Static model for the π and ρ trajectories*

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It is demonstrated, using the static-model crossing matrices, that the π- and ρ-meson Regge trajectories can support each other. Exchange of the π_j in the $\pi\rho_{j+1}$ channel (subscripts indicate spin) generates the π_{j+2}; exchange of the ρ_{j+1} in the $\pi\pi_{j+2}$ channel gives the ρ_{j+3}, etc.

An attractive idea within the framework of the S-matrix theory for strongly interacting particles is that to a sequence of external particles on a linearly rising Regge trajectory there corresponds a sequence of resonances on another such trajectory.[1-4] Given a channel consisting of a light particle and a given particle on Regge trajectory A, one finds that the dynamics is such as to give rise to a particle on trajectory B; when one considers the channel involving the light particle and the next particle on trajectory A, one obtains the next particle on trajectory B, etc. In this paper we present arguments, based on the crossing matrices in a simple static model, that it is possible for the π-meson and ρ-meson trajectories to support each other in such a manner. Not only does there exist the possibility of obtaining members of the ρ trajectory as resonances in channels containing members of the π trajectory, but one can exchange the rôle of these two families of particles.

To be specific, let π_j and $\rho_{j\pm1}$ (where the subscripts, with $j=0,2,4,\ldots$, indicate the spin) lie on the Regge trajectories that contain, respectively, the π-meson and the ρ-meson multiplets as their first members. Suppose one has somehow obtained π_j and ρ_{j+1} for some value of j and that a π_j is exchanged in the $\pi\rho_{j+1}$ channel, as shown in Fig. 1(a); a study of the isospin and angular-momentum crossing matrices indicates that the force can be strong enough to generate a π_{j+2} resonance, as illustrated in Fig. 1(b). One now investigates the $\pi\pi_{j+2}$ channel and finds that the exchange of a ρ_{j+1}, shown in Fig. 1(c), can give rise to the ρ_{j+3} in Fig. 1(d). This suggests that one can build up the two trajectories in a reciprocal way, with the lowest members, namely the π and the ρ, as input. In all cases the exchanged and derived particles occur in the p waves of their constituents; except for the first few j values, the pion mass is negligible compared to that of the other external particle, so the use of the static model is reasonable.

Our approach conforms to the ideas of Golowich,[2] who argued that high-spin states should be described by configurations with low values of the orbital angular momentum L. It is different from that of Carruthers,[5] who generated higher-spin positive-parity πN resonances by increasing the orbital angular momentum of the ground-state constituents. However, Golowich argued that the coupling of resonances should be strongest in states of low L and, consequently, Carruthers' model cannot explain the chains of particles with increasing mass and spin that one observes experimentally. Furthermore, in contrast to Golowich's model and ours, the singularities that Carruthers used for forces do not lie "nearby" the resonances, so it is difficult to justify the dynamics from the S-matrix viewpoint. Other authors who used p waves to generate higher-spin positive-parity baryons, which then served as external and exchanged particles for the next step in the chain, are Capps[6] and Abers, Balazs, and Hara,[7] but their higher-spin resonances correspond to unobserved SU(2) and SU(3) multiplets. In order to generate successive negative-parity baryons on a Regge trajectory, Carruthers and Nieto[1] employed a model in which low-L resonances are coupled to members of external positive-parity trajectories.

Since the π-meson mass is small compared to that of the other particles employed in our calculation, we use the static model, in which the well-known N/D equations for the scattering amplitude T_i in the s-channel i assume the form[7]

$$T_i(\omega)=\frac{N_i(\omega)}{D_i(\omega)}\,, \tag{1a}$$

$$N_i(\omega)=\sum_k \frac{C_{ik}\gamma_k D(-\omega_k)}{\omega+\omega_k}\,, \tag{1b}$$

$$D_i(\omega)=1-\frac{(\omega-\overline{\omega})^2}{\pi}\int_1^{\Lambda}\frac{(\omega'^2-1)^{3/2}N_i(\omega')}{(\omega'-\overline{\omega})(\omega'-\omega)}d\omega', \tag{1c}$$

where ω is the pion energy, $\overline{\omega}$ is a subtraction point, and ω_k and γ_k are, respectively, the pion

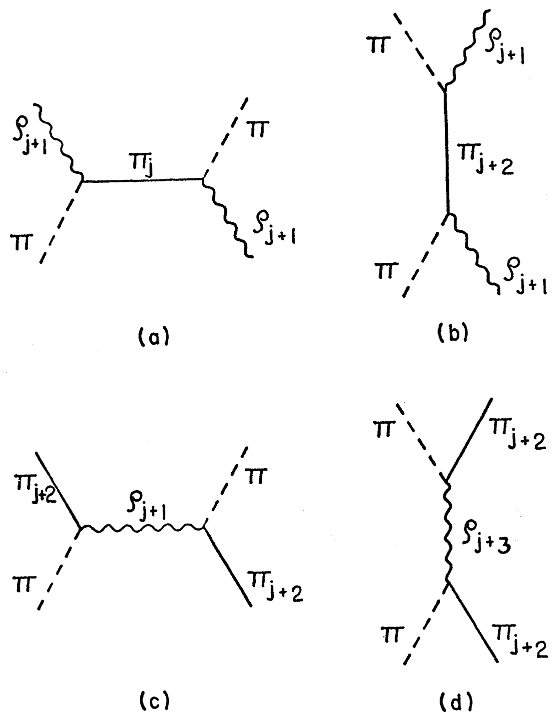

FIG. 1. The various exchange processes and resonances discussed in this paper.

energy and residue at the resonance in the u-channel k. The label i or k designates both the total and orbital angular momenta, J and L, and the isospin I. The crossing matrix, whose elements are designated by C_{ik}, is the product of the individual crossing matrices for angular momentum and isospin. Because of the dependence of the integral (1b) on the cut-off Λ, we make the usual effective-range approximation[7,8]

$$D_i(\omega)=\frac{\omega_i-\omega}{\omega_i-\overline{\omega}}\,, \tag{2}$$

so that the residue γ_i' at a zero of D_i, at which a resonance appears in the s channel, is

$$\gamma_i'=-\left.\frac{N_i(\omega)}{D_i'(\omega)}\right|_{\omega=\omega_i}=\sum_j C_{ij}\gamma_j. \tag{3}$$

Since the residues are always positive, the signs and magnitudes of the crossing-matrix elements determine whether a resonance is possible in a given channel. The static angular-momentum crossing matrix for the scattering of a light spinless particle off a heavy spin-S particle is diagonal in the orbital angular momentum, and the submatrix for $L=1$ is equal to $[S(S+1)(2S+1)]^{-1}$ times the matrix[9]

$$\begin{array}{cc|ccc|} & & & J_u & \\ & & S-1 & S & S+1 \\ \hline & S-1 & S+1 & -(S+1)(2S+1) & S(S+1)(2S+3) \\ J_s & S & -(2S-1)(S+1) & (2S+1)(S^2+S-1) & S(2S+3) \\ & S+1 & S(2S-1)(S+1) & S(2S+1) & S \end{array}\;. \tag{4}$$

It is remarkable that for $S\geq 4$ this submatrix can be approximated by

$$\begin{array}{cc|ccc|} & & & J_u & \\ & & S-1 & S & S+1 \\ \hline & S-1 & 0 & 0 & 1 \\ J_s & S & 0 & 1 & 0 \\ & S+1 & 1 & 0 & 0 \end{array}\;. \tag{5}$$

The isospin crossing matrix for two particles of isospin 1 is

$$\begin{array}{cc|ccc|} & & & I_u & \\ & & 0 & 1 & 2 \\ \hline & 0 & \frac{1}{3} & -1 & \frac{5}{3} \\ I_s & 1 & -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ & 2 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{array}\;. \tag{6}$$

According to (5) and (6), an exchanged particle in the state with $J_u=S-1$, $L_u=1$, and $I_u=1$, contributes a positive factor of approximately one-half its residue to the right-hand side of (3) when the direct channel state is given by $J_s=S+1$, $L_s=1$, and $I_s=1$; therefore, it is possible to obtain a resonance in the latter state when one exists in the former.

In other words, a spin-$(S+1)$ meson of unit isospin can be generated in a p wave in an s channel consisting of a pion and a spin-S meson, with the binding force resulting from a spin-$(S-1)$ meson in a p wave of the u channel. Since the pion has negative parity, the spin-S and spin-$(S\pm 1)$ particles all have the same parity; on the other hand, the spin-$(S\pm 1)$ particles have G parity opposite to that of the spin-S particle as a result of the negative G parity of the π. Thus, given as input the π and the negative parity ρ, with $G=-1$ and $G=+1$ respectively, there exists the possibility of obtaining successive members of their trajectories, using the chain illustrated in Fig. 1.

The usual assumption of linearly rising trajectories means that the π-trajectory masses m_j and the ρ-trajectory masses M_{j+1}, where $j=0,2,4,\ldots$, are related to their spins by

$$j=am_j^2+b\,, \tag{7a}$$

$$j+1=a'M_{j+1}^2+b'\,, \tag{7b}$$

where a and a' are the slopes of the π and ρ trajectories, respectively. There are theoretical reasons to justify a universal slope for hadron trajectories[3] and, except for that of the π, the value 0.9 $(\text{GeV})^{-2}$ gives a reasonable fit to experiment. Putting $b'=0.5$ for the ρ trajectory gives one good agreement with experiment by taking the $\rho(770)$ for the ρ_1, the $g(1680)$ for the ρ_3, and the $\rho(2275)$ for the ρ_5. If one also puts $a=0.9$ $(\text{GeV})^{-2}$ for the π trajectory and, in addition, $b=-0.4$, one gets the $A_3(1640)$ for the π_2 and a π_4 particle with mass around 2200 MeV for which there are several experimentally observed candidates.[10] The fact that one obtains a mass for the π_0 that is almost five times larger than that of the $\pi(140)$ is not surprising when one considers the fundamentally different role that the π plays from the other particles in a model such as ours; even when the calculation is made relativistic, the π mass must be inserted as a parameter. One may therefore assume that (7a) is valid for the π trajectory with the assigned values of a and b, except for $j=0$. Then, for $j\geq 2$ one has $m_j\approx M_{j+1}$, and for $j\geq 4$ the ratio $(m_j-M_{j-1})/M_{j-1}$ is small, so the singularities used as forces in our model do lie close to the resonances generated; the importance of this factor was stressed by Golowich.[2]

Of course, it is also not possible to obtain the ρ multiplet in our static model, but on the basis of crossing in a relativistic calculation one can obtain a self-consistent $I=1$ particle in the p-wave $\pi\pi$ state.[11] It is still an unsettled question as to whether one can actually "bootstrap" the ρ in the $\pi\pi$ channel, with some authors agreeing that one can[12] and with others claiming the contrary[13]; indeed, it has been questioned whether "bootstraps" of low-spin particles have any meaning at all.[14] Because of the low mass of the exchanged π that gives rise to the π_2 in the $\pi\rho$ channel and also because the ratio of the mass of the π_2 to the $\pi\rho$ threshold energy is 1.8, our static model is not valid in this case either, but a fully relativistic calculation should give the expected result. Similarly, one should make a more accurate study of the $\pi\pi_2$ channel, since the mass of the exchanged ρ is less than half of that of its constituents.

The isospin crossing matrix (6) predicts equally strong forces in the $I=1$ and $I=2$ channels as a result of the exchange of $I=1$ particles. An explanation of the absence of $I=2$ multiplets in nature probably requires an analysis of t-channel effects, for which the isospin crossing matrix is

$$I_s\;\begin{matrix} & \\ 0 \\ 1 \\ 2\end{matrix}\overset{\displaystyle I_t \atop \displaystyle 0 \quad\; 1 \quad\; 2}{\begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}}\,. \tag{8}$$

Comparison with (6) shows that, provided the angular-momentum crossing-matrix element between the t and s channels is positive, the forces arising from $I=1$ t-channel particles reinforce those of u-channel particles in the $I=1$ state, but oppose them in the $I=2$ state. The desired angular-momentum crossing matrix cannot, of course, be obtained from the static model.

The presence of forces due to t-channel singularities may also resolve another possible objection to our model. The crossing matrices (5) and (6), together with the relation (3) between the residues, predict a residue for the resonance in Fig. 1(b) that is approximately half of that for the resonance in Fig. 1(a). Since the same effect appears in Fig. 1(d) as a result of the exchange in Fig. 1(c) and since the coupling constants in Figs. 1(b) and 1(c) are the same, it might be argued that the coupling constant eventually becomes too small to generate a resonance in the next channel on our chain. This effect would not occur if the forces arising from t-channel singularities are attractive and strong enough; i.e., they would increase the size of each residue on the chain illustrated in Fig. 1.

To summarize, the analysis of this paper demonstrates that the π and ρ trajectories can play an important role in generating one another, provided one uses the first particle on each trajectory as input. Owing to the arbitrariness of the cut-off parameter Λ in (1b) one cannot, of course, predict any masses unless additional assumptions are introduced.

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