Infinite-component fields. II. The electromagnetic structure of bosons

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New closed forms of the inelastic form factors and structure functions $W_1(\nu,Q^2)$, $W_2(\nu,Q^2)$ as a function of two variables have been obtained in an infinite-multiplet model. The relativistic H atom and a model of the pion are special cases. The nonrelativistic limit in the former case agrees with the result of Massey and Mohr. For the latter case a special situation which leads to scaling is discussed in detail.

I. INTRODUCTION

This paper is a continuation of part I[1] and, in the same spirit, gives new results for infinite-multiplet towers with ground-state spin equal to zero, such as the pion, and the relativistic spinless H atom. We refer to part I for a general discussion of the background, motivation, and purpose of the paper.

II. ELECTROMAGNETIC FORM FACTORS

The most general electromagnetic current linear in the group generators of SO(4,2) and T_4 (space-time translation group) may be written as[1]

$$j_\mu(q)=\bar{\psi}^\dagger(p')\,(\alpha_1\Gamma_\mu+\alpha_2P_\mu+\alpha_3P_\mu S+\alpha_4L_{\mu\nu}q^\nu)\bar{\psi}(p), \tag{2.1}$$

where $\Gamma_\mu=L_{\mu 6}$, $S=L_{46}$, $P_\mu=(p'_\mu+p_\mu)$, and $q_\mu=(p'_\mu-p_\mu)$. If the particle is boosted in the third direction with rapidity η such that

$$P_\mu=(M_n\cosh\eta+M,0,0,M_n\sinh\eta),$$

$$q_\mu=(M_n\cosh\eta-M,0,0,M_n\sinh\eta),$$

then the boosted physical states in terms of the SO(4,2) group states are

$$|\bar{n}lmp\rangle=\frac{1}{N_n}e^{-i\eta L_{35}}e^{i\theta_nL_{45}}|nlm\rangle, \tag{2.2}$$

where θ_n is the tilt angle and N_n is a normalization factor that is to be determined. The quantum numbers n,l,m take the following values: $n=1,2,\ldots,\infty$, $l=0,1,\ldots,n-1$, and $-l\le m\le l$. The canonical states $|nlm\rangle$ form the discrete basis of the spin-0 representations of SO(4,2). The normalization factor N_n can be evaluated using the orthonormality condition

$$\langle\bar{n}'l'm',p'|\int j_0(x)d^3x\,|\bar{n}lm,p\rangle=2p_0\delta_{\bar{n}'\bar{n}}\delta_{l'l}\delta_{m'm}\delta^3(\vec{p}'-\vec{p}), \tag{2.2'}$$

and one obtains

$$N_n^2=(2M_n)^{-1}(\alpha_1n\cosh\theta_n+2\alpha_3nM_n\sinh\theta_n+2\alpha_2M_n). \tag{2.3}$$

The mass spectrum can be obtained from the infinite-component wave equation

$$[(\alpha_1\Gamma_\mu+\alpha_2p_\mu+\alpha_3p_\mu S)p_\mu-bS-c]\,|\bar{n}lm,p\rangle=0$$

by going to the rest frame, and one obtains

$$n^2=(\alpha_2M_n^2-c)^2/[\alpha_1^2M_n^2-(b-\alpha_3M_n^2)^2], \tag{2.4}$$

with

$$\tanh\theta_n=(b-\alpha_3M_n^2)/\alpha_1M_n, \tag{2.5}$$
$$\cosh\theta_n=\alpha_1M_nn/(c-\alpha_2M_n^2),$$
$$\sinh\theta_n=(b-\alpha_3M_n^2)n/(c-\alpha_2M_n^2).$$

In the O(4,2) boson ladder[2] the pion, say, is the ground state of the tower and hence it takes $n=1$, $l=0$, $m=0$. Using the electromagnetic current (2.1) explicitly we evaluate the three inelastic transition amplitudes for the ground state ($N'_n=\sqrt{2M_n}N_n$)

$$N'_nN'_1\langle\bar{n}l0,p|j_0(0)|\bar{1}00\rangle=(\alpha_1\cosh\theta_1+\alpha_3P_0\sinh\theta_1+\alpha_2P_0)I_{00}(0)+(\alpha_3P_0\cosh\theta_1+\alpha_1\sinh\theta_1)I_{46}(0)+\alpha_4q_3\cosh\theta_1I_{35}(0), \tag{2.6a}$$

$$N'_nN'_1\langle\bar{n}l1,p|j_1(0)|\bar{1}00\rangle=\alpha_1I_{16}(1)+\alpha_4q_0\cosh\theta_1I_{15}(1), \tag{2.6b}$$

$$N'_nN'_1\langle\bar{n}l-1,p|j_1(0)|\bar{1}00\rangle=\alpha_1I_{16}(-1)+\alpha_4q_0\cosh\theta_1I_{15}(-1), \tag{2.6c}$$

where

$$I_{AB}(m)\equiv\langle nlm|GL_{AB}|100\rangle,\quad A,B=1,2,3,4,5=0,6$$

and

$$I_{00}(m) \equiv \langle nlm | G | 100 \rangle ,$$

with

$$G = e^{-i\theta_n L_{45}} e^{i\eta L_{35}} e^{i\theta_1 L_{45}} .$$

The actions of the SO(4, 2) generators L_{AB} on the canonical basis are explicitly evaluated elsewhere.[3] We give the relevant expressions as follows:

$$\begin{aligned} L_{46}|100\rangle &= \frac{1}{\sqrt{2}}|200\rangle , \\ L_{35}|100\rangle &= \frac{1}{\sqrt{2}}|210\rangle , \\ L_{15}|100\rangle &= -\tfrac{1}{2}|211\rangle + \tfrac{1}{2}|21-1\rangle , \\ L_{16}|100\rangle &= -\tfrac{1}{2}i|211\rangle + \tfrac{1}{2}i|21-1\rangle . \end{aligned} \tag{2.7}$$

Then, as in Ref. 1, we use the following result[4] to evaluate the matrix elements $I_{AB}(m)$ and $I_{00}(m)$ of Eq. (2.6):

$$\langle n'l'm | G | nlm \rangle = \sum_{\tau=0}^{\mathrm{Min}\left\{ {n-|m|-1 \atop n'-|m|-1} \right\}} D^{[n'-1,0]}_{l',|m|+\tau,|m|}(-\alpha) V^{|m|+1+\tau}_{n',n}(\beta) D^{[n-1,0]}_{l,|m|+\tau,|m|}(-\gamma) , \tag{2.8}$$

where the D functions are the self-conjugate SO(4)-rotation functions which can be explicitly expressed in terms of Gegenbauer polynomials C as[4]

$$\begin{aligned} D^{[l_+,0]}_{lmm}(-\alpha) &= (-i)^{l-m} 2^l (2l+1)^{1/2} \Gamma(l+1) \left[\frac{\Gamma(m+\frac{3}{2})\Gamma(l_+-m+1)\Gamma(l_+-l+1)\Gamma(l+m+1)}{\Gamma(\frac{3}{2})\Gamma(l_++m+2)\Gamma(l_++l+2)\Gamma(l-m+1)\Gamma(m+1)} \right]^{1/2} (\sin\alpha)^{l-m} C^{l+1}_{l_+-l}(\cos\alpha) , \\ D^{[l_+,0]}_{l,m+1,m}(-\alpha) &= i \left[\frac{(2m+3)}{(l_++m+2)(l_+-m)} \right]^{1/2} \frac{d}{d\alpha} D^{[l_+,0]}_{lmm}(-\alpha) , \end{aligned} \tag{2.9}$$

where $l_+ = n-1$. The Bargmann functions in Eq. (2.8) can be conveniently expressed in terms of Jacobi polynomials, normalized to $P_0^{\mu,\nu}(x) = 1$, as

$$V^k_{mn}(\beta) = \left[\frac{(n-k)!\,(m+k-1)!}{(m-k)!\,(n+k-1)!} \right]^{1/2} (\tanh\tfrac{1}{2}\beta)^{m-n} (\cosh\tfrac{1}{2}\beta)^{-2k} P^{m-n,\,2k-1}_{n-k}(1-2\tanh^2\tfrac{1}{2}\beta) . \tag{2.10}$$

Furthermore, the Euler's angles α, β, and γ in Eq. (2.8) are related to the angles θ_1, θ_n, and η as given in Eqs. (3.3a) of part I.[5] Note that

$$\delta = \tfrac{1}{2}(\theta_1 - \theta_n), \quad \sigma = \tfrac{1}{2}(\theta_1 + \theta_n) . \tag{2.11}$$

Now it is quite straightforward to compute the matrix elements I_{AB} and I_{00} of Eqs. (2.6) using Eqs. (2.7) and (2.8). We give below the final results:

$$\begin{aligned} I_{00}(0) &= D^{[n-1,0]}_{l00}(-\alpha) V^1_{n1}(\beta) , \\ I_{46}(0) &= \frac{1}{\sqrt{2}} [\cos\gamma D^{[n-1,0]}_{l00}(-\alpha) V^1_{n2}(\beta) - i \sin\gamma D^{[n-1,0]}_{l10}(-\alpha) V^2_{n2}(\beta)] , \\ I_{35}(0) &= -\frac{1}{\sqrt{2}} [i \sin\gamma D^{[n-1,0]}_{l00}(-\alpha) V^1_{n2}(\beta) - \cos\gamma D^{[n-1,0]}_{l10}(-\alpha) V^2_{n2}(\beta)] , \\ -iI_{16}(1) &= I_{15}(1) = -\tfrac{1}{2} D^{[n-1,0]}_{l11}(-\alpha) V^2_{n2}(\beta) , \quad -iI_{16}(-1) = I_{15}(-1) = \tfrac{1}{2} D^{[n-1,0]}_{l11}(-\alpha) V^2_{n2}(\beta) , \end{aligned} \tag{2.12}$$

where we have used Eq. (2.9) to evaluate the SO(4) functions involving the angle γ. Substituting the above expressions in Eq. (2.6) we obtain

$$N'_nN'_1\langle\bar{n}l0,p|j_0|\bar{1}00\rangle=(\alpha_1\cosh\theta_1+\alpha_3P_0\sinh\theta_1+\alpha_2P_0)D^{[n-1,0]}_{l00}(-\alpha)V^1_{n1}(\beta)$$
$$+\frac{1}{\sqrt{2}}(\alpha_3P_0\cosh\theta_1+\alpha_1\sinh\theta_1)\cos\gamma D^{[n-1,0]}_{l00}(-\alpha)V^1_{n2}(\beta)$$
$$-\frac{i}{\sqrt{2}}(\alpha_3P_0\cosh\theta_1+\alpha_1\sinh\theta_1)\sin\gamma D^{[n-1,0]}_{l10}(-\alpha)V^2_{n2}(\beta)$$
$$-\frac{i}{\sqrt{2}}\alpha_4q_3\cosh\theta_1\sin\gamma D^{[n-1,0]}_{l00}(-\alpha)V^1_{n2}(\beta)+\frac{1}{\sqrt{2}}\alpha_4q_3\cosh\theta_1\cos\gamma D^{[n-1,0]}_{l10}(-\alpha)V^2_{n2}(\beta),\tag{2.13a}$$

$$N'_nN'_1\langle\bar{n}l1,p|j_1|\bar{1}00\rangle=-\tfrac{1}{2}(-i\alpha_1+\alpha_4q_0\cosh\theta_1)D^{[n-1,0]}_{l11}(-\alpha)V^2_{n2}(\beta),\tag{2.13b}$$

$$N'_nN'_1\langle\bar{n}l-1,p|j_1|\bar{1}00\rangle=\tfrac{1}{2}(-i\alpha_1+\alpha_4q_0\cosh\theta_1)D^{[n-1,0]}_{l11}(-\alpha)V^2_{n2}(\beta).\tag{2.13c}$$

The above expressions are the exact transition amplitudes and using them one can compute all form factors of spin-0 particles (say, pion). We will simplify the above for the special case where $l=n-1$, because in this case the Gegenbauer polynomial in (2.9) becomes unity $[C^\nu_0(x)=1]$ and the task is much simpler and furthermore from the final result we can easily deduce the appropriate elastic amplitudes. In order to evaluate this special case we need the following special values of SO(4) representation and Bargmann functions [Eqs. (2.9) and (2.10)]:

$$D^{[n-1,0]}_{n-1,0,0}(-\alpha)=(-i)^{n-1}2^{n-1}\left[\frac{(2n-1)\Gamma(n)\Gamma(n)}{n\Gamma(2n)}\right]^{1/2}\sin^{n-1}\alpha,$$
$$D^{[n-1,0]}_{n-1,1,0}(-\alpha)=(-i)^n2^{n-1}\left[\frac{3(n-1)(2n-1)\Gamma(n)\Gamma(n)}{n(n+1)\Gamma(2n)}\right]^{1/2}\sin^{n-2}\alpha\cos\alpha,$$
$$V^1_{n,1}(\beta)=\sqrt{n}\,\frac{\tanh^{n-1}(\frac{1}{2}\beta)}{\cosh^2(\frac{1}{2}\beta)},\tag{2.14}$$
$$V^1_{n,2}(\beta)=\frac{1}{\sqrt{2}}(n-1)\sqrt{n}\,\frac{\tanh^{n-1}(\frac{1}{2}\beta)}{\cosh^4(\frac{1}{2}\beta)}-(-1)^{\delta_{n1}}\sqrt{2}\,\sqrt{n}\,\frac{\tanh^n(\frac{1}{2}\beta)}{\cosh^2(\frac{1}{2}\beta)},$$
$$V_{n,2}{}^2(\beta)=\frac{1}{\sqrt{6}}[(n-1)n(n+1)]^{1/2}\frac{\tanh^{n-2}(\frac{1}{2}\beta)}{\cosh^4(\frac{1}{2}\beta)}.$$

Substituting the above in Eqs. (2.13) and after some straightforward manipulations we obtain

$$\langle\bar{n},n-1,0,p|j_0|\bar{1}00\rangle=\frac{\Gamma_nA_n}{N'_1N'_n}\frac{1}{\cosh^2\frac{1}{2}\beta}\Bigg\{e_n+\frac{\sinh\delta\cosh\delta}{\cosh^2\frac{1}{2}\beta}R_n-\frac{\sinh^2\frac{1}{2}\eta}{\cosh^2\frac{1}{2}\beta}[B_1-\sinh^2\eta(\cosh^2\sigma+\sinh^2\delta)B_2]$$
$$+\frac{(n-1)}{2\cosh^3\frac{1}{2}\beta\sinh\frac{1}{2}\beta}\Bigg[D_0-\sinh^2\tfrac{1}{2}\eta(D'_0-\sinh^2\tfrac{1}{2}\eta D''_0)$$
$$-\frac{1}{\sinh\frac{1}{2}\beta\cosh\frac{1}{2}\beta}(D_1-\sinh^2\tfrac{1}{2}\eta(D'_1-\sinh^2\tfrac{1}{2}\eta D''_1))\Bigg]\Bigg\},\tag{2.15}$$

where

$$e_n=\alpha_1\cosh\theta_1+\alpha_3(M_n+M)\sinh\theta_1+\alpha_2(M_n+M),$$
$$R_n=\alpha_3(M_n+M)\cosh\theta_1+\alpha_1\sinh\theta_1,$$
$$B_1=-2\alpha_3M_n\cosh\delta(\sinh\theta_1\cosh\delta+\cosh\theta_1\sinh\delta)+(-1)^{\delta_{n1}}2i\alpha_4M_n(\cosh^2\sigma+\sinh^2\delta)$$
$$-2M_n\alpha_2\cosh^2\delta-(-1)^{\delta_{n1}}R_n\cosh\theta_n\sinh\theta_1,$$
$$B_2=2M_n(\alpha_2-(-1)^{\delta_{n1}}i\alpha_4)+(1-\delta_{n1})4\alpha_3M_n\sinh\theta_1,$$
$$D_0=-R_n\sinh\delta\cosh\delta,$$
$$D'_0=2\alpha_3M_n\cosh\theta_1\sinh\delta\cosh\delta+R_n\sinh\theta_1\cosh\theta_n-2i\alpha_4M_n\cosh\theta_1\cosh\theta_n,$$
$$D''_0=-2\alpha_3M_n\sinh\theta_1\cosh\theta_1\cosh\theta_n+2i\alpha_4M_n\cosh\theta_1\cosh\theta_n,$$
$$D_1=\sinh\delta\cosh\delta(R_n\cosh\theta_n/\cosh\theta_1+2i\alpha_4M_n\sinh\delta\cosh\delta),$$

$$D_1' = -2\alpha_3 M_n \cosh\theta_n \sinh\delta\cosh\delta + R_n\cosh\theta_n\sinh\theta_n - 4i\alpha_4 M_n \sinh^2\delta\cosh^2\delta,$$

$$D_1'' = -2\alpha_3 M_n\cosh\theta_1\cosh\theta_n\sinh\theta_n - 2i\alpha_4 M_n\sinh\theta_1\sinh\theta_n\cosh\theta_1\cosh\theta_n,$$

$$\Gamma_n = (-1)^{n-1}\left[\frac{(2n-1)\Gamma(n)\Gamma(n)}{\Gamma(2n)}\right]^{1/2}, \quad A_n = \left(\frac{\cosh\theta_1\sinh\eta}{\cosh^2\frac{1}{2}\beta}\right)^{n-1},$$

and

$$\langle\bar{n}, n-1, 1|j_1|\bar{1}00\rangle = -\frac{\Gamma_n A_{n-1}}{2N_1'N_n'}\frac{[n(n-1)]^{1/2}}{\cosh^4\frac{1}{2}\beta}(\alpha_1 + i\alpha_4\cosh\theta_1(M_n - M) + 2i\alpha_4 M_n\cosh\theta_1\sinh^2\tfrac{1}{2}\eta), \tag{2.16}$$

$$\langle\bar{n}, n-1, -1|j_1|\bar{1}00\rangle = \frac{\Gamma_n A_{n-1}}{2N_1'N_n'}\frac{[n(n-1)]^{1/2}}{\cosh^4\frac{1}{2}\beta}(\alpha_1 + i\alpha_4\cosh\theta_1(M_n - M) + 2i\alpha_4 M_n\cosh\theta_1\sinh^2\tfrac{1}{2}\eta). \tag{2.17}$$

For elastic transitions we get ($n=1, \delta=0, \sigma=\theta_1$)

$$\langle\bar{1}00|j_0|\bar{1}00\rangle = \frac{1}{N_1'^2\cosh^2\frac{1}{2}\beta}\left[e_1 - \frac{\sinh^2\frac{1}{2}\eta}{\cosh^2\frac{1}{2}\beta}(B_1 - \sinh^2\tfrac{1}{2}\eta\cosh^2\theta_1 B_2)\right], \tag{2.18}$$

where ($M_1 = M$)

$$e_1 = \alpha_1\cosh\theta_1 + 2\alpha_3 M\sinh\theta_1 + 2\alpha_2 M,$$

$$B_1 = e_1\sinh^2\theta_1 - 2M(\alpha_2 + i\alpha_4)\cosh^2\theta_1,$$

$$B_2 = 2M(\alpha_2 + i\alpha_4),$$

and the other two amplitudes [Eqs. (2.16) and (2.17)] vanish. The spin-0 form factor $G(Q^2)$ is conventionally defined[2] as

$$\langle\bar{n}lm, p|j_\mu|\bar{1}00\rangle \equiv G(Q^2)P_\mu/2M_n, \quad Q^2 \equiv -q^2, \tag{2.18'}$$

and hence correspondingly the elastic electric form factor becomes

$$G_E(Q^2) = \cosh^{-2}\tfrac{1}{2}\eta\langle\bar{1}00, p|j_0|\bar{1}00\rangle$$
$$= \frac{1}{N_1'^2}\left(1 + \frac{Q^2}{4M^2}\cosh^2\theta_1\right)^{-1}\left\{2M(\alpha_2 + i\alpha_4) - [e_1 - 2M(\alpha_2 + i\alpha_4)]\left(1 + \frac{Q^2}{4M^2}\cosh^2\theta_1\right)^{-1}\right\}, \tag{2.18''}$$

where we have substituted

$$\cosh^2\tfrac{1}{2}\beta = \left(1 + \frac{Q^2}{4M^2}\cosh^2\theta_1\right).$$

Equations (2.15) and (2.18) show that the pion form factor has a leading single pole at $Q^2 \cong -4M_\pi^2$ and explicitly it is independent of the parameter α_3. One could also expect from Eq. (2.13a) the similar leading single-pole behavior for the inelastic form factors. The above expression seems to suggest that the convective term $\alpha_2 P_\mu$ in the electromagnetic current [Eq. (2.1)] is extremely crucial and it is this term which, in the absence of α_3, gives a linearly increasing mass spectrum for bosons. Furthermore, if we take $\alpha_4 = 0$ then

$$G_E(Q^2) = \frac{1}{N_1'^2}\left(1 + \frac{\cosh^2\theta_1}{4M^2}Q^2\right)^{-1}$$
$$\times\left[2\alpha_2 M + \left(1 + \frac{\cosh^2\theta_1}{4M^2}Q^2\right)^{-1}(e_1 - 2\alpha_2 M)\right]. \tag{2.18'''}$$

Also, the above expressions show that the presence of the saturation term $\alpha_3 P_\mu$ in the current (2.1) is not essential to obtain a single-pole behavior for the pion form factor, and it seems that the current with $\alpha_3 = \alpha_4 = 0$ would be much suitable for pions . More experimental evidence is needed to establish this conclusion. As in the case of the hydrogen atom the choice of $\alpha_4 = 0$ (α_4 is the anomalous magnetic moment term) for mesons seems to be quite reasonable. But the vanishing of α_3 depends on whether mesons satisfy a nonsaturating mass spectrum or not. When $\alpha_2 = 0$ (α_3 need not be zero) the pion form factor has exact dipole behavior. [Remember the definition (2.18'). The form factors $G(Q^2)$ are related by different factors to the matrix element of j_μ for spin-0 and spin-$\frac{1}{2}$ cases).]

In the case of the hydrogen atom one knows exactly the values of the constants $\alpha_1 \cdots \alpha_4$ and the constants b and c appearing in the infinite-component wave equation [Eq. (2.4).] They are $\alpha_1 = 1$, $\alpha_2 = -\alpha/2m_p$, $\alpha_3 = 1/2m_p$, $\alpha_4 = 0$, $b = (m_e^2 - m_p^2)/2m_p$, and $c = -(\alpha/2m_p)(m_p^2 + m_e^2)$, where α is the fine-structure constant and m_p and m_e are the masses of the proton and the electron, respectively. The mass spectrum from Eq. (2.4) becomes

$$M_n^2 = m_p^2 + m_e^2 + 2m_p m_e \left(1 - \frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}$$

$$\approx (m_p + m_e)^2\left[1 - \frac{\alpha^2}{n^2}\frac{m_p m_e}{(m_p+m_e)^2} + O(\alpha^4)\right]. \quad (2.19)$$

Also, from Eqs. (2.5) and (2.3)

$$\cosh\theta_n \approx \frac{n}{\alpha m_e}(m_p+m_e)\left[1+\frac{\alpha^2}{2n^2}\left(1-\frac{m_p m_e}{(m_p+m_e)^2}\right)+O(\alpha^4)\right],$$

$$\sinh\theta_n \approx \frac{-n}{\alpha m_e}(m_p+m_e)\left[1+\frac{\alpha^2}{2n^2}\frac{m_p}{m_p+m_e}+O(\alpha^4)\right],$$

$$N_n'^2 \approx \frac{n^2}{\alpha m_e}(m_p+m_e)\left[\frac{1}{2}\frac{\alpha^2}{n^2}\frac{m_e^2}{(m_p+m_e)^2} - \frac{m_e}{m_p}\left(1+\frac{\alpha^2}{n^2}\right)\right].$$

After substituting the above values in Eq. (2.13a) we obtain

$$\frac{\alpha m_e N_1' N_n'}{(m_p+m_e)}\langle \bar{n}l0,p|j_0|\bar{1}00\rangle = \left(1-\frac{P_0}{2m_p}\right)\left\{\left(1+\tfrac{1}{2}\alpha^2\left[1-\frac{m_p m_e}{(m_p+m_e)^2}-\frac{P_0}{2m_p}\left(1+\frac{m_e}{m_p+m_e}\right)\right]\left(1-\frac{P_0}{2m_p}\right)^{-1}\right)D_{l00}^{[n-1,0]}(-\alpha)V_m^1(\beta)\right.$$

$$-\frac{1}{\sqrt{2}}\left[1-\tfrac{1}{2}\alpha^2\left(\frac{P_0}{2m_p}\right)\left(1-\frac{m_p m_e}{(m_p+m_e)^2}-\frac{m_p}{m_p+m_e}\right)\left(1-\frac{P_0}{2m_p}\right)^{-1}\right]$$

$$\left.\times\left[\cos\gamma D_{l00}^{[n-1,0]}(-\alpha)V_{n2}^1(\beta) - i\sin\gamma D_{l10}^{[n-1,0]}(-\alpha)V_{n2}^2(\beta)\right]\right\}. \quad (2.20)$$

In the nonrelativistic limit [$\alpha \to 0$, or essentially $\theta_n = \ln(n)$, $\theta_1 = 0$, and $P_0 \approx 2(m_p + m_e)$] the above relation will reduce to the following expression:

$$\langle \bar{n}l0,p|j_0|\bar{1}00\rangle = \frac{1}{n}\left[D_{l00}^{[n-1,0]}(-\alpha)V_{n1}^1(\beta) - \frac{1}{\sqrt{2}}\cos\gamma D_{l00}^{[n-1,0]}(-\alpha)V_{n2}^1(\beta) + \frac{i}{\sqrt{2}}\sin\gamma D_{l10}^{[n-1,0]}(-\alpha)V_{n2}^2(\beta))\right]. \quad (2.21)$$

This is precisely the same classical nonrelativistic expression derived by Massey and Mohr.[6] Therefore, in Eq. (2.20) the coefficients of α_2 are indeed the first-order (α^2) relativistic corrections to the Massey-Mohr formula. Furthermore, the elastic transition form factor for the hydrogen atom can be obtained from Eq. (2.18″) with $\alpha^2 = -\alpha/2m_p$. For large Q^2 the form factor shows a single-pole behavior. However, when $\alpha_2 = 0$ the form factor shows an exact dipole behavior.

III. ELECTROMAGNETIC STRUCTURE FUNCTIONS

In order to calculate the usual electromagnetic structure functions[1] MW_1 and νW_2, we have to evaluate first the tensor components W_{11} and W_{33} defined in the zero-width approximation as

$$W_{11} = \sum_{nlm} \delta((p+q)^2 - M_n^2)\,|\langle \bar{n}lm,p|j_1(0)|\bar{1}00\rangle|^2, \quad (3.1a)$$

$$W_{33} = \sum_{nlm} \delta((p+q)^2 - M_n^2)\,|\langle \bar{n}lm,p|j_3(0)|\bar{1}00\rangle|^2. \quad (3.1b)$$

We substitute the current (2.1) and do the summation over m explicitly and obtain in terms of the matrix elements $I_{AB}(m)$ and $I_{00}(m)$ of Eq. (2.6) [cross terms include complex conjugates also, for example, $I_{16}(1)I_{15}^*(1)$ is $I_{16}(1)I_{15}^*(1)+I_{15}(1)I_{16}^*(1)$]

$$W_{11} = \sum_{nl} \frac{\delta((p+q)^2 - M_n^2)}{N_1^2 N_n^2}\{\alpha_1^2[I_{16}(1)I_{16}^*(1)+I_{16}(-1)I_{16}^*(-1)] + \alpha_4^2 q_0^2\cosh^2\theta_1[I_{15}(1)I_{15}^*(1)+I_{15}(-1)I_{15}^*(-1)]$$

$$+\alpha_1\alpha_4 q_0\cosh\theta_1[I_{16}(1)I_{15}^*(1)+I_{16}(-1)I_{15}^*(-1)]\}, \quad (3.2a)$$

$$W_{33}=\sum_{nl}\frac{\delta((p+q)^2-M_n^2)}{N_1^2N_n^2}[\alpha_1^2I_{36}(0)I_{36}^*(0)+\alpha_3^2P_3^2\cosh^2\theta_1I_{46}(0)I_{46}^*(0)$$
$$+(\alpha_2P_3+\alpha_3P_3\sinh\theta_1)^2I_{00}(0)I_{00}^*(0)+\alpha_4^2q_0^2\cosh^2\theta_1I_{35}(0)I_{35}^*(0)$$
$$+\alpha_1\alpha_3P_3\cosh\theta_1I_{36}(0)I_{46}^*(0)+(\alpha_1\alpha_2P_3+\alpha_1\alpha_3P_3\sinh\theta_1)I_{36}(0)I_{00}^*(0)$$
$$+\alpha_1\alpha_4q_0\cosh\theta_1I_{36}(0)I_{35}^*(0)+\alpha_3P_3\cosh\theta_1(\alpha_2P_3+\alpha_3P_3\sinh\theta_1)I_{46}(0)I_{00}^*(0)$$
$$+\alpha_3\alpha_4P_3q_0\cosh^2\theta_1I_{46}(0)I_{35}^*(0)+\alpha_4q_0\cosh\theta_1(\alpha_2P_3+\alpha_3P_3\sinh\theta_1)I_{00}(0)I_{35}^*(0)]. \quad (3.2b)$$

In the above two equations, the summation over l can be done by using the orthonormality condition of the SO(4)-rotation functions, since each matrix element is expressed [Eq. (2.12)] in terms of SO(4) and Bargmann functions, and l is built only into the former. Therefore, using

$$\sum_l D_{ll'm}^{[l_+,l_-]}(\alpha)D_{ll''m}^{[l_+,l_-]}(-\alpha)=\delta_{l'l''}$$

we obtain exactly

$$\sum_l I_{36}(0)I_{36}^*(0)=\tfrac{1}{2}\sin^2\gamma[V_{n2}^1(\beta)]^2+\tfrac{1}{2}\cos^2\gamma[V_{n2}^2(\beta)]^2,$$

$$\sum_l I_{35}(0)I_{35}^*(0)=\tfrac{1}{2}\sin^2\gamma[V_{n,2}^1(\beta)]^2+\tfrac{1}{2}\cos^2\gamma[V_{n,2}^2(\beta)]^2,$$

$$\sum_l I_{46}(0)I_{46}^*(0)=\tfrac{1}{2}\cos^2\gamma[V_{n,2}^1(\beta)]^2+\tfrac{1}{2}\cos^2\gamma[V_{n2}^2(\beta)]^2,$$

$$\sum_l I_{00}(0)I_{00}^*(0)=[V_{n,1}^1(\beta)]^2,$$

$$\sum_l I_{36}(0)I_{46}^*(0)=\tfrac{1}{2}\sin2\gamma\{[V_{n,2}^1(\beta)]^2-[V_{n,2}^2(\beta)]^2\},$$

$$\sum_l I_{36}(0)I_{00}^*(0)=\sqrt{2}\,\sin\gamma V_{n,1}^1(\beta)V_{n,2}^1(\beta), \quad (3.3)$$

$$\sum_l I_{46}(0)I_{00}^*(0)=\sqrt{2}\,\cos\gamma V_{n,1}^1(\beta)V_{n,2}^1(\beta),$$

$$\sum_l I_{16}(1)I_{16}^*(1)=\tfrac{1}{4}[V_{n,2}^2(\beta)]^2,$$

$$\sum_l I_{15}(1)I_{15}^*(1)=\tfrac{1}{4}[V_{n,2}^2(\beta)]^2,$$

$$\sum_l I_{16}(-1)I_{16}^*(-1)=\tfrac{1}{4}[V_{n,2}^2(\beta)]^2,$$

$$\sum_l I_{15}(-1)I_{15}^*(-1)=\tfrac{1}{4}[V_{n,2}^2(\beta)]^2.$$

We then substitute Eqs. (3.3) in Eqs. (3.2a) and (3.2b).

Then in order to compute W_{11} and W_{33} completely we have to do only the summation over n and this can be done using the identity

$$\delta(g(n))=\sum_i\frac{\delta(n-n_i)}{|g'(n)|_{n=n_i}},\quad g(n_i)=0,\quad i=1,2,\ldots\,. \quad (3.4)$$

If we take into account only the positive root of n from Eq. (2.4) then

$$\delta(g(n))=\frac{\delta(n-N)}{|g'(n)|_{n=N}},$$

where

$$g(n)=(p+q)^2-M_n^2,\quad N=\frac{(\alpha_2W-c)}{[\alpha_1^2W-(b-\alpha_3W)^2]^{1/2}},\quad W=(p+q)^2$$
$$|g'(n)|_{n=N}=\mathrm{Mod}\left\{\frac{2[\alpha_1^2W-(b-\alpha_3W)^2]^{3/2}}{W[\alpha_2(\alpha_1^2+2\alpha_3b)-2\alpha_3^2c]+[c(\alpha_1^2+2\alpha_3b)-2b^2\alpha_2]}\right\}. \quad (3.5)$$

Thus once the tensor components W_{11} and W_{33} are explicitly evaluated, then the electromagnetic structure functions may be obtained by using the relations[1]

$$MW_1(Q^2,\nu)=MW_{11}(p,Q),$$
$$\nu W_2(Q^2,\nu)=\frac{\nu}{(1+\nu^2/Q^2)}W_{11}(p,Q)+\frac{Q^2}{\nu(1+\nu/Q^2)}W_{33}(p,Q).$$

Substituting W_{11} and W_{33} we get

$$MW_1(Q^2,\nu)=\frac{M}{N_1^2}\frac{1}{N_N^2}\frac{1}{|g'(N)|}b_{11}^{(1)}(\nu,Q^2)g_{11}^{(1)}(N,W),$$

$$\nu W_2(Q^2,\nu)=N_1^{-2}N_N^{-2}|g'(N)|^{-1}\left[\frac{\nu}{(1+\nu^2/Q^2)}b_{11}^{(1)}g_{11}^{(1)}+\frac{Q^2}{\nu(1+\nu^2/Q^2)}\sum_{i=1}^{4}b_{33}^{(i)}g_{33}^{(i)}\right], \tag{3.6}$$

where N and $g'(N)$ have been given in Eq. (3.5) and

$$N_N^2=\frac{[\alpha_1^2+2\alpha_3(b-\alpha_3W)](c-\alpha_2W)}{2[\alpha_1^2W-(b-\alpha_3W)^2]}+\alpha_2,$$

$$b_{11}^{(1)}(\nu,Q^2)=\tfrac{2}{3}(\alpha_1^2+\alpha_4^2\cosh^2\theta_1\nu^2),$$

$$b_{33}^{(1)}(\nu,Q^2)=4\left(\frac{-\alpha_1^2}{c-\alpha_2W}+\alpha_3\cosh^2\theta_1\frac{b-\alpha_3W}{c-\alpha_2W}-\alpha_1\alpha_3\cosh\theta_1\sinh\theta_1\frac{\nu+M}{c-\alpha_2W}\right)^2(\nu^2+Q^2)+4\alpha_1^2\alpha_4^2\cosh^2\theta_1\frac{\nu^2(\nu^2+Q^2)}{(c-\alpha_2W)^2},$$

$$b_{33}^{(2)}(\nu,Q^2)=\frac{4}{3}\left(\alpha_1\cosh\theta_1\frac{b-\alpha_3W}{c-\alpha_2W}-\alpha_1^2\sinh\theta_1\frac{\nu+M}{c-\alpha_2W}+\alpha_1\alpha_3\cosh\theta_1\frac{\nu^2+Q^2}{c-\alpha_2W}\right)^2$$
$$-\tfrac{4}{3}\alpha_4^2\cosh^2\theta_1\left(\cosh\theta_1\frac{b-\alpha_3W}{c-\alpha_2W}-\alpha_1\sinh\,\theta_1\frac{\nu+M}{c-\alpha_2W}\right)^2\nu^2,$$

$$b_{33}^{(3)}(\nu,Q^2)=8(\nu^2+Q^2)\left(\frac{-\alpha_1^2}{c-\alpha_2W}+\alpha_3\cosh^2\theta_1\frac{b-\alpha_3W}{c-\alpha_2W}-\alpha_1\alpha_3\cosh\theta_1\sinh\theta_1\frac{\nu+M}{c-\alpha_2W}\right)(\alpha_2+\alpha_3\sinh\theta_1),$$

$$b_{33}^{(4)}(\nu,Q^2)=4(\nu^2+Q^2)(\alpha_2+\alpha_3\sinh\theta_1)^2,$$

$$g_{11}^{(1)}(N,W)=\left(\frac{\kappa N-1}{\kappa N+1}\right)^N\frac{N(N-1)(N+1)}{(\kappa N+1)^2(\kappa N-1)^2},$$

$$g_{33}^{(1)}(N,W)=\left(\frac{\kappa N-1}{\kappa N+1}\right)^N\frac{N^3}{(\kappa N+1)^3(\kappa N-1)}\left[1-\frac{2(N-1)}{(\kappa^2N^2-1)^{1/2}}+\frac{(N-1)^2}{\kappa^2N^2-1}\right],$$

$$g_{33}^{(2)}(N,W)=\left(\frac{\kappa N-1}{\kappa N+1}\right)^N\frac{N^3(N-1)(N+1)}{(\kappa N+1)^3(\kappa N-1)^3},$$

$$g_{33}^{(3)}(N,W)=\left(\frac{\kappa N-1}{\kappa N+1}\right)^N\frac{N^2}{(\kappa N+1)^2(\kappa N-1)}\left[\frac{N-1}{(\kappa^2N^2-1)^{1/2}}-1\right],$$

$$g_{33}^{(4)}(N,W)=\left(\frac{\kappa N-1}{\kappa N+1}\right)^N\frac{N}{(\kappa N+1)(\kappa N-1)},$$

$$\kappa\equiv\alpha_1\cosh\theta_1\frac{\nu+M}{c-\alpha_2W}-\sinh\theta_1\frac{b-\alpha_3W}{c-\alpha_2W},$$

and the kinematical quantities

$$\nu=\frac{Q^2}{2M\xi}-\tfrac{1}{2}M,\quad Q^2=\xi(M^2+2M\nu),\quad 0<\xi\leq 1$$
$$W=Q^2(1-\xi)/\xi.$$

Equations (3.6a) and (3.6b) give the exact final results for the structure functions MW_1 and νW_2. In the next section we will consider a special case for the pion and will evaluate the structure functions in the scaling limit ($\nu,Q^2\to\infty$, ξ fixed).

IV. A SPECIAL CASE

In this section we confine ourself to a specific model in which the mesons ($I=1$) lie on a linear mass trajectory.[2] We choose $\alpha_4=\alpha_3=0$ in the electromagnetic current (2.1) and the corresponding mass spectrum from Eq. (2.4) will be

$$n^2=(\alpha_2M_n^2-c)^2/(\alpha_1^2M_n^2-b^2). \tag{4.1}$$

In the scaling limit ($M_n^2\to\infty$), $n^2\approx(\alpha_2^2/\alpha_1^2)W$, or $n\approx(\alpha_2/\alpha_1)\sqrt{W}$. We give below all the relevant quantities needed to calculate the structure functions [Eqs. (3.6a) and (3.6b)] in the scaling limit:

$$N^2\to W\frac{\alpha_2^2}{\alpha_1^2},$$

$$N_N^2\to\alpha_2/2+O\left(\frac{1}{W}\right),$$

$$|g'(N)|\to\sqrt{W}\,\frac{2\alpha_1}{\alpha_2}+O\left(\frac{1}{W}\right),$$

$$\kappa N\to\sqrt{W}\frac{\cosh\theta_1}{2M(1-\xi)},$$

$$b_{11}^{(1)}(\nu,Q^2)\to\tfrac{2}{3}\alpha_1^2,$$

$$b_{33}^{(1)}(\nu,Q^2)\to\frac{\alpha_1^4}{\alpha_2^2}/M^2(1-\xi)^2,$$

$$b_{33}^{(2)}(\nu,Q^2)\to\frac{1}{3}\frac{\alpha_1^4}{\alpha_2^2}\sinh\theta_1/M^2(1-\xi)^2,$$

$$b_{33}^{(3)}(\nu,Q^2)\to 4\nu\frac{\alpha_1^2}{M(1-\xi)},$$

$$b_{33}^{(4)}(\nu,Q^2)\to 4\nu\alpha_2^2,$$

$$g_{11}^{(1)}(N,W)\to W^{-1/2}\frac{\alpha_1}{\alpha_2}B^4,$$

$$B=\frac{\alpha_2}{\alpha_1}\frac{2M(1-\xi)}{\cosh\theta_1}$$

$$g_{33}^{(1)}(N,W)\to W^{-1/2}\frac{\alpha_1}{\alpha_2}B^4(1-B)^2,$$

$$g_{33}^{(2)}(N,W)\to W^{-1/2}\frac{\alpha_1^2}{\alpha_2^2}B^6,$$

$$g_{33}^{(3)}(N,W)\to W^{-1/2}\frac{\alpha_1}{\alpha_2}B^3(B-1),$$

$$g_{33}^{(4)}(N,W)\to W^{-1/2}\frac{\alpha_1}{\alpha_2}B^2.$$

Substituting the above asymptotic limits in Eqs. (3.6b) we obtain

$$MW_1(Q^2,\nu)=O(1/\nu), \quad (4.2a)$$

$$\nu W_2(Q^2,\nu)=\frac{4}{N_1^2}\frac{\alpha_2^3}{\alpha_1^2}\frac{(2M)^3}{\cosh^2\theta_1}\xi^2(1-\xi). \quad (4.2b)$$

Thus we find, in this specific case, that the structure function MW_1 vanishes but νW_2 remains finite in the scaling limit. The vanishing MW_1 is characteristic of a boson target[7] (pion). The threshold behavior of $\nu W_2 \approx (1-\xi)$ is consistent with the general prediction[8] that if the spin-0 form factor $G(Q^2)\sim(Q^2)^{-m}$, as $Q^2\to\infty$, then more precisely $\nu W_2\sim(1-\xi)^p$ with $p=2m-2, m\geq 2; p=m, m\leq 2$. In our model (with $\alpha_3=\alpha_4=0$) the pion form factor has a leading single-pole behavior [Eq. (2.19)] and hence $m=1$. This means that the threshold behavior of $\nu W_2\sim(1-\xi)$, in agreement with our result.

V. CONCLUSIONS AND COMMENTS

Our model for the determination of the pion form factors and structure functions is, in principle, very simple: We have used the simplest boson representation of $O(4,2)$ suitable to the hydrogen atom and consequently the multiplet structure of mesons (say for $I=1$) is that of the hydrogen atom. Therefore, the spin parity assignments are $(0^-)_{l=0}^{n=1}$, $(0^-,1^+)_{l=0,1}^{n=2}$, $(0^-,1^+,2^-)_{l=0,1,2}^{n=3}$, ... Hence, 1^- vector mesons lie in a different tower of multiplets. However, within the framework of such an assignment we have succeeded in obtaining a closed expression for the inelastic transition form factors and structure functions by using the most general conserved electromagnetic current [Eq. (2.1)]. Perhaps such a current might be too general for mesons and only through experimental inputs can one determine which terms in the current should be retained and which ones should be deleted. But unfortunately only very little and uncertain experimental evidence is available now regarding the electromagnetic structure of the pion and it is insufficient to determine anything conclusively concerning the current. Nevertheless, the specific model [$\alpha_3=\alpha_4=0$ in the current (2.1)] we have considered shows that the inelastic transition form factors have a leading single-pole behavior (at $Q^2=-4M_\pi^2/\cosh^2\theta_1$) and this seems to be in agreement with many theoretical models and experimental fits.[9] Furthermore, the structure function $M_\pi W_1$ vanishes in the scaling limit as $1/\nu$ just as predicted by Callan and Gross and Landshoff, Polkinghorne and Short[7] for spin-0 targets (more precisely for spin-0 partons). Also, the structure function νW_2 scales and its threshold behavior ($\xi\to 1$) is in complete agreement with the predictions made by Hughes[8] concerning the relation between the threshold behavior of νW_2 and the power of falloff in Q^2 of the excitation form factors. Hence we believe that the present model[5] by itself might be a good approximate model for pions or at least it would give enough clues to construct a better model within the framework of O(4, 2) theory in the future when more clear experimental evidence would be available.

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