

where R_{ik} is defined in (1.5). By use of (3.1), (3.2), (3.3), and (3.5), the invariance of the Lagrangian (3.6), under an isotopic gauge transformation as well as under a λ-gauge transformation, may be verified by straightforward computation. It is these remarkable symmetry properties of the curvature tensor, that it admits both an Abelian and a non-Abelian gauge transformation, which make it possible to construct a unified theory containing both the Yang-Mills field and the electromagnetic field.

By independent variation of (3.6) with respect to g'^{ik} and Γ^i_{kj} we obtain the system of field equations

$$R^*_{ik}=0, \tag{3.7}$$

$$g'^{jk}{}_{,i}+\Gamma^j_{ai}g'^{ak}+g'^{ja}\Gamma^k_{ia}-g'^{jk}\Gamma^t_{it}+\tfrac{2}{3}g'^{jb}\delta^k_i\Gamma^t_{[bt]}=0, \tag{3.8}$$

where R^*_{ik} is defined by

$$R^*_{ik}=R_{ik}+I_{ik}, \tag{3.9}$$

$$I_{ik}=\frac{1}{p^2}(g_{ia}g^{[ab]}g_{bk}+\tfrac{1}{2}g_{ik}g_{ab}g^{[ab]}+g_{[ik]}). \tag{3.10}$$

In order to disclose the physical content of the field equations (3.7) and (3.8), we decompose the fields g_{ik} and Γ^i_{kj} according to

$$g_{ik}=g_{(ik)}+p(\tau_0F_{ik}+\mu E_{ik}), \tag{3.11}$$

$$\Gamma^i_{kj}=\Gamma^{*i}_{kj}+\delta^i_k\left(-\frac{2}{p}A_j\tau_0-i\kappa B_j\right), \tag{3.12}$$

where the decomposition is fully defined by

$$\Gamma^{*a}_{[ka]}=0, \tag{3.13}$$

$$E_{ik}=\vec{\tau}\cdot\vec{f}_{ik}, \tag{3.14}$$

$$B_k=\vec{\tau}\cdot\vec{b}_k, \tag{3.15}$$

and where we have used the fact that every 2×2 matrix can be uniquely expressed as a linear combination of the four matrices $\tau_0, \vec{\tau}$. With the decomposition (3.11) and (3.12), the gauge transformations (2.3) and (2.5) becomes entirely equivalent to the gauge transformations (3.1) and (3.2). Also, the gauge transformation of A_i,

$$\tau_0A_i\to\tau_0A_i-\tfrac{1}{2}p\lambda_{,i}\tau_0, \tag{3.16}$$

becomes equivalent to the λ-gauge transformation (3.3). The decomposition (3.12) is thus not only a decomposition with respect to tensor symmetries and matrix properties, but also with respect to gauge properties: The three fields Γ^*, A_i, and B_i generating, respectively, general covariance, an Abelian gauge, and an isotopic gauge, correspond to the three natural forces: gravitation, electromagnetism, and the Yang-Mills field.

We have not written Γ^* and $g_{(ik)}$ in terms of the four Pauli matrices. In the following we shall only be concerned with the possibility

$$\Gamma^{*i}_{kj}=\tfrac{1}{2}\mathrm{Tr}(\Gamma^{*i}_{kj})\tau_0,\quad g_{(ik)}=\tfrac{1}{2}\mathrm{Tr}(g_{(ik)})\tau_0, \tag{3.17}$$

which leads to Einstein's theory of gravitation. The inclusion of traceless parts of Γ^{*i}_{kj} and $g_{(ik)}$ is equivalent to the introduction of new, electrically neutral, isotopically charged fields, the investigation of which will be left for future considerations.

Inserting the decomposition (3.11) and (3.12) in the field equations (3.7) and (3.8) splits these into the system of equations

$$R^*_{(ik)}(\Gamma^*)=0, \tag{3.18}$$

$$R^*_{[ik]}(\Gamma^*)+\frac{4}{p}A_{[i,k]}\tau_0+i\kappa[B_{i,k}-B_{k,i}+i\kappa(B_iB_k-B_kB_i)]=0, \tag{3.19}$$

$$g'^{jk}{}_{,i}+\Gamma^{*j}_{ai}g'^{ak}+\Gamma^{*k}_{ib}g'^{jb}-\Gamma^{*t}_{(it)}g'^{jk}-i\kappa(B_ig'^{jk}-g'^{jk}B_i)=0, \tag{3.20}$$

$$\Gamma^{*a}_{[ka]}=0, \tag{3.21}$$

where $R_{ik}(\Gamma^*)$ is related to Γ^* in the usual way by means of (1.5). Antisymmetrizing (3.20) in j and k and thereafter contracting in k and i, the Γ^*-dependent terms cancel identically, and we obtain

$$g'^{[ja]}{}_{,a}-i\kappa(B_ag'^{[ja]}-g'^{[ja]}B_a)=0. \tag{3.22}$$

If analogous to (3.11) we decompose $g^{[ik]}$ according to[8]

$$\frac{1}{p}g^{[ik]}=\tau_0F^{ik}+\mu E^{ik}, \tag{3.23}$$

$$E^{ik}=\vec{\tau}\cdot\vec{f}^{ik}, \tag{3.24}$$

and insert in (3.22), we obtain the following on separating into parts with vanishing and nonvanishing trace:

$$(wF^{ia})_{,a}=0, \tag{3.25}$$

$$(w\vec{f}^{ja})_{,a}+2\kappa\vec{b}_a\times\vec{f}^{ja}w=0. \tag{3.26}$$

Equations (3.25) and (3.26) are seen to correspond to the "current parts" of Maxwell's equations and the Yang-Mills equations, respectively. We observe that the Yang-Mills fields do not carry any electric charge, a result which already might have been foreseen from the unbroken unitary symmetry of the theory. The unitary symmetry considered can therefore not be connected with electric charge such as is the case for ordinary isospin. An interpretation which suggests itself is color SU(3) in the quark model, where the quanta of the $\vec{b}_i$ field are then the gluons which keep the quarks together. Another possibility would be to introduce a symmetry-breaking term in the Lagrangian—a possibility which we shall not consider for the time being.

IV. THE EINSTEIN-MAXWELL-YANG-MILLS LIMIT

It remains to find the content of the field equations (3.18) and (3.19). This is most conveniently done by investigating the correspondence principle of the unified theory, which, as in the theory of Moffat and Boal, consists in taking the formal limit of the system of field equations (3.7) and (3.8) as $p\to 0$. This limit we shall call the Einstein-Maxwell–Yang-Mills limit, or for short, the EMYM limit.

From the inversion relation (3.4c) we derive

$$\frac{1}{p}g_{[ab]}g^{ka}g^{ib}=\frac{1}{p}g^{[ki]}. \tag{4.1}$$

As F_{ik} and E_{ik} remain finite, it follows from (3.11) and (3.4c) that $g_{[ik]}$ and $g^{[ik]}$ vanish as $p\to 0$. Thereby we get from (4.1)

$$\begin{aligned} &F^{ik}\to F_{ab}g^{(ai)}g^{(bk)} \text{ for } p\to 0,\\ &E^{ik}\to E_{ab}g^{(ai)}g^{(bk)} \text{ for } p\to 0. \end{aligned} \tag{4.2}$$

The symmetric and skew-symmetric parts of I_{ik}, defined in (3.10), are given by[2]

$$\begin{aligned} I_{(ik)}&=\frac{1}{p^2}\big(g_{(ia)}g^{[ab]}g_{[bk]}+g_{(ka)}g^{[ab]}g_{[bi]}\\ &\quad+\tfrac{1}{2}g_{(ik)}g_{[ab]}g^{[ab]}\big),\\ I_{[ik]}&=\frac{1}{p^2}\big(g_{[ia]}g^{[ab]}g_{[bk]}+g_{(ia)}g^{[ab]}g_{(bk)}\\ &\quad+\tfrac{1}{2}g_{[ik]}g_{[ab]}g^{[ab]}+g_{[ik]}\big), \end{aligned}$$

which lead to the limits

$$\begin{aligned} \tfrac{1}{2}\lim_{p\to 0}\mathrm{Tr}(I_{(ik)})=&-8\pi\left(\frac{1}{4\pi}\left(F_{ia}F^{a}_{\ k}-\tfrac{1}{4}g_{(ik)}F_{ab}F^{ab}\right)\right)\\ &-8\pi\left(\frac{1}{4\pi}\left(\vec{f}_{ia}\cdot\vec{f}^{\,a}_{\ k}-\tfrac{1}{4}g_{(ik)}\vec{f}_{ab}\cdot\vec{f}^{\,ab}\right)\right), \end{aligned} \tag{4.3}$$

$$\lim_{p\to 0} pR^{*}_{[ik]}=2(\tau_0F_{ik}+\mu E_{ik}). \tag{4.4}$$

Split into parts with vanishing and nonvanishing trace, (3.19) then becomes in the EMYM limit

$$F_{ik}=-2A_{[i,k]}, \tag{4.5}$$

$$\vec{f}_{ik}=\vec{b}_{i,k}-\vec{b}_{k,i}-2\kappa\vec{b}_i\times\vec{b}_k, \tag{4.6}$$

where by comparison with (2.10) we have fixed the constant μ as

$$\mu=-\tfrac{1}{2}i\kappa p. \tag{4.7}$$

Equations (4.5) and (4.6), together with (3.25) and (3.26), form Maxwell's equations and the Yang-Mills equations. By use of (4.3), Eqs. (3.18) and (3.19) in the EMYM limit become

$$\begin{aligned} &\tfrac{1}{2}\mathrm{Tr}[R_{(ik)}(\Gamma^*)]\\ &\qquad=8\pi\left[M_{ik}+\mu^2\left(\frac{1}{4\pi}\left(\vec{f}_{ia}\cdot\vec{f}^{\,a}_{\ k}-\tfrac{1}{4}g_{(ik)}\vec{f}_{ab}\cdot\vec{f}^{\,ab}\right)\right)\right],\\ &g'^{(jk)}_{\ \ ,i}+g'^{(jb)}\Gamma^{*k}_{ib}+g'^{(ak)}\Gamma^{*j}_{ai}-g'^{(jk)}\Gamma^{*t}_{(it)}=0, \end{aligned}$$

which is Einstein's equations with M_{ki} as Maxwell's stress-energy tensor, and Γ^* as the Christoffel symbols formed from the metric tensor $g_{(ik)}$.

V. GAUGE-COVARIANT FORMULATION OF ELECTRICALLY AND ISOTOPICALLY CHARGED SPIN-½ FIELDS

The field equations in the presence of electric and external isotopic currents can be obtained by adding to the Lagrangian (3.6) a "phenomenological" interaction term of the form

$$\mathfrak{L}_{\rm int}=\mathrm{Tr}\left[\Gamma^{a}_{[ba]}(AJ^{b}_{E}+BJ^{b}_{I})\right], \tag{5.1}$$

where A and B are suitable constants, and J^{k}_{E} and J^{k}_{I} denote the electric and isotopic current densities of the sources, respectively:

$$\begin{aligned} &J^{k}_{E}=\tfrac{1}{2}\tau_0\,\mathrm{Tr}(J^{k}_{E}),\\ &J^{k}_{I}=\vec{\tau}\cdot\vec{j}^{\,k}_{I}. \end{aligned}$$

In a nonphenomenological context, that is, when some definite source fields are considered, the interaction is obtained by the construction of suitable covariant gauge derivatives. We shall therefore proceed to find such a suitable gauge derivative in the unified field theory.

The contracted curvature tensor (1.5) may be written in the form

$$R_{ik}=D^{s}_{ts}\Gamma^{t}_{ik}-D^{t}_{sk}\Gamma^{s}_{it}, \tag{5.2}$$

with the derivative D^{i}_{kj} defined by

$$D^{i}_{kj}=\delta^{i}_{k}\partial_j+\Gamma^{i}_{kj}. \tag{5.3}$$

Using the decomposition (3.12), D^{i}_{kj} becomes

$$D^{i}_{kj}=\delta^{i}_{k}\left(\partial_j-\frac{2}{p}A_j-i\kappa B_j\right)+\Gamma^{*i}_{kj}. \tag{5.4}$$

D^{i}_{kj} is the gauge derivative we are looking for. It contains three gauge fields: the Abelian gauge field A, the isotopic gauge field B, and the field Γ^* securing general covariance—all three fields united in a field Γ. If the derivative $\partial_j-(2/p)A_j-i\kappa B_j$ should correspond to the gauge derivative used in electrodynamics and in the Yang-Mills field theory, the constants p and κ must be fixed as

$$p = -2i\hbar/e\,, \tag{5.5}$$

$$\kappa = \epsilon/\hbar\,, \tag{5.6}$$

by which the constant μ in (4.7) becomes

$$\mu = -\epsilon/e\,, \tag{5.7}$$

and the gauge derivative D^i_{kj} becomes

$$D^i_{kj} = \delta^i_k\left(\tau_0\partial_j - \frac{ie}{\hbar}A_j\tau_0 - \frac{i\epsilon}{\hbar}B_j\right) + \Gamma^{*i}_{kj}\,, \tag{5.8}$$

where e and ϵ are the elementary quanta of electric and isotopic charge. As an example we shall consider the general gauge-covariant formulation of an electrically and isotopically charged spin-$\frac{1}{2}$ field.

In the special theory of relativity, such a field satisfies Dirac's equation, which in spinor form may be written[9,10]

$$\begin{aligned}\sqrt{2}\,\sigma^{k\dot{\nu}}_{\rho}(\partial_k\chi^\rho) &= \frac{m}{\hbar}\xi^{\dot{\nu}}\,,\\ \sqrt{2}\,\sigma^{k\nu}_{\dot{\rho}}(\partial_k\xi^{\dot{\rho}}) &= \frac{m}{\hbar}\chi^\nu\,,\end{aligned} \tag{5.9}$$

where Greek indices take on the values 1, 2 and Latin indices as usual take on the values 0, 1, ..., 3. $\sigma^{k\dot{\alpha}}_{\beta}$ are the generalized Pauli matrices which connect the ordinary metric $g_{(ik)}$ and the skew-symmetric spinor metric $\gamma_{\mu\nu}$,

$$g_{(kl)}\sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\rho}\sigma} = \gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\sigma}\,, \tag{5.10}$$

from which we derive the orthogonality relation

$$\sigma^{k\dot{\lambda}\mu}\sigma_{k\dot{\alpha}\beta} = \delta^{\dot{\lambda}}_{\dot{\alpha}}\delta^{\mu}_{\beta}\,. \tag{5.11}$$

As the σ's transform as mixed quantities,[11] we see from (5.9) that the problem of general covariant formulation consists in finding a mixed connection $\Gamma^\alpha_{\beta s}$ for which the covariant derivatives

$$\begin{aligned}\chi^\rho_{|k} &= \partial_k\chi^\rho + \Gamma^\rho_{\alpha k}\chi^\alpha\,,\\ \chi_{\rho|k} &= \partial_k\chi_\rho - \Gamma^\alpha_{\rho k}\chi_\alpha\end{aligned} \tag{5.12}$$

transform as mixed quantities. This requirement is expressed in the transformation rules

$$\Lambda^\rho_\sigma\Gamma'^{\alpha}_{\rho s} = \Lambda^\alpha_\rho\Gamma^\rho_{\sigma s} - \partial_s\Lambda^\alpha_\sigma \tag{5.13}$$

for a spinor transformation with transformation coefficients Λ^ρ_σ, and as a covariant vector in the indices s, under an ordinary coordinate transformation. In order that the generally covariant Dirac equation

$$\begin{aligned}\sqrt{2}\,\sigma^{k\dot{\nu}}_{\rho}(\chi^\rho_{|k}) &= \frac{m}{\hbar}\xi^{\dot{\nu}}\,,\\ \sqrt{2}\,\sigma^{k\dot{\rho}\nu}(\xi_{\dot{\rho}|k}) &= -\frac{m}{\hbar}\chi^\nu\end{aligned} \tag{5.14}$$

have a correspondence to the gauge-invariant Dirac equation in special relativity

$$\begin{aligned}\tau_0\sqrt{2}\,\sigma^{k\dot{\nu}}_{\rho}\left(\partial_k - \frac{ie}{\hbar}A_k - \frac{ie}{\hbar}B_k\right)\chi^\rho &= \frac{m}{h}\xi^{\dot{\nu}}\,,\\ \tau_0\sqrt{2}\,\sigma^{k\nu}_{\dot{\rho}}\left(\partial_k - \frac{ie}{\hbar}A_k - \frac{ie}{\hbar}B_k\right)\xi^{\dot{\rho}} &= \frac{m}{h}\chi^\nu\,,\end{aligned} \tag{5.15}$$

we must demand that the mixed connection $\Gamma^\alpha_{\beta s}$, in a local inertial system, reduce to $-(ie/\hbar)\delta^\alpha_\beta A_s\tau_0 - (i\epsilon/\hbar)\delta^\alpha_\beta B_s$, in the EMYM limit

$$\Gamma^\alpha_{\beta s} \to -\frac{ie}{\hbar}\delta^\alpha_\beta A_s\tau_0 - \frac{i\epsilon}{\hbar}\delta^\alpha_\beta B_s \text{ for } p\to 0\,, \tag{5.16}$$

whenever

$$\partial_i\sigma^{k\dot{\alpha}\beta} = 0, \quad \partial_i\gamma_{\mu\nu} = 0, \quad \partial_i g_{(kj)} = 0\,.$$

It may be verified by straightforward computation that the mixed connection defined by

$$\Gamma^\mu_{\beta s} = -\tfrac{1}{2}(\gamma)^{-1/4}\sigma_{k\dot{\alpha}\beta}\bar{D}^{+k}_{rs}(\gamma)^{1/4}\sigma^{r\dot{\alpha}\mu}, \quad \gamma = \gamma_{12}\gamma_{\dot{1}\dot{2}} \tag{5.17}$$

satisfies the transformation rule (5.13), as well as the condition that it transform as a covariant vector in the indices s. In order to verify the gauge invariance with respect to Abelian and non-Abelian gauge transformations, as well as the correspondence condition (5.16), we decompose $\Gamma^\mu_{\beta s}$, using (5.8),

$$\begin{aligned}\Gamma^\mu_{\beta s} = &-\frac{ie}{\hbar}\delta^\mu_\beta A_s\tau_0 - \frac{i\epsilon}{\hbar}\delta^\mu_\beta B_s\\ &-\tfrac{1}{2}(\gamma)^{-1/4}\sigma_{k\dot{\alpha}\beta}(\delta^k_r\partial_s\tau_0 + \Gamma^{*k}_{rs})\sigma^{r\dot{\alpha}\mu}(\gamma)^{1/4}\,,\end{aligned} \tag{5.18}$$

where we have used the relations (5.11). By comparison with (5.15), we see that (5.18) is gauge-invariant, as Γ^* does not change under Abelian and non-Abelian gauge transformations. Using the fact that Γ^* reduces to the Christoffel symbols in the EMYM limit, it follows that the correspondence condition (5.16) is also satisfied.

VI. CONCLUSION

The unified theory for gravitation, electromagnetism, and the Yang-Mills field presented in the foregoing is characterized by a linear connection Γ^i_{kj} and a fundamental Hermitian tensor field g_{ik}, the components of which form matrices related to some unitary symmetry. The field equations of the theory can be derived from the Lagrangian

$$\mathfrak{L} = \mathrm{Tr}\left[g'^{ab}(D^s_{ts}\Gamma^t_{ab} - D^t_{sb}\Gamma^s_{at}) + \frac{1}{p^2}g'^{[ab]}g_{[ab]}\right] \tag{6.1}$$

by independent variation of g'^{ik} and Γ^i_{kj}. D^i_{kj} is a gauge derivative defined by

$$D^i_{kj} = \delta^i_k \partial_j + \Gamma^i_{kj}. \tag{6.2}$$

The universal constant p is given by

$$p = -2i\hbar/e \quad (c = G = 1),$$
$$|p| = 3.8 \times 10^{-32} \text{ cm}. \tag{6.3}$$

The theory is invariant under three different kinds of gauge transformations: general (coordinate) covariance, and Abelian gauge transformation, and a non-Abelian gauge transformation. The system of field equations is split up into the Einstein-Maxwell equations and the Yang-Mills equations if we take the formal limit $p \to 0$. This principle of correspondence is deepened by comparing (6.3) with the metric of the Moffat-Boal solution

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)\left(1 + \frac{p^2 Q^2}{r^4}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{6.4}$$

We would expect deviations from the EMYM theory to become significant for phenomena characterized by a distance R,

$$R \lesssim \sqrt{\hbar}\,,$$

of the order of a Planck length. At these distances quantum field effects must be expected to dominate. A positive test of the theory therefore presumably awaits its quantization.

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[8]It is important to notice that F^{ik} and E^{ik} are *defined* by (3.23) and not by raising the indices of F_{ik} and E_{ik} with the help of the metric tensor $g_{(ik)}$. We shall later see that these two definitions become identical in the limit $p \to 0$.

[9]L. Infeld and B. L. Van Der Waerden, Sitzber. Preuss. Akad. Wiss., Phys.—math. Kl. 9, 380 (1933).

[10]W. L. Bade and H. Jehle, Rev. Mod. Phys. 25, 714 (1953).

[11]A mixed quantity is a quantity transforming as a tensor in the Latin indices and as a spinor in the Greek indices.