Electrodynamics and the electron equation of motion

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We show the extent to which Lorentz-Dirac equations of motion for the electron follow from Maxwell's equations. The standard derivations depend on cutoff procedures and other prescriptions. The conclusion is that the Lorentz force is cutoff-independent but radiation-reaction forces are not.

I. INTRODUCTION

The purpose of this paper is to show the extent to which the equations of motion for charged particles, including radiation-reaction forces, actually follow from Maxwell's equations. The main difficulty encountered in deriving these equations is that the electron's self-energy is infinite.

Since Dirac,[1-3] the way around this problem has been to isolate the electron's world line (EWL) by a timelike tube (Σ) and to evaluate the energy integral outside Σ only. Once the trick of mass renormalization[1,2] is performed Σ is shrunk to the EWL. We study the dependence of the equation of motion on the tube Σ and other necessary prescriptions. The conclusion we arrive at is that the Lorentz force emerges independent of the cutoff procedures while radiation-reaction forces do depend on them. The dependence is explicitly calculated. Ambiguities are removed in Sec. V.

In the following we describe the way the problem is solved, leaving the details of calculation for the sections to come.

Classical electrodynamics is a very good theory far away from the charge, but it breaks down somewhere on the way as we approach the charge since physically important variables such as the four-momentum diverge. For regions close to the electron all we have is a missing theory. On the other hand, we know that very little detail of this missing theory actually shows up. At the classical level it is just the electron's mass which is accounted for in an *ad hoc* manner.

We begin by assuming that Maxwell's equations are valid for distances greater than some length ϵ away from the electron, that is, outside some world tube Σ surrounding the EWL.

As is well known,[2] it is the conservation of four-momentum that gives the equation of motion most easily. An action principle plus translation invariance[4] ensure that the total four-momentum

$$P^\mu = \int_\sigma T^{\mu\nu} d\sigma_\nu \tag{1}$$

does not depend on the hypersurface σ as long as it is spacelike at spatial infinity and the physical system is closed. This property is also valid for radiating systems if the total radiated four-momentum is finite. $T^{\mu\nu}$ is the energy-momentum tensor.[4] This conservation theorem cannot be applied to the electron without modification because P^μ is divergent. What is usually done is to evaluate integral (1) outside the tube Σ postulating that its contribution inside Σ is just mv^μ, where m is the mechanical mass of the electron and v^μ its four-velocity.[2]

Let $f^{\mu\nu}$ denote the external electromagnetic field driving the electron and $F^{\mu\nu}$ denote the retarded potential.[4] The total electromagnetic field outside Σ is then $f^{\mu\nu} + F^{\mu\nu}$ and the corresponding energy-momentum tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = T^{\mu\nu}_{\text{mix}} + T^{\mu\nu}_{e} + T^{\mu\nu}_{\text{ex}}, \tag{2}$$

where

$$T^{\mu\nu}_{\text{mix}} = F^{\mu\sigma} f_\sigma{}^\nu + f^{\mu\sigma} F_\sigma{}^\nu + \tfrac{1}{2}\eta^{\mu\nu} F^{\alpha\beta} f_{\alpha\beta}, \tag{3}$$

$$T^{\mu\nu}_{e} = F^{\mu\sigma} F_\sigma{}^\nu + \tfrac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{4}$$

and

$$T^{\mu\nu}_{\text{ex}} = f^{\mu\sigma} f_\sigma{}^\nu + \tfrac{1}{4}\eta^{\mu\nu} f^{\alpha\beta} f_{\alpha\beta}. \tag{5}$$

The contribution P^μ_{mix} of $T^{\mu\nu}_{\text{mix}}$ to P^μ is best calculated by applying Gauss's integral theorem on $T^{\mu\nu}_{\text{mix},\nu} = 0$ in the space-time region bounded by σ, Σ, the remote past, and spatial infinity (see Fig. 1). We assume the external field $f^{\mu\nu}$ to vanish at spatial infinity and at the remote past and thus see that the integral $P^\mu_{\text{mix}} = \int T^{\mu\nu}_{\text{mix}} d\sigma_\nu$ may be evaluated entirely on the tube Σ. The advantage is that when Σ is shrunk to the EWL values of $f^{\mu\nu}$ on the EWL only contribute to P^μ_{mix}.

The contribution P^μ_e of $T^{\mu\nu}_e$ to P^μ can be calculated explicitly and consists of two parts P^μ_B and P^μ_R which require separate computation. The following geometrical construction is essential in defining P^μ_B and P^μ_R. Let S denote the closed two-surface of intersection between Σ and σ as indicated in Fig. 1. C is the three-surface generated by null rays joining points of S and the EWL. Σ^∞

is a timelike tube surrounding the EWL which eventually will tend to spatial infinity. In applying the Gauss integral theorem on $T^{\mu\nu}_{e,\nu}=0$ in the space-time region bounded by C, σ, and Σ^∞ we see from Fig. 1 that the contribution to P^μ_e due to the portion AB of σ may be decomposed into an integral over region AD of C and another over region BD of Σ^∞. We now see that P^μ_e consists of two integrals, one over C up to S and one over Σ^∞ up to its intersection with C. The contribution P^μ_B due to C is called the bound four-momentum[2] because after Σ is shrunk to the EWL it depends on the kinematical variables of the EWL at the point of intersection of σ and the EWL, at $\tau=\bar{\tau}$ in Fig. 1. P^μ_R, the radiated four-momentum up to proper time $\bar{\tau}$, depends on the entire history of the EWL since it is the contribution to P^μ_e due to Σ^∞. The existence of P^μ_R is the only asymptotic condition for the infinite past needed in our calculation.[2,3]

P^μ_{ex} is conserved independently since $f^{\mu\nu}$ is assumed to satisfy Maxwell's equation for a vacuum and to have no singularities.

Anticipating the results (22), (35), and (36) for P^μ_{mix}, P^μ_B, and P^μ_R, respectively, we write the four-momentum conservation law as

$$\frac{d}{d\bar{\tau}}\,(mv^\mu+P^\mu_{\text{mix}}+P^\mu_B+P^\mu_R)=0$$

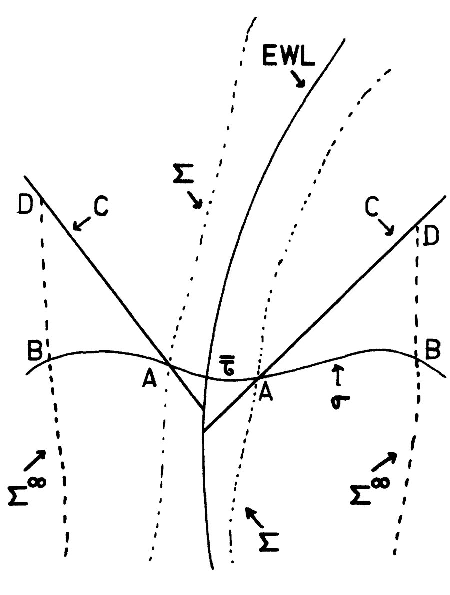

FIG. 1. The world diagram describing various hypersurfaces used in evaluating the total electron four-momentum. σ is a spacelike hypersurface cutting the electron world line (EWL) at proper time $\bar{\tau}$. Σ is a timelike tube surrounding the EWL. C is a three-surface generated by light rays joining points on the EWL and points in the intersection of Σ and σ. This intersection is referred to by S.

by ignoring all cutoff-dependent quantities, which we know should not appear in a "good" theory anyway. Lorentz-Dirac equations then obtain:

$$m\dot{v}^\mu=ef^{\mu\nu}v_\nu+\tfrac{2}{3}e^2(\dot{v}^2v^\mu-\ddot{v}^\mu).$$

Strictly speaking, these equations do not follow from electrodynamics unless extra assumptions are made. The Lorentz force is reliable since it is cutoff-independent.

II. PRELIMINARY GEOMETRY

The main mathematical problem to be solved is the evaluation of the integral P_e for small ϵ. This is best accomplished through the use of the retarded coordinates introduced by Newman and Penrose,[5] which we describe briefly below. Let $x^\mu=z^\mu(\tau)$ be a timelike world line parametrized by its proper time τ. To any given point x^μ in Minkowski space we associate four labels $(\tau,\kappa,\theta,\phi)$ as follows: τ is the retarded proper time of x^μ, that is,

$$[x^\mu-z^\mu(\tau)][x_\mu-z_\mu(\tau)]=0; \tag{6}$$

κ is defined to be

$$\kappa=v_\mu(\tau)[x^\mu-z^\mu(\tau)], \tag{7}$$

where $v_\mu(\tau)=\dot{z}_\mu(\tau)=dz_\mu/d\tau$; and θ and ϕ are the polar angles of the null vector $x^\mu-z^\mu(\tau)$ as referred to an arbitrary inertial system with a constant time axis along t^μ.

Coordinate transformation formulas are best written in terms of the ray vector $k^\mu=\kappa^{-1}(x^\mu-z^\mu)$ which does not depend on κ, that is, $k^\mu=k^\mu(\tau,\theta,\phi)$. By definition of θ and ϕ we have[5] $\dot{k}^\mu=\psi k^\mu$, since the direction of k^μ is fixed for constant θ and ϕ, which combined with $v_\mu k^\mu=1$ gives

$$\frac{\partial k^\mu}{\partial\tau}=-(\dot{v}\cdot k)k^\mu, \tag{8}$$

where dots denote differentiation with respect to τ and $a\cdot b=a_\mu b^\mu$. From $x^\mu=z^\mu+\kappa k^\mu$ and (8) one obtains

$$dx^\mu=[v^\mu-\kappa(\dot{v}\cdot k)k^\mu]d\tau+k^\mu d\kappa+\kappa k^\mu_\theta d\theta+\kappa k^\mu_\phi d\phi\,, \tag{9}$$

where $\kappa^\mu_\theta=\partial k^\mu/\partial\theta$ and $k^\mu_\phi=\partial k^\mu/\partial\phi$. The vectors v^μ, k^μ, k^μ_θ, and k^μ_ϕ satisfy nice orthogonality relations. We first notice that from $k\cdot k=0$ and $v\cdot k=1$ we have $k_\theta\cdot k=k_\phi\cdot k=k_\theta\cdot v=k_\phi\cdot v=0$. We can set $k_\theta\cdot k_\phi=0$ too because from (8)

$$\frac{\partial}{\partial\tau}(k_\theta\cdot k_\phi)=-2(\dot{v}\cdot k)(k_\theta\cdot k_\phi)$$

follows, and therefore if $k_\theta\cdot k_\phi=0$ for any given τ it vanishes for all τ.

Now let σ denote a spacelike hypersurface that cuts the EWL $z^\mu(\tau)$ at $\bar{\tau}$ and let Σ denote a timelike tube surrounding the EWL but never touching it. Let $\epsilon>0$ be a distance scale such that when $\epsilon\to 0$ Σ degenerates to the EWL; S denotes the intersection of σ and Σ. We represent S analytically by two functions of four variables as

$$\tau = T(\theta\phi;\bar{\tau};\epsilon), \tag{10}$$

$$\kappa = R(\theta\phi;\bar{\tau};\epsilon), \tag{11}$$

such that when $\bar{\tau}$ is eliminated from these equations we get the tube Σ given by

$$\kappa=\Sigma(\tau\theta\phi;\epsilon), \tag{12}$$

and when ϵ is eliminated we get the hypersurface σ given by

$$\tau=\sigma(\kappa\theta\phi;\bar{\tau}). \tag{13}$$

Equation (10) is of central importance by itself; it describes the three-surface C generated by null rays emanating from the EWL and piercing S. As seen in the Introduction, C contains the domain of integration for the integral P^μ_B.

It is very easy now to calculate the volume elements for C and Σ. We give the details for C only since for Σ the calculation is the same. Let dC_μ be the volume element of C, that is,[6]

$$dC_\mu = \epsilon_{\mu\nu\lambda\rho} d_\kappa x^\nu\, d_\theta x^\lambda\, d_\phi x^\rho,$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the Levi-Civita permutation symbol and $d_\kappa x^\mu$, $d_\theta x^\lambda$, and $d_\phi x^\rho$ are three independent displacements within C. $d_\kappa x^\mu$ is the displacement when κ alone varies, $d_\theta x^\mu$ and $d_\phi x^\mu$ are displacements when θ and ϕ alone vary, respectively. From (9) we have

$$d_\kappa x^\mu = k^\mu d\kappa,$$
$$d_\theta x^\mu = [(v^\mu - \kappa\dot{v}\cdot k\, k^\mu)T_\theta + \kappa k^\mu_\theta]d\theta,$$
$$d_\phi x^\mu = [(v^\mu - \kappa\dot{v}\cdot k\, k^\mu)T_\phi + \kappa k^\mu_\phi]d\phi,$$

which gives for dC_μ the following:

$$dC_\mu = (\kappa^2\epsilon_{\mu\nu\lambda\rho}k^\nu k^\lambda_\theta k^\rho_\phi + \kappa T_\theta \epsilon_{\mu\nu\lambda\rho}k^\nu v^\lambda k^\rho_\phi + \kappa T_\phi \epsilon_{\mu\nu\lambda\rho}k^\nu k^\lambda_\theta v^\rho)d\kappa\, d\theta\, d\phi.$$

The first term in this expression is orthogonal to k^ν, k^λ_θ, and k^ρ_ϕ and is therefore parallel to k_μ because of the orthogonality relations given below Eq. (9).

For the same reason the second and third terms are parallel to k^μ_θ and k^μ_ϕ respectively. The proportionality factors are calculated by contraction with v^μ, k^μ_θ, and k^μ_ϕ. The result is

$$dC^\mu = \kappa^2 d\kappa\, d\Omega_0\left(k^\mu - \frac{T_\theta}{\kappa}\frac{k^\mu_\theta}{k_\theta{}^2} - \frac{T_\phi}{\kappa}\frac{k^\mu_\phi}{k_\phi{}^2}\right), \tag{14}$$

where

$$d\Omega_0 = \epsilon_{\lambda\mu\nu\rho}v^\lambda k^\mu k^\nu_\theta k^\rho_\phi d\theta\, d\phi, \tag{15}$$

which is the solid-angle element for the inertial frame with time axis v^λ.

The volume element $d\Sigma_\mu$ for the tube Σ is given by

$$d\Sigma^\mu = \kappa^2 d\tau\, d\Omega_0\left[(1+\Sigma_\tau - \kappa\dot{v}\cdot k)k^\mu_\perp + (\Sigma_\tau - \kappa\dot{v}\cdot k)v^\mu + \frac{\Sigma_\theta}{\Sigma}\frac{k^\mu_\theta}{k_\theta{}^2} + \frac{\Sigma_\phi}{\Sigma}\frac{k^\mu_\phi}{k_\phi{}^2}\right], \tag{16}$$

where $k^\mu_\perp = k^\mu - v^\mu$ is the projection of k^μ onto the hyperplane orthogonal to v^μ.

III. LORENTZ FORCE

In this section we compute the contribution of $T^{\mu\nu}_{\text{mix}}$ to the total four-momentum P^μ as described in the Introduction.

The integral under consideration is

$$P^\mu_{\text{mix}} = \int_\sigma T^{\mu\nu}_{\text{mix}} d\sigma_\nu, \tag{17}$$

where $T^{\mu\nu}_{\text{mix}}$ is given by (3). Reasonable physical assumptions on the asymptotic behavior of the external field $f^{\mu\nu}$, as discussed in the Introduction, allow this integral to be evaluated on the tube Σ, that is,

$$P^\mu = -\int_\Sigma T^{\mu\nu} d\Sigma_\nu\,. \tag{18}$$

The minus sign is due to the fact that σ is spacelike and Σ is timelike.[6] The limits of integration are from the infinite past up to σ. The advantage of putting (17) in the form (18) is that when Σ shrinks to the EWL the external field $f^{\mu\nu}$ contributes to P^μ on the EWL only. In this limit the nonvanishing contribution to (18) is obtained by inserting for the electron field

$$F^{\mu\nu} \simeq \frac{e^2}{4\pi\kappa^2}(k^\mu v^\nu - k^\nu v^\mu) = \frac{e^2}{4\pi\kappa^2}(k^\mu_\perp v^\nu - k^\nu_\perp v^\mu), \tag{19}$$

which is obtained from Ref. 4 as the leading term when $\kappa\to 0$. In the same limit, from (16),

$$d\Sigma^\nu = \kappa^2 d\tau\, d\Omega_0\left(k^\nu_\perp + \frac{\gamma_\theta}{\gamma}\frac{k^\mu_\theta}{k_\theta{}^2} + \frac{\gamma_\phi}{\gamma}\frac{k^\mu_\phi}{k_\phi{}^2}\right) \tag{20}$$

is obtained when the regularity condition $\Sigma(\tau\theta\phi;\epsilon) = \gamma(\tau\theta\phi)\epsilon + O(\epsilon^2)$ is assumed for (12). From (3), (18), (19), and (20) we see that the angular integral to be calculated is

$$I^{\alpha\delta} = \int \frac{d\Omega_0}{4\pi}k^\alpha\left(k^\delta_\perp + \frac{\gamma_\theta}{\gamma}\frac{k^\delta_\theta}{k_\theta{}^2} + \frac{\gamma_\phi}{\gamma}\frac{k^\delta_\phi}{k_\phi{}^2}\right),$$

which is simplified to get

$$I^{\alpha\delta} = -\left[h^{\alpha\delta}\frac{1}{3}\left(1+3\int\frac{d\Omega}{4\pi}\ln\gamma\right) + 3\int\frac{d\Omega}{4\pi}\ln\gamma k_\perp^\alpha k_\perp^\delta\right] \tag{21}$$

by use of the partial integration formulas (A2) developed in Appendix A and Eqs. (A4). We have defined $h^{\alpha\delta} = \eta^{\alpha\delta} - v^\alpha v^\delta$ to be the standard projector onto the hyperplane orthogonal to the four-velocity v^α.

Once (3), (19), and (21) are put together into (18) we get the final result:

$$P^\mu_{\text{mix}} = -e\int_{-\infty}^{\bar\tau} f^{\mu\nu}v_\nu d\tau. \tag{22}$$

The arbitrary function $\gamma = \gamma(\tau\theta\phi)$ cancels out. This shows that P^μ_{mix} is independent of the way Σ is shrunk to the EWL.

IV. BOUND FOUR-MOMENTUM

In this section we calculate the electron bound four-momentum as defined in the Introduction, that is,

$$P^\mu_B = \int_C T_e^{\mu\nu}dC_\nu. \tag{23}$$

We quote $T_e^{\mu\nu}$ from Ref. 6:

$$T_e^{\mu\nu} = \frac{e^2}{4\pi}[-\kappa^{-2}((k\cdot\dot v)^2 + \dot v^2)k^\mu k^\nu + \kappa^{-3}(2(k\cdot\dot v)k^\mu k^\nu - (k\cdot\dot v)(k^\mu v^\nu + k^\nu v^\mu) + k^\mu\dot v^\nu + k^\nu\dot v^\mu) + \kappa^{-4}(-k^\mu k^\nu + k^\mu v^\nu + k^\nu v^\mu - \tfrac{1}{2}\eta^{\mu\nu})]. \tag{24}$$

In computing (23) we need the following relations:

$$T_e^{\mu\nu}k_\nu = \frac{e^2}{8\pi\kappa^4}k^\mu,$$

$$T_e^{\mu\nu}k_\nu^\theta = \frac{e^2}{8\pi\kappa^4}[2\kappa(k_\theta\cdot\dot v)k^\mu - k_\theta^\mu], \tag{25}$$

$$T_e^{\mu\nu}k_\nu^\phi = \frac{e^2}{8\pi\kappa^4}[2\kappa(k_\phi\cdot\dot v)k^\mu - k_\phi^\mu].$$

From (14) and (25), once the integration in κ is done we get

$$P^\mu_B = \frac{e^2}{2}\int\frac{d\Omega_0}{4\pi R}\left[\left(1 - \frac{2(k_\theta\cdot\dot v)T_\theta}{k_\theta^2} - \frac{2(k_\phi\cdot\dot v)T_\phi}{k_\phi^2}\right)k^\mu + \frac{1}{2}\frac{T_\theta}{R}\frac{k_\theta^\mu}{k_\theta^2} + \frac{1}{2}\frac{T_\phi}{R}\frac{k_\phi^\mu}{k_\phi^2}\right], \tag{26}$$

where the functions T and R are defined in (10) and (11). A simple expression can be found for this integral in the limit $\epsilon \to 0$. We consider the first two leading terms that go like $1/\epsilon$ and a constant; all other terms vanish when Σ is shrunk to the EWL.

Expansions in ϵ are carried out by assuming the regularity conditions on T and R,

$$T = T(\theta\phi;\bar\tau;\epsilon) = \bar\tau - \alpha(\theta\phi;\bar\tau)\epsilon + \beta(\theta\phi;\bar\tau)\epsilon^2 + O(\epsilon^3), \tag{27}$$

$$R = R(\theta\phi;\bar\tau;\epsilon) = \gamma(\theta\phi;\bar\tau)\epsilon + \delta(\theta\phi;\bar\tau)\epsilon^2 + O(\epsilon^3), \tag{28}$$

which prevent wild behavior of the closed surface S. The existence of the normal and the second fundamental form for the hypersurface σ at the EWL gives some relations among the functions α, β, γ, and δ. These relations are most easily expressed by representing the surface σ by $\phi(x^\mu;\bar\tau) = 0$, where $\bar\tau$ is the proper time at which σ intercepts the EWL. We now differentiate ϕ twice with respect to ϵ and use (9) to get

$$\frac{\alpha}{\gamma} = 1 + \frac{\phi_{,\mu}k_\perp^\mu}{\phi_{,\mu}v^\mu} \tag{29}$$

and

$$-\frac{\delta}{\gamma^2} = k\cdot\dot v + \frac{\beta}{\gamma^2} + \tfrac{1}{2}\phi_{,\mu,\nu}k_\perp^\mu k_\perp^\nu, \tag{30}$$

where all quantities are evaluated at $\tau = \bar\tau$ and $\epsilon = 0$. In Eq. (30) we have assumed the normal $\phi_{,\mu}$ to be parallel to $v^\mu(\bar\tau)$ since no use of it is made otherwise.

The leading term for P^μ_B is read off from (26) by using (27) and (28) to get

$$P^\mu_B \simeq \frac{e^2}{2\epsilon}\int\frac{d\Omega_0}{4\pi\gamma}v^\mu + \frac{e^2}{2\epsilon}\int\frac{d\Omega_0}{4\pi}\left(\frac{1}{2}k_\perp^\mu - \frac{1}{2}\frac{\alpha_\theta}{\gamma^2}\frac{k_\theta^\mu}{k_\theta^2} - \frac{1}{2}\frac{\alpha_\phi}{\gamma^2}\frac{k_\phi^\mu}{k_\phi^2}\right). \tag{31}$$

v^μ and k^μ are evaluated at $\tau = \bar\tau$. We see that P^μ_B has a divergent component orthogonal to v^μ in the limit $\epsilon \to 0$. This makes mass renormalization impossible in the usual sense.[1,2] The second integral in (31) vanishes when σ cuts the EWL orthogonally. This can be seen from (29) when (A3) is applied to $\Gamma = 1/\gamma = 1/\alpha$.

From now on we shall assume that σ cuts the EWL orthogonally ($\alpha = \gamma$), and therefore the leading term for P^μ_B is

$$P_B^\mu \simeq \frac{e^2}{2\epsilon}\int \frac{d\Omega_0}{4\pi\gamma} v^\mu . \tag{32}$$

In calculating the next term, the constant term when $\epsilon \to 0$, it is necessary to normalize the angles θ, ϕ to a fixed inertial frame with time axis along a unit four-vector t^μ. For this we define a lightlike vector $\eta^\mu = \eta^\mu(\theta, \phi)$ parallel to k^μ such that $\eta \cdot t = 1$. The solid angle as seen by the observer t^μ is given by

$$d\Omega = \epsilon_{\lambda\mu\nu\rho} t^\lambda \eta^\mu \eta_\theta^\nu \eta_\phi^\rho d\theta d\phi$$

just as $d\Omega_0$ is given by (15). Both are related by

$$d\Omega_0 = \frac{d\Omega}{(\eta \cdot v)^2} . \tag{33}$$

In writing (26) in terms of η^μ we need the following conversion formulas:

$$\begin{aligned}
&k^\mu = \eta^\mu / \eta \cdot v,\\
&k_\theta^\mu = \left(\eta_\theta^\mu - \frac{v\cdot\eta_\theta}{v\cdot\eta}\eta^\mu \right) \Big/ \eta \cdot v,\\
&k_\phi^\mu = \left(\eta_\phi^\mu - \frac{v\cdot\eta_\phi}{v\cdot\eta}\eta^\mu \right) \Big/ \eta \cdot v,\\
&k_\theta{}^2 = \eta_\theta{}^2/(\eta\cdot v)^2,\\
&k_\phi{}^2 = \eta_\phi{}^2/(\eta\cdot v)^2.
\end{aligned} \tag{34}$$

After a simple but long computation, we get

$$P_B^\mu = \frac{e^2}{2\epsilon}\int \frac{d\Omega}{4\pi\gamma} v^\mu + \frac{e^2}{2}\int \frac{d\Omega}{4\pi\gamma} \Bigg| \left[3\alpha k\cdot\dot{v} - \frac{\delta}{\gamma} + \left(2 - \frac{\alpha}{2\gamma}\right)\left(\frac{\alpha_\theta k_\theta \cdot \dot{v}}{k_\theta{}^2} + \frac{\alpha_\phi k_\phi \cdot \dot{v}}{k_\phi{}^2} \right)\right] k^\mu + \left(\beta_\theta + \frac{2\alpha_\theta\delta}{\gamma} - \alpha\alpha_\theta \dot{v}\cdot k\right)\frac{k_\theta^\mu}{2\gamma k_\theta{}^2} + \left(\beta_\phi + \frac{2\alpha_\phi\delta}{\gamma} - \alpha\alpha_\phi \dot{v}\cdot k\right)\frac{k_\phi^\mu}{2\gamma k_\phi{}^2} \Bigg| + O(\epsilon).$$

Severe simplification results when (30) and the partial integration formulas developed in Appendix A are used. The final result is

$$P_B^\mu = \left\{ \frac{e^2}{2\epsilon}\int\frac{d\Omega}{4\pi\gamma} + \frac{e^2}{2}\int\frac{d\Omega}{4\pi}\left[\frac{\beta}{\gamma^2} - 3\ln\gamma(\dot{v}\cdot k)\right] - \frac{e^2}{12}\phi_{,\sigma,\tau}h^{\sigma\tau}\right\} v^\mu - \tfrac{2}{3}e^2\dot{v}^\mu + e^2\phi_{,\sigma,\tau}(J^{\sigma\tau\mu} - \tfrac{1}{2}J_\nu^{\nu\sigma}h^{\tau\mu}), \tag{35}$$

where

$$J^{\sigma\tau\mu} = \int \frac{d\Omega}{4\pi}\ln\gamma k_\perp^\sigma k_\perp^\tau k_\perp^\mu .$$

All quantities are evaluated at $\tau = \bar{\tau}$, that is, at the intersection of σ and the EWL.

Equation (35) tells us that P_B^μ, as contrasted with P_{mix}^μ is highly dependent on the way we approach the singularity of the electron. The last term can be made parallel to $\dot{v}^\mu$ so as to modify the Schott[2,3] term, $-\frac{2}{3}e^2\dot{v}^\mu$, at will.

The radiated four-momentum P_R^μ is calculated in a standard way, so we just quote the result:

$$P_R^\mu = \tfrac{2}{3}e^2\int_{-\infty}^{\bar{\tau}} \dot{v}^2 v^\mu d\tau, \tag{36}$$

which is the well-known Larmor formula.

V. HOW TO REMOVE AMBIGUITIES

In this section we give a prescription in order to remove the ambiguities that show up as cutoff-dependent quantities. The following two assumptions are made:

(1) Infinities may be mass-renormalized.

(2) The total four-momentum P_μ is as independent as possible on the arbitrary spacelike surface σ used in calculating P_μ.

In a theory without infinities, assumption (1) does not arise, and assumption (2) is satisfied since P_μ is independent of σ as long as the total radiated energy-momentum remains finite.

The timelike tube Σ that surrounds the EWL is assumed to be the surface outside of which electrodynamics is valid; in this context the shape of Σ is not entirely at our disposal. Hypothesis (2) will actually restrict Σ, as will be shown below.

The total four-momentum P^μ of the electron and the external field is

$$P^\mu = P_{\text{mix}}^\mu + P_B^\mu + P_R^\mu + P_{\text{ex}}^\mu ,$$

as given by formulas (22), (35), and (36). The external four-momentum P_{ex}^μ will not enter in our considerations because it is conserved independently. The expression (35) for P_B^μ is valid only when σ cuts the EWL orthogonally. This restriction of σ is now shown to be related to the hypothesis of mass renormalization.

Mass renormalization is possible in the usual sense[1,2] only when the second integral of formula (31), defining the divergent term of P_B^μ, vanishes. That is when

$$I^\mu = \int \frac{d\Omega_0}{4\pi}\left(\frac{1}{\gamma}k^\mu - \frac{1}{2}\frac{\alpha_\theta}{\gamma^2}\frac{k_\theta^\mu}{k_\theta{}^2} - \frac{1}{2}\frac{\alpha_\phi}{\gamma^2}\frac{k_\phi^\mu}{k_\phi{}^2} \right) = 0.$$

This is so because I^μ is orthogonal to v^μ. This condition is equivalent to

$$I^\mu = -\frac{1}{\phi_\lambda v^\lambda} M^{\mu\nu}(\gamma)\,\phi_{,\nu} = 0,$$

where

$$M^{\mu\nu}(\gamma) = 2\int \frac{d\Omega_0}{4\pi\gamma} k_\perp^\mu k_\perp^\nu + h^{\mu\nu}\int \frac{d\Omega_0}{4\pi\gamma}.$$

The calculation of I^μ is done by use of Eqs. (29), (A1), and the identity $h^{\mu\nu} = -k_\perp^\mu k_\perp^\nu + (1/k_\theta{}^2)k_\theta^\mu k_\theta^\nu + (1/k_\phi{}^2)k_\phi^\mu k_\phi^\nu$. We notice that I^μ cannot vanish for arbitrary $\phi_{,\nu}$ because this would imply that $M^{\mu\nu}(\gamma)=0$, which is not possible since $n_{\mu\nu}M^{\mu\nu}(\gamma) = \int d\Omega_0/4\pi\gamma > 0$. The inequality follows from the fact that Σ never touches the EWL. We therefore conclude that mass renormalization is not possible for an arbitrary hypersurface σ, no matter what Σ may be. $\phi_{,\nu}$ is therefore chosen to be parallel to v_ν and thus I^μ vanishes for any tube Σ. Notice that γ defines Σ in order ϵ. We see that if σ cuts the EWL orthogonally mass renormalization is automatically ensured.

Once mass renormalization is performed the total four-momentum P^μ takes the form

$$P^\mu = mv^\mu - e\int_{-\infty}^{\tau} f^{\mu\nu} v_\nu\, d\tau - \tfrac{2}{3}e^2\dot{v}^\mu + e^2\phi_{,\sigma,\tau}(J^{\sigma\tau\mu} - \tfrac{1}{2}J_\nu^{\nu\sigma}h^{\tau\mu}) + P_{\text{ex}}^\mu,$$

where m is the observed electron mass. This expression depends on the shape of σ in order ϵ^2 away from the EWL, through the second fundamental form $\phi_{,\sigma,\tau}$, and on the tube Σ in order ϵ through the term $J^{\sigma\tau\mu} - \frac{1}{2}J_\nu^{\nu\sigma}h^{\tau\mu}$. Application of hypothesis (2) implies that P^μ should be independent of $\phi_{,\sigma,\tau}$ for some tube Σ if possible. This is achieved only when $J^{\sigma\tau\mu}=0$, which is a restriction on Σ given by $\int d\Omega_0(\ln\gamma)k^\sigma k^\tau k^\mu = 0$. $\gamma=\gamma(\overline{\tau})$ is the simplest candidate.

Now P^μ takes the form

$$P^\mu = mv^\mu - e\int_{-\infty}^{\tau} f^{\mu\nu} v_\nu\, d\tau - \tfrac{2}{3}e^2\dot{v}^\mu + P_{\text{ex}}^\mu,$$

and is entirely independent of σ as long as it cuts the EWL orthogonally. The Lorentz-Dirac equation now follows, as indicated in the Introduction.

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APPENDIX A

Repeated use has been made of partial integration on a two-sphere for expressions of the form

$$I^\mu = \int \frac{d\Omega_0}{4\pi}\psi\left(\Gamma_\theta \frac{k_\theta^\mu}{k_\theta{}^2} + \Gamma_\phi \frac{k_\phi^\mu}{k_\phi{}^2}\right),$$

where k_θ^μ and k_ϕ^μ are defined in Sec. II. Γ and ψ are functions of θ and ϕ. Since k_θ^μ and k_ϕ^μ are orthogonal to the electron four-velocity v^μ this integral is best evaluated in the electron instantaneous rest frame. In this frame we easily see that $\Gamma_\theta k_\theta^\mu/k_\theta{}^2 + \Gamma_\phi k_\phi^\mu/k_\phi{}^2$ is just the gradient of Γ evaluated on a unit sphere. We can therefore write

$$I^\mu = \int \frac{d\Omega_0}{4\pi}\psi\partial^\mu\Gamma = \int \frac{d^3x}{2\pi}\psi\partial^\mu\Gamma,$$

where the last integral is evaluated over the interior of a unit sphere. After partial integration and application of Stokes's theorem one gets

$$I^\mu = -\int \frac{d\Omega_0}{2\pi} k_\perp^\mu \psi\Gamma - \int \frac{d\Omega_0}{4\pi}\Gamma\left(\psi_\theta \frac{k_\theta^\mu}{k_\theta{}^2} + \psi_\phi \frac{k_\phi^\mu}{k_\phi{}^2}\right). \tag{A1}$$

This result when applied to $\psi = \kappa_\perp^{\alpha_1}\cdots k_\perp^{\alpha_n}$ gives

$$\int \frac{d\Omega_0}{4\pi} k_\perp^{\alpha_1}\cdots k_\perp^{\alpha_n}\left(\Gamma_\theta \frac{k_\theta^\mu}{k_\theta{}^2} + \Gamma_\phi \frac{k_\phi^\mu}{k_\phi{}^2}\right) = -(n+2)\int \frac{d\Omega_0}{4\pi} k_\perp^{\alpha_1}\cdots k_\perp^{\alpha_n} k^\mu \Gamma$$
$$-h^{\mu\alpha_1}\int \frac{d\Omega_0}{4\pi} k_\perp^{\alpha_2}\cdots k_\perp^{\alpha_n}\Gamma - \cdots - h^{\mu\alpha_n}\int \frac{d\Omega_0}{4\pi} k_\perp^{\alpha_1}\cdots k_\perp^{\alpha_{n-1}}\Gamma, \tag{A2}$$

which in turn, when used for the case $n=0$, gives the much used identity

$$\int \frac{d\Omega_0}{4\pi}\left(\Gamma k_\perp^\mu + \frac{1}{2}\Gamma_\theta \frac{k_\theta^\mu}{k_\theta^2} + \frac{1}{2}\Gamma_\phi \frac{k_\phi^\mu}{k_\phi^2}\right) = 0. \tag{A3}$$

For completeness we give the following integrals:

$$\int \frac{d\Omega_0}{4\pi} k_\perp^\alpha = 0,$$
$$\int \frac{d\Omega_0}{4\pi} k_\perp^\alpha k_\perp^\beta = -\tfrac{1}{3}h^{\alpha\beta}, \tag{A4}$$
$$\int \frac{d\Omega_0}{4\pi} k_\perp^\alpha k_\perp^\beta k_\perp^\gamma = 0,$$

where $k_\perp^\alpha = k^\alpha - v^\alpha$ and $h^{\alpha\beta} = \eta^{\alpha\beta} - v^\alpha v^\beta$.

APPENDIX B

For points on the two-surface S (Fig. 1) defined by (10) and (11) one gets the following formulas by series expansions:

$$z^\mu(T) = z^\mu(\bar{\tau}) - \epsilon\bar{\boldsymbol{\alpha}}\bar{\boldsymbol{v}}^\mu + \epsilon^2(\bar{\boldsymbol{\beta}}\bar{v}^\mu + \tfrac{1}{2}\bar{\alpha}^2\bar{\dot{v}}^\mu) + O(\epsilon^3),$$

$$v^\mu(T) = \bar{v}^\mu - \epsilon\bar{\alpha}\bar{\dot{v}}^\mu + \epsilon^2(\bar{\beta}\bar{\dot{v}}^\mu + \tfrac{1}{2}\bar{\boldsymbol{\alpha}}\bar{\dot{v}}^\mu) + O(\epsilon^3),$$

$$k^\mu(T, \theta\phi) = \bar{k}^\mu + \epsilon\bar{\alpha}(\bar{k}\cdot\bar{v})\bar{k}^\mu$$
$$-\epsilon^2[\bar{\boldsymbol{\beta}}\bar{k}\cdot\bar{\dot{v}} + \tfrac{1}{2}\bar{\alpha}^2(\bar{k}\cdot\bar{\dot{v}}) - \bar{\alpha}^2(\bar{k}\cdot\bar{\dot{v}})^2]$$

once use of (27) and (28) is made. The bars indicate that we evaluate the symbols at $\tau = \bar{\tau}$.

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