Infinite-component fields. I. Electromagnetic inelastic form factors and structure functions of the proton

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New closed forms of the inelastic form factors and structure functions $W_1(\nu,Q^2)$, $W_2(\nu,Q^2)$ as a function of two variables have been obtained in an infinite-multiplet model in order to study the onset of scaling. Summations over intermediate states have been carried out before going to the scaling limit. Assumptions of the phenomenological models of scaling about form factors are explicitly derived in the present theory.

I. INTRODUCTION

This work is a further development of the calculation of the electromagnetic structure functions of the proton from the infinite-multiplet model of the proton. The new features and results are as follows:

(i) Spin is taken care of whereas previously, for simplicity, a spinless infinite-component proton was considered.

(ii) New closed exact forms of inelastic form factors are given.

(iii) The summation over intermediate states is done before taking the scaling limit so that we have actually closed expressions for $W_1(\nu, Q^2)$ and $\nu W_2(\nu, Q^2)$ as a function of ν and Q^2 valid in all regions (averaged over the resonances, because we do not attach widths to the resonances); hence no approximations are involved. As a result much simpler formulas are obtained than previously. One can therefore study explicitly the possible onset of scaling.

(iv) All terms in the infinite-component wave equations are considered.

Our result confirms most of the successful phenomenological assumptions made in the resonance models of scaling concerning the behavior of inelastic form factors, that they are essentially of the dipole form as the ground state with slight variations which we give. Further, the dominance of magnetic form factors in the structure functions is established. We also derive the correct threshold behavior, the Drell-Yan-West condition, and the Callan-Gross relation exactly.

II. KINEMATICS, SCALING, AND THE MODEL

In inelastic electron-proton scattering[1] an energetic incoming electron scatters off a proton target of mass M and four-momentum p, and consequently a jet of particles characterized by the over-all mass M_n and four-momentum p_n is produced, such that $M_n{}^2 = M^2 + 2M\nu - Q^2$ $(Q^2 = -q^2)$, where the momentum q is assumed to be transferred through one spacelike photon $(q^2 < 0)$ during the electromagnetic interactions. The quantities $q^2 = (p_n - p)^2$ and $M\nu = p_\mu q^\mu$ are Lorentz-invariants. In terms of the Bloom-Gilman scaling parameter ξ the kinematical region in which the above inclusive reation takes place is restricted by $0 < \xi \leq 1$, where $\xi \equiv Q^2(2M\nu + M^2)^{-1}$. The total cross section corresponding to the hadronic part is provided in terms of locally conserved current by the covariant tensor

$$W_{\mu\nu}(p,q) = \frac{1}{2}\frac{1}{4\pi^2\alpha}\sum_{|n\rangle}\delta((p+q)^2 - M_n{}^2) \times \langle p|j_\mu|n\rangle\langle n|j_\nu|p\rangle, \quad (2.1)$$

where the states are covariantly normalized by

$$\langle p|p'\rangle = (2\pi)^3 2p_0\delta^3(\vec{p} - \vec{p}').$$

According to the nature of the intermediate states $|n\rangle$, $W_{\mu\nu}$ may be cluster-decomposed into the sum of the contributions due to connected, semidisconnected, and pair diagrams. But the kinematical constraint $0 < \xi \leq 1$ (or $q^2 < 0$, $M_n{}^2 \geq 0$) precludes the contributions from the latter two types of diagrams and allows only the contribution from the connected diagram as relevant to the inelastic scattering. This means that the intermediate states $|n\rangle$ in Eq. (2.1) are all timelike states (discrete as well as continuum). Furthermore, $W_{\mu\nu}$ as given by Eq. (2.1) is essentially the absorptive part of the spin-averaged virtual forward Compton scattering amplitude.

The tensor $W_{\mu\nu}$ is usually expressed (Gehlen-Gourdin-Bjorken theorem), taking into account the conservation of electromagnetic current, in terms of the two invariant functions, called structure functions $W_1(\nu, q^2)$ and $W_2(\nu, q^2)$, as

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{1}{q^2}q_\mu q_\nu\right)W_1(\nu, q^2) + \frac{1}{M^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)W_2(\nu, q^2). \quad (2.2)$$

We now introduce two other invariant functions W^μ_μ and $p^\mu p^\nu W_{\mu\nu}$ and invert Eq. (2.2) as

$$W_1(\nu, Q^2) = \frac{1}{2M^2}\left(1+\frac{\nu^2}{Q^2}\right)^{-1} \times \left[-M^2\left(1+\frac{\nu^2}{Q^2}\right)W^\mu_\mu + p^\mu p^\nu W_{\mu\nu}\right] \quad (2.3a)$$

and

$$W_2(\nu, Q^2) = \frac{1}{2M^2}\left(1+\frac{\nu^2}{Q^2}\right)^{-2} \times \left[-M^2\left(1+\frac{\nu^2}{Q^2}\right)W^\mu_\mu + 3p^\mu p^\nu W_{\mu\nu}\right]. \quad (2.3b)$$

These four invariant functions satisfy the following conditions[2] (with ξ fixed):

(i) They are odd with respect to ν.

(ii) They depend only on ν and Q^2 (radially symmetric).

(iii) They vanish in the domain $\xi > 1$ (spectrality condition).

(iv) They are positive-definite for $q^2 < 0$.

In the laboratory frame, which is especially convenient for the kinematical analysis, the four-momenta take the value $p_\mu = (p_0, 0, 0, 0)$ and the four-momentum transfer $q_\mu = (\nu, 0, 0, q_3)$, and therefore Eqs. (2.3) become (using the gauge condition $q^\mu W_{\mu\nu} = 0 = q^\nu W_{\mu\nu}$)

$$W_1(\nu, Q^2) = W_{11}(p, q), \quad (2.4a)$$

$$\begin{aligned} W_2(\nu, Q^2) &= \left(1+\frac{\nu^2}{Q^2}\right)^{-1} W_{11}(p,q) + \frac{Q^2}{\nu^2}\left(1+\frac{\nu^2}{Q^2}\right)^{-1} W_{33}(p,q), \\ &= \left(1+\frac{\nu^2}{Q^2}\right)^{-1} W_{11}(p,q) + \left(1+\frac{\nu^2}{Q^2}\right)^{-2} W_{00}(p,q). \end{aligned} \quad (2.4b)$$

Substituting Eq. (2.1) in the above we obtain

$$W_1(\nu, Q^2) = \frac{1}{2}\frac{1}{4\pi^2\alpha}\sum_{|n\rangle} \delta((p+q)^2 - M_n{}^2) h_1{}^2 [G_1(Q^2)]^2, \quad (2.5a)$$

$$W_2(\nu, Q^2) = \frac{1}{2}\frac{1}{4\pi^2\alpha}\sum_{|n\rangle} \delta((p+q)^2 - M_n{}^2)\left\{h_0{}^2\frac{[G_0(Q^2)]^2}{(1+\nu^2/Q^2)^2} + h_1{}^2\frac{[G_1(Q^2)]^2}{(1+\nu^2/Q^2)}\right\}, \quad (2.5b)$$

where we have defined the proton inelastic form factors $G_1(Q^2)$ and $G_0(Q^2)$ as

$$\langle n | j_1(0) | p\rangle = h_1 G_1(Q^2), \qquad \langle n | j_0(0) | p\rangle = h_0 G_0(Q^2). \quad (2.5c)$$

One may also introduce "universal" excitation form factor

$$[G(Q^2)]^2 = \left\{h_0{}^2\frac{[G_0(Q^2)]^2}{(1+\nu^2/Q^2)^2} + h_1{}^2\frac{[G_1(Q^2)]^2}{(1+\nu^2/Q^2)}\right\}, \quad (2.5d)$$

so that

$$W_2(\nu, Q^2) = \frac{1}{2}\frac{1}{4\pi^2\alpha}\sum_{|n\rangle} \delta[(p+q)^2 - M_n{}^2] \times [G(Q^2)]^2. \quad (2.5e)$$

The functions h_1 and h_0 we have introduced in Eqs. (2.5c) depend on the nature of the initial and final states (for example, the boost angle, mass, etc.) and also include coefficients related to the normal and abnormal parity transitions. If the final state has also spin $\frac{1}{2}$ (say proton) then h_0 and h_1 can be explicitly evaluated using the Dirac current. But, in general, for final states with any intrinsic semi-integer spin, determination[3] of these quantities is a matter of convention and definition of G's and is nontrivial. The situation would be more complicated if the final states were infinite-component states with unlimited semi-integer spins, as we will see later. However, the general expresssions for the excitation form factors we have derived in the next section seem to suggest that h_1 and h_0 might assume a value either $\cosh\frac{1}{2}\eta$ or $\sinh\frac{1}{2}\eta$, where η is the boost angle, depending on the parity and the nature (odd or even) of the quantum numbers of the state $|n\rangle$. In the elastic case one obtains[4]

$$\begin{aligned} h_0^{\text{el}} &= \cosh\tfrac{1}{2}\eta = \left(1+\frac{\nu^2}{Q^2}\right)^{1/2} = \left(1+\frac{Q^2}{4M^2}\right)^{1/2}, \\ h_1^{\text{el}} &= \sinh\tfrac{1}{2}\eta = \left(\frac{\nu^2}{Q^2}\right)^{1/2} = \left(\frac{Q^2}{4M^2}\right)^{1/2}, \end{aligned} \quad (2.6a)$$

and hence the elastic "excitation" form factor becomes[1]

$$[G(Q^2)]^2_{\text{el}} = \left(1+\frac{Q^2}{4M^2}\right)^{-1} \times \left([G_E(Q^2)]^2 + \frac{Q^2}{4M^2}[G_M(Q^2)]^2\right), \quad (2.6b)$$

where G_E and G_M are the usual Sachs form factors. Experimentally, up to the largest momentum transfer so far studied, the form factors G_M and G_E are monotonically decreasing functions of Q^2 and it appears that $G_M(Q^2)$, normalized to $G_M(0)=1$, is reasonably approximated by the dipole formula[5]

$$G_M(Q^2) = \left(1+\frac{Q^2}{t_1}\right)^{-2}, \quad (2.7)$$

where the proton dipole mass $t_1 \sim 0.71\ \mathrm{GeV}^2$. Furthermore, for large Q^2, if $G_E(Q^2) \sim G_M(Q^2)$ then according to Eq. (2.6b) the magnetic form factor gives the dominant contribution to the elastic excitation form factor $G_{\mathrm{el}}(Q^2)$.

Bjorken predicted that in the scaling limit ($\nu, Q^2 \to \infty$ and ξ fixed), the structure functions $MW_1(Q^2, \nu)$ and $\nu W_2(Q^2, \nu)$ given by Eqs. (2.5a) and (2.5b) cease to be functions of the two variables ν and Q^2 but instead become nontrivial functions of the ratio ξ, i.e.,

$$\begin{aligned} MW_1(\nu, Q^2) &= F_1(\xi), \\ \nu W_2(\nu, Q^2) &= F_2(\xi). \end{aligned} \tag{2.8}$$

Such a "universal" dependence of the structure functions on only the dimensionless variable ξ has also been observed experimentally.[6] Callan and Gross[7] related the scaling behavior of the structure functions to the constitution of the electromagnetic current. They found that

$$F_1(\xi) = 0 \text{ (for spin-0 fields),}$$

and

$$F_2(\xi) = 2\xi F_1(\xi) \text{ (for semi-integer spin fields).}$$

This may be seen very naively from Eqs. (2.5a), (2.5d), and (2.5e). If one assumes that in the scaling limit the factors $h_0{}^2$ and $h_1{}^2$ (presumably they are known exactly) become some constants and $G_0(Q^2) \sim G_1(Q^2)$ then $[G(Q^2)]^2$ of Eq. (2.5d) is contributed only by the "magnetic" part [second term of Eq. (2.5d)], and consequently from Eqs. (2.5a) and (2.5e) one obtains the Callan-Gross condition [Eq. (2.9a)]. For the spin-0 case the "magnetic" form factor vanishes as $1/\nu$ [also $G_M(Q^2) = 0$]. This means that the Callan-Gross condition would be exactly satisfied only if only the "magnetic" part [W_{11} term of Eq. (2.4b)] contributes to the structure functions. This conclusion seems to be true in many resonance models[8] and Regge-pole-model analysis due to Moffat and Snell.[9] In the sealing region this conclusion is consistent with the experimental result that the ratio $R = \sigma_L/\sigma_T$ is very small (almost equal to zero). Here $\sigma_{L(T)}$ is the virtual longitudinal (transverse) photoabsorption cross section. Furthermore, Drell and Yan and West[1,10] correlated the threshold behavior of the function $F_2(\xi)$ near $\xi = 1$ with the rate of decrease of the elastic form factor [Eq. (2.5d)] for large momentum transfer. Their argument goes as follows: Let, for large Q^2, the form factor fall off as some power, say, $G(Q^2) \to (1/Q^2)^{d/2}$. If one identifies the intermediate states $|n\rangle$ as nonexotic s-channel resonances, then it is clear from Eq. (2.5e) that the nth resonance peaks around $(p+q)^2 = M_n{}^2$ or $1/\xi = 1 + M_n{}^2/Q^2$. As $Q^2 \to \infty$ the nth resonance moves toward the region $\xi \to 1$. Since in this region each and every resonance has the same power of falloff in Q^2 (i.e., all resonances lie on the same scaling-limit curve) and since the structure function $F_2(\xi)$ is regarded as the average of the squares of all resonance form factors, it is possible to relate, at the region $\xi \to 1$, the behavior of $F_2(\xi)$ to the falloff in Q^2 of the form factors. It is found[1,10] that

$$F_2(\xi) \underset{\xi \to 1}{=} \text{constant} \times (1-\xi)^{d-1}. \tag{2.9b}$$

Thus the dipole behavior ($d = 4$) of form factor $G(Q^2)$ will give a threshold behavior (near $\xi \to 1$) of $(1-\xi)^3$ to $F_2(\xi)$. Various analytic fits[11] to the experimental data for e-p scattering seem to suggest that in the scaling limit $F_2(\xi)$ is given by (preferably in the region $0 \cdot 1 < \xi < 0 \cdot 8$)

$$F_2(\xi) = \sum_{i=3}^{5} c_i (1-\xi)^i, \tag{2.10}$$

$$c_3 = 0.6453, \quad c_4 = 1.902, \quad c_5 = -2.343$$

and as $\xi \to 1$, $F_2(\xi) \sim 0.6453(1-\xi)^3$. Thus the experimental data seem to satisfy the conclusions of Drell, Yan, and West. This means that all resonance form factors (including the elastic one) must have the same power of falloff in Q^2 for large Q^2 in order that the resonances satisfy scaling. It is also observed from the experimental data that the function $\nu W_2(\nu, Q^2)$ attains its limit $F_2(\xi)$ even at the small value $Q^2 \sim 1\ (\mathrm{GeV})^2$ (usually referred to as precocious scaling). This means that even at this energy the resonances seem to start overlapping each other and hence the continuum might already be replacing the discrete resonances.

The algebraic model[12,13] we use to explain the above experimental observations is based on the assumption that the intermediate states $|n\rangle$ are all one-particle resonance states with varying mass and arbitrary semi-integer spin (with isospin $\frac{1}{2}$). These states are timelike parity eigenstates $|nj^{\pm}m\rangle$ transforming according to the most degenerate irreducible representation of the dynamical group[14] $O(4,2)$. One can go to continuum if it is required by making a simple analytic continuation with respect to the quantum number n ($n^2 \to -\lambda^2$, $0 \le \lambda^2 \le \infty$) within the timelike region.[14] If we consider $O(4,2) \times T_4$ as the dynamical group (T_4 is translation) then a simple locally conserved electromagnetic current $j_\mu(q)$, linear in the generators of SO(4, 2) and T_4, may be constructed as

$$j_\mu(q) = \tilde{\psi}^\dagger(p')(\alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S + \alpha_4 L_{\mu\nu} q^\nu)\tilde{\psi}(p), \tag{2.11}$$

where $\Gamma_\mu = L_{\mu 6}$, $S = L_{46}$ (L_{AB} are the generators of $O(4,2)$, $A, B = 1, 2, 3, 4, 5 = 0, 6$), $P_\mu = (p'_\mu + p_\mu)$, and $q_\mu = (p'_\mu - p_\mu)$. The physical tilted states in the rest frame $\tilde{\psi}(0)$ ($=|\tilde{n}j^{\pm}m\rangle$) are obtained from the group states $|nj^{\pm}m\rangle$ through "scale" (or tilt) transformation. Thus if the particle is boosted in the third direction then the physical states become

$$|\tilde{n}j^{\pm}m,p\rangle=\frac{1}{N_{n'}}e^{-i\eta L_{35}}e^{i\theta_{n'}L_{45}}|nj^{\pm}m\rangle, \tag{2.12}$$

where η and $\theta_{n'}$ are the boost and tilt angles, respectively, L_{45} is the tilt (or "dilatation") operator in the SO(4,2) algebra, and $N_{n'}$ is some normalization factor that is to be determined. The quantum numbers take the following values: $n=1,2,\ldots,\infty$, $j=\frac{1}{2},\frac{3}{2},\ldots(n-\frac{1}{2})$, $-j\le m\le j$, and $n'=n+\frac{1}{2}$. The group states satisfy the usual orthonormality and completeness conditions, i.e.,

$$\langle n''j^{\pm\prime\prime}m''|nj^{\pm}m\rangle=\delta_{n''n}\delta_{j''j}\delta_{m''m},$$
$$\sum_{n,j,m}|nj^{\pm}m\rangle\langle nj^{\pm}m|=1,$$

whereas the inner products of the physical states are made only with respect to the metric[15] of $j_0(q^2=0)$, i.e.,

$$\langle\tilde{n}''j^{\pm\prime\prime}m'',p'|\int d^3x\,j_0(x)|\tilde{n}j^{\pm}m,p\rangle=2p_0\delta_{\tilde{n}''\tilde{n}}\delta_{j''j}\delta_{m''m}\delta^3(\vec{p}'-\vec{p})(2\pi)^3,$$
$$\sum_{njm}\int_{-\infty}^{\infty}d^4p\,\delta(p_\mu p^\mu-M_n{}^2)\theta(p_0)|\tilde{n}j^{\pm}m,p\rangle\langle\tilde{n}j^{\pm}m,p|j_0(q^2=0)=(2\pi)^3. \tag{2.13}$$

Furthermore, in Eq. (2.11), the anomalous term $\alpha_4L_{\mu\nu}q^\nu$ of the current is all by itself conserved, but the conservation of the remaining part is imposed by the infinite-component wave equation,

$$[(\alpha_1\Gamma_\mu+\alpha_2p_\mu+\alpha_3p_\mu s)p^\mu-bs-c]|\tilde{n}j^{\pm}m,p\rangle=0. \tag{2.14}$$

In the rest frame, using Eq. (2.12), one can derive the mass spectrum[14]

$$n'^2=(\alpha_2M_{n'}{}^2-c)^2[\alpha_1{}^2M_{n'}{}^2-(b-\alpha_3M_{n'}{}^2)^2]^{-1},\quad n'=n+\tfrac{1}{2} \tag{2.15}$$

with

$$\tanh\theta_{n'}=\frac{(b-\alpha_3M_{n'}{}^2)}{\alpha_1M_{n'}},$$
$$\cosh\theta_{n'}=\frac{\alpha_1M_{n'}n'}{(c-\alpha_2M_{n'}{}^2)}, \tag{2.16}$$
$$\sinh\theta_{n'}=\frac{(b-\alpha_3M_{n'}{}^2)n'}{(c-\alpha_2M_{n'}{}^2)}.$$

One can go from the discrete energy levels to continuum by analytically continuing the discrete basis $|nj^{\pm}m\rangle$ to the continuous basis $|\lambda j^{\pm}m\rangle$ ($-i\infty\le\lambda\le i\infty$) through the substitutions $n'^2=-\lambda^2$ and $\cosh^2\theta_{n'}=-\sinh^2\theta(\lambda)$. However, initially we restrict ourselves only to positive discrete values of n'. Furthermore, the normalization factor $N_{n'}$ of Eq. (2.12) can be explicitly evaluated using the orthonormality condition (2.13). One obtains[14]

$$N_{n'}{}^2(2M_{n'})=\alpha_1n'\cosh\theta_{n'}+2M_{n'}\alpha_2+2\alpha_3M_{n'}n'\sinh\theta_{n'}$$
$$2N_{n'}{}^2=\left\{\frac{[\alpha_1{}^2+2\alpha_3(b-\alpha_3M_{n'}{}^2)](c-\alpha_2M_{n'}{}^2)}{[\alpha_1{}^2M_{n'}{}^2-(b-\alpha_3M_{n'}{}^2)^2]}+2\alpha_2\right\}. \tag{2.17}$$

Finally, in order to evaluate the matrix elements of Eq. (2.5c) one conveniently expresses the normalized fermion basis $|nj^{\pm}m\rangle$ in terms of the parabolic representation basis as[14,16]

$$\begin{aligned}|nj^{\pm}m\rangle=(-1)^{m_-+1}[\tfrac{1}{2}(2j+1)]^{1/2}\sum_{n_1,n_2}\Bigg[&\begin{pmatrix}\frac{1}{2}(n-1)+\frac{1}{2} & \frac{1}{2}(n-1) & j\\ \frac{1}{2}(n_2-n_1+m_-)+\frac{1}{2} & \frac{1}{2}(n_1-n_2+m_-) & -(m_-+\frac{1}{2})\end{pmatrix}\left|\begin{matrix}n_2+m_-+1,n_1,n_1+m_-,n_2\\ n_2+1,n_1-m_-,n_1,n_2-m_-\end{matrix}\right\rangle\\ &\pm i(-1)^{n-1}\begin{pmatrix}\frac{1}{2}(n-1)+\frac{1}{2} & \frac{1}{2}(n-1) & j\\ \frac{1}{2}(n_1-n_2+m_-)+\frac{1}{2} & \frac{1}{2}(n_2-n_1+m_-) & -(m_-+\frac{1}{2})\end{pmatrix}\\ &\times\left|\begin{matrix}n_2+m_-,n_1,n_1+m_-+1,n_2\\ n_2,n_1-m_-,n_1+1,n_2-m_-\end{matrix}\right\rangle\Bigg],\end{aligned} \tag{2.18a}$$

where the upper states are for $m_-(=m-\frac{1}{2})\ge0$ and the lower ones for $m_-\le0$. For proton with $n=1$, $j^+=\frac{1}{2}^+$, $m=\frac{1}{2}$ we get

$$|1\tfrac{1}{2}^+\tfrac{1}{2}\rangle=\frac{1}{\sqrt{2}}(|1000\rangle\pm i|0010\rangle). \tag{2.18b}$$

Thus, by substituting Eqs. (2.18), (2.12), and (2.11) in (2.4) and (2.5c) we evaluate general expressions for the proton inelastic form factors and structure functions.

III. ELECTROMAGNETIC FORM FACTORS

The proton electromagnetic transition amplitude may be easily evaluated using the current (2.11) and the physical states (2.12). Explicitly we obtain $[N_{n'}' = (2M_{n'})^{1/2} N_{n'}]$

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm \frac{1}{2}}, p | j_0 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = \langle n j^{\pm\frac{1}{2}} | G[(\alpha_1 \cosh\theta_1 + \alpha_3 P_0 \sinh\theta_1) L_{56} + (\alpha_3 P_0 \cosh\theta_1 + \alpha_1 \sinh\theta_1) L_{46} + \alpha_4 q_3 \cosh\theta_1 L_{35} + \alpha_4 q_3 \sinh\theta_1 L_{34} + \alpha_2 P_0] | 1 \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle , \tag{3.1a}$$

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm \frac{3}{2}}, p | j_1 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = \langle n j^{\pm\frac{3}{2}} | G(\alpha_1 L_{16} + \alpha_4 q_0 \cosh\theta_1 L_{15} + \alpha_4 q_0 \sinh\theta_1 L_{14} - \alpha_4 q_3 L_{13}) | 1 \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle , \tag{3.1b}$$

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm}, -\tfrac{1}{2}, p | j_1 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = \langle n j^{\pm} - \tfrac{1}{2} | G[\alpha_1 L_{16} + \alpha_4 q_0 \cosh\theta_1 L_{15} + \alpha_4 q_0 \sinh\theta_1 L_{14} - \alpha_4 q_3 L_{13}) | 1 \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle , \tag{3.1c}$$

where we have used the Lie commutation relations

$$[L_{AB}, L_{CD}] = i(g_{AD} L_{BC} - g_{AC} L_{BD} + g_{BC} L_{AD} - g_{BD} L_{AC}),$$

and the operator identity

$$e^{\theta A} B e^{-\theta A} = B + \theta [A, B] + \frac{\theta^2}{2!} [A, [A, B]] + \cdots .$$

We have taken

$$P_\mu = (M_{n'} \cosh\eta + M, 0, 0, M_{n'} \sinh\eta), \quad q_\mu = (M_{n'} \cosh\eta - M, 0, 0, M_{n'} \sinh\eta),$$
$$G = e^{-i\theta_{n'} L_{45}} e^{i\eta L_{35}} e^{i\theta_1 L_{45}} .$$

Also, for convenience we define

$$\begin{aligned} I_{AB}(m) &= \langle n j^{\pm} m | G L_{AB} | 1 \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle , \\ I_{00}(m) &= \langle n j^{\pm} m | G | 1 \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle . \end{aligned} \tag{3.1d}$$

Using these definitions in Eqs. (3.1a), (3.1b), and (3.1c) we obtain

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm \frac{1}{2}}, p | j_0 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = (\alpha_1 \cosh\theta_1 + \alpha_3 P_0 \sinh\theta_1) I_{56}(\tfrac{1}{2}) + (\alpha_3 P_0 \cosh\theta_1 + \alpha_1 \sinh\theta_1) I_{46}(\tfrac{1}{2}) + \alpha_4 q_3 \cosh\theta_1 I_{35}(\tfrac{1}{2}) + \alpha_4 q_3 \sinh\theta_1 I_{34}(\tfrac{1}{2}) + \alpha_2 P_0 I_{00}(\tfrac{1}{2}) , \tag{3.2a}$$

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm \frac{3}{2}}, p | j_1 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = \alpha_1 I_{16}(\tfrac{3}{2}) + \alpha_4 q_0 \cosh\theta_1 I_{15}(\tfrac{3}{2}) + \alpha_4 q_0 \sinh\theta_1 I_{14}(\tfrac{3}{2}) - \alpha_4 q_3 I_{13}(\tfrac{3}{2}) , \tag{3.2b}$$

$$N_{n'}' N_1' \langle \tilde{n} j^{\pm}, -\tfrac{1}{2}, p | j_1 | \tilde{1} \tfrac{1}{2}{}^{+\frac{1}{2}} \rangle = \alpha_1 I_{16}(-\tfrac{1}{2}) + \alpha_4 q_0 \cosh\theta_1 I_{15}(-\tfrac{1}{2}) + \alpha_4 q_0 \sinh\theta_1 I_{14}(-\tfrac{1}{2}) - \alpha_4 q_3 I_{13}(-\tfrac{1}{2}) . \tag{3.2c}$$

The matrix elements (3.1d) can be evaluated as follows[14,16]: We substitute the group states (2.18a) and (2.18b) in (3.1d) and explicitly evaluate the actions of the generators L_{AB} on the parabolic basis states. In terms of the usual ladder operators $\{a_1, a_2\}$ and $\{b_1, b_2\}$ the relevant generators are expressed as follows:

$$\begin{aligned} L_{56} &= \tfrac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + b_1^+ b_1 + b_2^+ b_2 + 2), \quad L_{13} = \tfrac{1}{2} i (a_1^+ a_2 - a_2^+ a_1 + b_1^+ b_2 - b_2^+ b_1) , \\ L_{46} &= \tfrac{1}{2}(a_1^+ b_2^+ - a_2^+ b_1^+ + a_1 b_2 - a_2 b_1), \quad L_{14} = -\tfrac{1}{2}(a_1^+ a_2 + a_2^+ a_1 - b_1^+ b_2 - b_2^+ b_1) , \\ L_{34} &= -\tfrac{1}{2}(a_1^+ a_1 - a_2^+ a_2 - b_1^+ b_1 + b_2^+ b_2), \quad L_{15} = \tfrac{1}{2}(a_1^+ b_1^+ - a_2^+ b_2^+ + a_1 b_1 - a_2 b_2) , \\ L_{35} &= -\tfrac{1}{2}(a_1^+ b_2^+ + a_2^+ b_1^+ + a_1 b_2 + a_2 b_1), \quad L_{16} = \tfrac{1}{2} i (a_1^+ b_1^+ - a_2^+ b_2^+ - a_1 b_1 + a_2 b_2) . \end{aligned} \tag{3.2d}$$

Then the matrix elements of G alone can be obtained from the relation

$$\langle \phi_1'' \phi_2'' \phi_3'' \phi_4'' | G | \phi_1 \phi_2 \phi_3 \phi_4 \rangle = e^{-i\alpha(\phi_2'' + \phi_3'' + 1)/2} V^{(|\phi_2 - \phi_3| + 1)/2}_{(\phi_2'' + \phi_3'' + 1)/2, (\phi_2 + \phi_3 + 1)/2}(\beta) e^{-i\gamma(\phi_2 + \phi_3 + 1)/2} \times e^{i\alpha(\phi_1'' + \phi_4'' + 1)/2} V^{(|\phi_1 - \phi_4| + 1)/2}_{(\phi_1'' + \phi_4'' + 1)/2, (\phi_1 + \phi_4 + 1)/2}(-\beta) e^{i\gamma(\phi_1 + \phi_4 + 1)/2} ,$$

where $V^k_{m,n}(\beta)$ are Bargmann functions for $D_k^{(+)}$ representations of SO(2, 1) and α, β, and γ are Eulerian angles and they are related to the angles θ_1, $\theta_{n'}$, and η as follows $[\delta = \tfrac{1}{2}(\theta_1 - \theta_{n'}),\ \sigma = \tfrac{1}{2}(\theta_1 + \theta_{n'})$, and the arrow indicates the limit when $n = 1]$:

$$
\begin{aligned}
&\cosh\tfrac{1}{2}\beta \cos\left(\frac{\alpha+\gamma}{2}\right) = \cosh\tfrac{1}{2}\eta \cosh\delta \to \cosh\tfrac{1}{2}\eta\,,\\
&\sinh\tfrac{1}{2}\beta \cos\left(\frac{\alpha-\gamma}{2}\right) = -\cosh\tfrac{1}{2}\eta \sinh\delta \to 0 \ (\text{or } \alpha=\pi+\gamma)\,,\\
&\sinh\tfrac{1}{2}\beta \sin\left(\frac{\alpha-\gamma}{2}\right) = \sinh\tfrac{1}{2}\eta \cosh\sigma \to \sinh\tfrac{1}{2}\eta \cosh\theta_1\,,\\
&\cosh\tfrac{1}{2}\beta \sin\left(\frac{\alpha+\gamma}{2}\right) = -\sinh\tfrac{1}{2}\eta \sinh\sigma \to -\sinh\tfrac{1}{2}\eta \sinh\theta_1\,,\\
&\sinh\beta \cos\alpha = (\cosh\theta_1 \sinh\theta_{n'} \cosh\eta - \sinh\theta_1 \cosh\theta_{n'})\\
&\qquad = 2(\cosh\theta_1 \sinh\theta_{n'} \sinh^2\tfrac{1}{2}\eta - \sinh\delta \cosh\delta)\\
&\qquad \to 2\sinh\theta_1 \cosh\theta_1 \sinh^2\tfrac{1}{2}\eta\,,\\
&\sinh\beta \cos\gamma = (\cosh\theta_1 \sinh\theta_{n'} - \sinh\theta_1 \cosh\theta_{n'} \cosh\eta)\\
&\qquad = -2(\sinh\theta_1 \cosh\theta_{n'} \sinh^2\tfrac{1}{2}\eta + \sinh\delta \cosh\delta)\\
&\qquad \to -2\sinh\theta_1 \cosh\theta_1 \sinh^2\tfrac{1}{2}\eta\,,\\
&\sinh\beta \sin\alpha = \cosh\theta_1 \sinh\eta \to \cosh\theta_1 \sinh\eta\,,\\
&\sinh\beta \sin\gamma = -\cosh\theta_{n'} \sinh\eta \to -\cosh\theta_1 \sinh\eta\,,\\
&2\cosh^2\tfrac{1}{2}\beta = (\cosh\theta_1 \cosh\theta_{n'} \cosh\eta - \sinh\theta_1 \sinh\theta_{n'} + 1)\\
&\qquad = 2[\cosh^2\delta + (\cosh^2\sigma + \sinh^2\delta)\sinh^2\tfrac{1}{2}\eta]\\
&\qquad \to 2(1+\cosh^2\theta_1 \sinh^2\tfrac{1}{2}\eta)\,,\\
&2\sinh^2\tfrac{1}{2}\beta = (\cosh\theta_1 \cosh\theta_{n'} \cosh\eta - \sinh\theta_1 \sinh\theta_{n'} - 1)\\
&\qquad = 2[\sinh^2\delta + (\cosh^2\sigma + \sinh^2\delta)\sinh^2\tfrac{1}{2}\eta]\\
&\qquad \to 2\cosh^2\theta_1 \sinh^2\tfrac{1}{2}\eta\,.
\end{aligned}
\tag{3.3a}
$$

The Bargmann functions are well known and are given as

$$
\begin{aligned}
V^k_{m,n}(\beta) &= \frac{1}{(m-n)!}\left[\frac{(m-k)!\,(m+k-1)!}{(n-k)!\,(n+k-1)!}\right]^{1/2} (\tanh\tfrac{1}{2}\beta)^{m-n}(\cosh\tfrac{1}{2}\beta)^{-2n}\, {}_2F_1\left[\begin{matrix} k-n,\ 1-n-k, \\ 1+m-n,\ -\sinh^2\tfrac{1}{2}\beta \end{matrix}\right], \quad m \geq n\\
&= \left[\frac{(n-k)!(m+k-1)!}{(m-k)!\,(n+k-1)!}\right]^{1/2} (\tanh\tfrac{1}{2}\beta)^{m-n}(\cosh\tfrac{1}{2}\beta)^{-2k} P^{m-n,2k-1}_{n-k}(1-2\tanh^2\tfrac{1}{2}\beta)\,,
\end{aligned}
$$

where

$$
P^{\mu,\nu}_{\rho}(x) = \sum_{\epsilon=0}^{\rho} \binom{\rho+\mu}{\rho-\epsilon}\binom{\rho+\nu}{\epsilon}\left(\frac{x-1}{2}\right)^{\epsilon}\left(\frac{x+1}{2}\right)^{\rho-\epsilon}, \qquad P^{\mu,\nu}_0(x) = 1\,. \tag{3.3b}
$$

Then we express the product of the two Bargmann V functions in terms of one V function by using the relations[16] ($n = n_1 + n_2 + |\lambda| + 1$)

$$
V^{(|\lambda|+1)/2}_{n_1''+(|\lambda|+1)/2,\, n_1+(|\lambda|+1)/2}(\beta)\, V^{(|\lambda|+1)/2+\mu}_{n_2''+(|\lambda|+1)/2+|\mu|,\, n_2+(|\lambda|+1)/2+|\mu|}(-\beta) = \sum_{\tau} C^{\mu}_{n'',n_1'',\tau} C^{\mu}_{n,n_1,\tau} V^{|\lambda|+1+\mu+\tau}_{n''+|\mu|,\, n+|\mu|}(\beta)\,,
$$

$$
V^{(|\lambda|+1)/2+\mu}_{n_1''+(|\lambda|+1)/2+|\mu|,\, n_1+(|\lambda|+1)/2+|\mu|}(\beta)\, V^{(|\lambda|+1)/2}_{n_2''+(|\lambda|+1)/2,\, n_2+(|\lambda|+1)/2}(-\beta) = \sum_{\tau} D^{\mu}_{n'',n_1'',\tau} D^{\mu}_{n,n_1,\tau} V^{|\lambda|+1+\mu+\tau}_{n''+|\mu|,\, n+|\mu|}(\beta)\,,
$$

where the SO(2, 1) Clebsch-Gordan coefficients are

$$
C^{\mu}_{n,n_1,\tau} = \begin{bmatrix} \tfrac{1}{2}(n-1)+\tfrac{1}{2}(\mu+|\mu|) & \tfrac{1}{2}(n-1)+\tfrac{1}{2}(|\mu|-\mu) & |\lambda|+\mu+\tau \\ \tfrac{1}{2}(n_2-n_1+|\lambda|)+\tfrac{1}{2}(\mu+|\mu|) & \tfrac{1}{2}(n_1-n_2+|\lambda|)+\tfrac{1}{2}(\mu-|\mu|) & (|\lambda|+\mu) \end{bmatrix},
$$

$$
D^{\mu}_{n,n_1,\tau} = \begin{bmatrix} \tfrac{1}{2}(n-1)+\tfrac{1}{2}(|\mu|-\mu) & \tfrac{1}{2}(n-1)+\tfrac{1}{2}(|\mu|+\mu) & |\lambda|+\mu+\tau \\ \tfrac{1}{2}(n_2-n_1+|\lambda|)+\tfrac{1}{2}(\mu-|\mu|) & \tfrac{1}{2}(n_1-n_2+|\lambda|)+\tfrac{1}{2}(\mu+|\mu|) & (|\lambda|+\mu) \end{bmatrix},
$$

and $\mu=\pm\frac{1}{2}$. The range of τ is

$$0\le\tau\le \mathrm{Min}\{n-|\lambda|-1+|\mu|-\mu,\, n''-|\lambda|-1+|\mu|-\mu\}.$$

Thus we have introduced one more summation, over τ, but it is harmless in our case since for the initial proton state it takes only two values, 0 and 1. We are then left with only one summation and that is over the parabolic quantum numbers (n_1, n_2). Essentially this summation is built into the SO(4) rotation functions defined as (the angle α here is $-\alpha$ in Ref.16)

$$D^{[j_+ j_-]}_{j,j',m}(\alpha)=\sum_{m_1}\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j'\\ m_1 & m_2 & m\end{pmatrix} e^{i\alpha(m_1-m_2)},$$

where $j_+=j_1+j_2$ and $j_-=j_1-j_2$. For our purpose, we need only some special cases of the above functions, i.e.,

$$D^{[j_+,\pm 1/2]}_{j,m,m}(\alpha)=(-i)^{j-m-1}a(j_+,j,m)(\sin\alpha)^{j-m-1}[\pm(j_++j+1)e^{\mp i\alpha/2}C^{j+1/2}_{j_+-j}(\cos\alpha)\mp(j_+-j+1)e^{\pm i\alpha/2}C^{j+1/2}_{j_+-j+1}(\cos\alpha)],$$

where

$$a(j_+,j,m)=2^{j-1/2}\Gamma(j+\tfrac{1}{2})\left[\frac{\Gamma(m+1)\Gamma(j_+-m+1)\Gamma(j_+-j+1)\Gamma(j+m+1)}{\Gamma(\frac{3}{2})\Gamma(m+\frac{1}{2})\Gamma(j_++j+2)\Gamma(j-m+1)\Gamma(j_++m+2)}\right]^{1/2} \tag{3.4a}$$

and the identity

$$D^{[j_+,\pm 1/2]}_{j,m+1,m}(\alpha)=\left[\frac{2(m+1)^2}{(m+\frac{1}{2})(j_++m+2)(j_+-m)}\right]^{1/2}\left(-i\frac{d}{d\alpha}\pm\frac{1}{2}\,\frac{(j_++1)}{(m+1)}\right)D^{[j_+,\pm 1/2]}_{j,m,m}(\alpha). \tag{3.4b}$$

In Eq. (3.4a) $C^{\nu}_n(t)$ are the Gegenbauer polynomials. Some of the special cases are

$$\begin{aligned} &C^{\nu}_0(t)=1,\quad C^{\nu}_1(t)=2\nu t,\quad C^{\nu}_2(t)=2\nu(\nu+1)\left(t^2-\frac{1}{2\nu+2}\right),\\ &\frac{d}{d\alpha}C^{\nu}_n(\cos\alpha)=-2\nu\sin\alpha\, C^{\nu+1}_{n-1}(\cos\alpha). \end{aligned} \tag{3.4c}$$

Thus we can explicitly evaluate the matrix elements (3.1d) in terms of Bargmann functions and SO(4) rotation functions. We below quote the relevant results [the exponentials are indeed special cases of SO(4) functions]:

$$\begin{aligned}
&I_{00}(\tfrac{1}{2})=\tfrac{1}{2}[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{i\gamma/2}\pm \mathrm{c.c.}],\\
&I_{56}(\tfrac{1}{2})=\tfrac{3}{4}[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{i\gamma/2}\pm \mathrm{c.c.}],\\
&-iI_{36}(\tfrac{1}{2})=I_{35}(\tfrac{1}{2})=\left[\frac{1}{2\sqrt{3}}D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)(e^{i\gamma 3/2}-\tfrac{1}{2}e^{-i\gamma/2})\mp \mathrm{c.c.}\right]\\
&\qquad\qquad-\left[\frac{1}{2\sqrt{6}}D^{[n-1/2,1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)(e^{i\gamma 3/2}+e^{-i\gamma/2})\mp \mathrm{c.c.}\right],\\
&I_{46}(\tfrac{1}{2})=\left[\frac{-1}{2\sqrt{3}}D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)(e^{i\gamma 3/2}+\tfrac{1}{2}e^{-i\gamma/2})\pm \mathrm{c.c.}\right]\\
&\qquad\qquad+\left[\frac{1}{2\sqrt{6}}D^{[n-1/2,1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)(e^{i\gamma 3/2}-e^{-i\gamma/2})\pm \mathrm{c.c.}\right],\\
&I_{34}(\tfrac{1}{2})=-\tfrac{1}{4}[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{i\gamma/2}\mp \mathrm{c.c.}],\\
&-iI_{16}(\tfrac{3}{2})=I_{15}(\tfrac{3}{2})=\frac{1}{2\sqrt{2}}[D^{[n-1/2,1/2]}_{j,3/2,3/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp \mathrm{c.c.}],\\
&-iI_{16}(-\tfrac{1}{2})=I_{15}(-\tfrac{1}{2})=\left[\frac{1}{4\sqrt{3}}D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp \mathrm{c.c.}\right]\\
&\qquad\qquad-\left[\frac{1}{2\sqrt{6}}D^{[n-1/2,-1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp \mathrm{c.c.}\right],\\
&I_{14}(-\tfrac{1}{2})=\tfrac{1}{4}[D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{-i\gamma/2}\mp \mathrm{c.c.}],\\
&I_{13}(-\tfrac{1}{2})=\tfrac{1}{4}i[D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{-i\gamma/2}\pm \mathrm{c.c.}],
\end{aligned} \tag{3.5}$$

and $I_{14}(\frac{3}{2})$ and $I_{14}(-\frac{1}{2})$ vanish [obvious from Eqs. (3.2d), (3.1d), and (2.18b)]. We then substitute these matrix elements in Eqs. (3.2a)–(3.2c) and obtain the following transition amplitudes:

$$\begin{aligned} N'_{n'}N'_1\langle\tilde{n}j^{\pm}\tfrac{1}{2},p|j_0|\tilde{1}\tfrac{1}{2}\tfrac{1}{2}\rangle =& [\tfrac{3}{4}(\alpha_1\cosh\theta_1+\alpha_3P_0\sinh\theta_1)+\tfrac{1}{2}\alpha_2P_0][D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{i\gamma/2}\pm c.c.] \\ &-\tfrac{1}{4}\alpha_4q_3\sinh\theta_1[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{i\gamma/2}\mp c.c.] \\ &-\frac{1}{2\sqrt{3}}(\alpha_3P_0\cosh\theta_1+\alpha_1\sinh\theta_1)[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)(e^{i\gamma3/2}+\tfrac{1}{2}e^{-i\gamma/2})\pm c.c.] \\ &+\frac{1}{2\sqrt{3}}\alpha_4q_3\cosh\theta_1[D^{[n-1/2,1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)(e^{i\gamma3/2}-\tfrac{1}{2}e^{-i\gamma/2})\mp c.c.] \\ &+\frac{1}{2\sqrt{6}}(\alpha_3P_0\cosh\theta_1+\alpha_1\sinh\theta_1)[D^{[n-1/2,1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)(e^{i\gamma3/2}-e^{-i\gamma/2})\pm c.c.] \\ &-\frac{1}{2\sqrt{6}}\alpha_4q_3\cosh\theta_1[D^{[n-1/2,1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)(e^{i\gamma3/2}+e^{-i\gamma/2})\mp c.c.], \end{aligned} \tag{3.6a}$$

$$N'_{n'}N'_1\langle\tilde{n}j^{\pm}\tfrac{3}{2},p|j_1|\tilde{1}\tfrac{1}{2}{+}\tfrac{1}{2}\rangle=\frac{1}{2\sqrt{2}}(-i\alpha_1+\alpha_4q_0\cosh\theta_1)[D^{[n-1/2,1/2]}_{j,3/2,3/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp c.c.], \tag{3.6b}$$

$$\begin{aligned} N'_{n'}N'_1\langle\tilde{n}j^{\pm},-\tfrac{1}{2},p|j_1|\tilde{1}\tfrac{1}{2}{+}\tfrac{1}{2}\rangle =& \frac{1}{4\sqrt{3}}(-i\alpha_1+\alpha_4q_0\cosh\theta_1)[D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp c.c.] \\ &-\frac{1}{2\sqrt{6}}(-i\alpha_1+\alpha_4q_0\cosh\theta_1)[D^{[n-1/2,-1/2]}_{j,3/2,1/2}(\alpha)V^{5/2}_{n+1/2,5/2}(\beta)e^{i\gamma/2}\mp c.c.] \\ &+\tfrac{1}{4}\alpha_4q_0\sinh\theta_1[D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{-i\gamma/2}\mp c.c.] \\ &+\tfrac{1}{4}i\alpha_4q_3[D^{[n-1/2,-1/2]}_{j,1/2,1/2}(\alpha)V^{3/2}_{n+1/2,3/2}(\beta)e^{-i\gamma/2}\pm c.c.]. \end{aligned} \tag{3.6c}$$

From Eqs. (3.6a)–(3.6c) one can compute all proton form factors. However, we evaluate below the form factors for the special case when $j^+=(n-\frac{1}{2})^+$, where n is an odd integer, due to the following reasons: (i) Mathematically it is comparatively simpler in this case since the Gegenbauer polynomials in Eqs. (3.4a) and (3.4b) take simple values [Eq. (3.4c)]. (ii) From the final results we can easily deduct the corresponding expression for the elastic form factors ($n=1$) for which standard expressions derived using this O(4,2) dynamical group framework are available.[17] This provides a neat test of our general expressions [Eqs. (3.6a)–(3.6c)]. (iii) From the final expressions we can see how they factorize according to Eq. (2.5c) and formally ascertain (although it may not be exactly) what values the factors h_1 and h_0 might take for inelastic transitions. (iv) Finally if the final results show leading dipole behavior, then we can see how the dipole mass $t_{n'}$ [Eq. (2.7)] varies with n' and possibly test the conclusion with experimental observations. Having these motivations in mind we proceed to compute this special case. For this we need the following special values of Bargmann and SO(4)-rotation functions [Eqs. (3.3b) and (3.4a)–(3.4c)]:

$$\begin{aligned} V^{3/2}_{n+1/2,5/2}(\beta) &= \frac{1}{\sqrt{6}}(n-1)[n(n+1)]^{1/2}(\tanh\tfrac{1}{2}\beta)^{n-2}(\cosh\tfrac{1}{2}\beta)^{-5}-(-1)^{\delta_{n1}}\frac{3}{\sqrt{6}}[n(n+1)]^{1/2}(\tanh\tfrac{1}{2}\beta)^{n}(\cosh\tfrac{1}{2}\beta)^{-3}, \\ V^{3/2}_{n+1/2,3/2}(\beta) &= \frac{1}{\sqrt{2}}[n(n+1)]^{1/2}(\tanh\tfrac{1}{2}\beta)^{n-1}(\cosh\tfrac{1}{2}\beta)^{-3}, \\ V^{5/2}_{n+1/2,5/2}(\beta) &= \frac{1}{\sqrt{24}}[(n-1)n(n+1)(n+2)]^{1/2}(\tanh\tfrac{1}{2}\beta)^{n-2}(\cosh\tfrac{1}{2}\beta)^{-5}, \end{aligned} \tag{3.6d}$$

$$\begin{aligned} D^{[n-1/2,1/2]}_{n-1/2,1/2,1/2}(\alpha) &= (-i)^{n-1}2^n\left[\frac{\Gamma(n+1)\Gamma(n+1)}{(n+1)\Gamma(2n+1)}\right]^{1/2}e^{i\alpha/2}(\sin\alpha)^{n-1}, \\ D^{[n-1/2,1/2]}_{n-1/2,3/2,1/2}(\alpha) &= (-i)^{n-3}2^{n-1/2}\left[\frac{(n-1)\Gamma(n+1)\Gamma(n+1)}{(n+2)(n+1)\Gamma(2n+1)}\right]^{1/2}[2e^{i\alpha/2}(\sin\alpha)^{n-1}-3ie^{-i\alpha/2}(\sin\alpha)^{n-2}], \\ D^{[n-1/2,1/2]}_{n-1/2,3/2,3/2}(\alpha) &= (-i)^{n}2^n\left[\frac{3\Gamma(n+1)\Gamma(n+1)}{2(n+2)\Gamma(2n+1)}\right]^{1/2}e^{i\alpha/2}(\sin\alpha)^{n-2} \\ D^{[j_+,1/2]*}_{j,m,m}(\alpha) &= (-1)^{j-m}D^{[j_+,-1/2]}_{j,m,m}(\alpha). \end{aligned} \tag{3.6e}$$

Substituting the above special values in Eqs. (3.6a)–(3.6c) and using Eqs. (3.3a) for the angles α and γ we obtain, after a laborious but quite straightforward algebra,

$$\begin{aligned}\langle\tilde{n}(n-\tfrac{1}{2})^{+}\tfrac{1}{2},p|j_0|\tilde{1}\tfrac{1}{2}^{+}\tfrac{1}{2}\rangle = &\left(\frac{\Gamma_n A_n}{N_1' N_{n'}'}\right)\frac{\cosh\frac{1}{2}\eta}{\cosh^4\frac{1}{2}\beta}\\ &\times\left\{e_{n'}-(-1)^{\delta_{n1}}\frac{\sinh\delta}{\cosh^2\frac{1}{2}\beta}\,(\tfrac{3}{2}\cosh^2\delta R_{n'})\right.\\ &-\frac{\sinh^2\frac{1}{2}\eta}{\cosh^2\frac{1}{2}\beta}\,[B_{n'}-\sinh^2\tfrac{1}{2}\eta(\cosh^2\sigma+\sinh^2\delta)B_{n'}']\\ &-(n-1)\left(\frac{1}{\sinh^2\frac{1}{2}\beta}\left[D_{n'}-\frac{\sinh\delta}{\cosh^2\frac{1}{2}\beta}\,(\tfrac{2}{3}\cosh^2\delta R_{n'})\right.\right.\\ &\left.\left.\left.-\frac{\sinh^2\frac{1}{2}\eta}{\cosh^2\frac{1}{2}\beta}\,(D_{n'}'-\sinh^2\tfrac{1}{2}\eta(\cosh^2\sigma+\sinh^2\delta)D_{n'}'')\right]\right)\right\},\end{aligned} \tag{3.7a}$$

where

$$\begin{aligned}
\Gamma_n &= \left[\frac{n\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+1)}\right]^{1/2},\\
A_n &= \sqrt{2}\left(\frac{i\cosh\theta_1\sinh\eta}{\cosh^2\frac{1}{2}\beta}\right)^{n-1},\\
(N_{n'}')^2 &= \alpha_1 n'\cosh\theta_{n'}+2\alpha_2 M_{n'}+2\alpha_3 n' M_{n'}\sinh\theta_{n'},\\
e_{n'} &= [\tfrac{3}{2}\alpha_1\cosh\theta_1+\alpha_2(M_{n'}+M)+\tfrac{3}{2}\alpha_3(M_{n'}+M)\sinh\theta_1]\cosh\delta,\\
R_{n'} &= \alpha_1\sinh\theta_1+\alpha_3(M_{n'}+M)\cosh\theta_1,\\
B_{n'} &= -e_{n'}'\cosh^3\delta-\mu_{n'}(3\cosh^2\delta+4\sinh^2\sigma)\\
&\quad+i\alpha_4 M_{n'}\cosh\delta[2\sinh^2\sigma-\sinh\theta_1\sinh\sigma\cosh\delta+(-1)^{\delta_{n1}}\cosh\theta_1\cosh\sigma\cosh\delta\\
&\qquad+\cosh^2\delta-(-1)^{\delta_{n1}}2\cosh\theta_1\sinh\sigma\sinh\delta]\\
&\quad+\cosh\delta\sinh(2\sigma)(R_{n'}-\alpha_1\sinh\sigma)+\sinh\delta\{R_{n'}[2\cosh^2\delta-\tfrac{1}{2}(\cosh^2\sigma+\sinh^2\delta)]+3\alpha_3 M_{n'}\cosh\theta_1\cosh^2\delta\},\\
e_{n'}' &= \tfrac{3}{2}\alpha_1\cosh\sigma+2\alpha_2 M_{n'}+3\alpha_3 M_{n'}\sinh\theta_1,\\
\mu_{n'} &= (-\tfrac{1}{2}\alpha_1\cosh\sigma+i\alpha_4 M_{n'})\cosh\delta,\\
B_{n'}' &= e_{n'}'\cosh\delta+3\mu_{n'}-(-1)^{\delta_{n1}}4\alpha_3 M_{n'}\sinh\sigma-\alpha_3 M_{n'}\sinh\delta\cosh\theta_1-2(-1)^{\delta_{n1}}\alpha_3 M_{n'}\sinh\delta\cosh\theta_1\\
&\quad+i\alpha_4 M_{n'}[(-1)^{\delta_{n1}}(2\sinh\sigma\sinh\theta_1-2\cosh\delta-\cosh\theta_1\cosh\sigma)+\sinh\theta_1\sinh\sigma-3\cosh\delta],\\
D_{n'} &= \tfrac{1}{2}\sinh\delta R_{n'}\left(\tfrac{1}{3}+\frac{\cosh\theta_{n'}}{\cosh\theta_1}\right)+i\alpha_4 M_{n'}\cosh^3\delta,\\
D_{n'}' &= \tfrac{1}{3}\cosh\theta_{n'}R_{n'}(2\sinh\sigma+\sinh\delta)-2\alpha_3 M_{n'}\cosh^2\delta\sinh^2\delta\sinh\sigma\\
&\quad-2i\alpha_4 M_{n'}\cosh\delta(\cosh^2\sigma+\sinh^2\delta)+i\alpha_4 M_{n'}\cosh^3\delta(\cosh^2\sigma-\sinh^2\delta),\\
D_{n'}'' &= -\tfrac{2}{3}M_{n'}[2\sinh\theta_1(\alpha_3\cosh\delta+i\alpha_4\sinh\sigma)+(\alpha_3\sinh\sigma-i\alpha_4\cosh\delta)],
\end{aligned} \tag{3.7b}$$

$$\begin{aligned}\langle\tilde{n}(n-\tfrac{1}{2})^{+}\tfrac{3}{2},p|j_1|\tilde{1}\tfrac{1}{2}^{+}\tfrac{1}{2}\rangle = \frac{1}{2}\left(\frac{\Gamma_n A_{n-1}}{N_1' N_{n'}'}\right)(n^2-1)^{1/2}\frac{\cosh\frac{1}{2}\eta\cosh\delta}{\cosh^6\frac{1}{2}\beta}\,&[(\alpha_1+i\alpha_4\cosh\theta_1(M_{n'}-M))\\ &+\sinh^2\tfrac{1}{2}\eta(2\alpha_4 M_{n'}\cosh\theta_1)],\end{aligned}$$

$$\begin{aligned}\langle\tilde{n}(n-\tfrac{1}{2})^{\pm},-\tfrac{1}{2},p|j_1|\tilde{1}\tfrac{1}{2}^{+}\tfrac{1}{2}\rangle = &\left(\frac{\Gamma_n A_n}{N_1' N_{n'}}\right)\\ &\times\left\{-(-1)^{\delta_{n1}}\tfrac{1}{4}(-i\alpha_1+\alpha_4 q_0\cosh\theta_1)\frac{\sinh\frac{1}{2}\beta}{\cosh^4\frac{1}{2}\beta}\begin{bmatrix}-2i\sin[(\alpha-\gamma)/2]\\ 2\cos[(\alpha-\gamma)/2]\end{bmatrix}\right.\\ &+\tfrac{1}{4}\alpha_4 q_0\sinh\theta_1\frac{1}{\cosh^3\frac{1}{2}\beta}\begin{bmatrix}-2i\sin[(\alpha+\gamma)/2]\\ 2\cos[(\alpha+\gamma)/2]\end{bmatrix}+\tfrac{1}{4}i\alpha_4 q_3\frac{1}{\cosh^3\frac{1}{2}\beta}\begin{bmatrix}2\cos[(\alpha+\gamma)/2]\\ -2i\sin[(\alpha+\gamma)/2]\end{bmatrix}\\ &\left.-(n-1)\tfrac{1}{4}i(-i\alpha_1+\alpha_4 q_0\cosh\theta_1)\frac{1}{\cosh\theta_1\sinh\eta\cosh^3\frac{1}{2}\beta}\begin{bmatrix}2\cos[(\alpha+\gamma)/2]\\ 2i\sin[(\alpha+\gamma)/2]\end{bmatrix}\right\}.\end{aligned}$$

If the final states have positive parity, then the above expression becomes, after some manipulations,

$$\langle\tilde{n}(n-\tfrac{1}{2})^{+},-\tfrac{1}{2},p\,|j_1|\,\tilde{1}\tfrac{1}{2}{}^{+}\tfrac{1}{2}\rangle=\left(\frac{\Gamma_n A_n}{N'_{n'}N'_1}\right)\frac{\sinh\frac{1}{2}\eta}{\cosh^4\frac{1}{2}\beta}\times\left\{\mu'_{n'}-(n-1)\frac{\cos\delta}{2\cosh\theta_1\sinh^2\frac{1}{2}\eta}\left[(\alpha_1+i\alpha_4\cosh\theta_1(M_{n'}-M))+\sinh^2\tfrac{1}{2}\eta(2i\alpha_4 M_{n'}\cosh\theta_1)\right]\right\},\tag{3.7c}$$

where

$$\mu'_{n'}=(-1)^{\delta_{n1}}\tfrac{1}{2}\alpha_1\cosh\sigma+i\alpha_4M_{n'}\cosh\delta+(-1)^{\delta_{n1}}M_{n'}\sinh^2\tfrac{1}{2}\eta\cosh\theta_1\cosh\sigma+i\alpha_4\tfrac{1}{2}(M_{n'}-M)[\sinh\theta_1\sinh\sigma+(-1)^{\delta_{n1}}\cosh\theta_1\cosh\sigma].$$

From the above expressions we conclude that the transition amplitudes behave as

$$\langle\tilde{n}(n-\tfrac{1}{2})^{+}\tfrac{1}{2},p\,|j_0|\,\tilde{1}\tfrac{1}{2}{}^{+}\tfrac{1}{2}\rangle\sim\left(\frac{\sinh\eta}{\cosh^2\frac{1}{2}\beta}\right)^{n-1}\frac{\cosh\frac{1}{2}\eta}{\cosh^4\frac{1}{2}\beta},\quad\langle\tilde{n}(n-\tfrac{1}{2})^{+},-\tfrac{1}{2},p\,|j_0|\,\tilde{1}\tfrac{1}{2}{}^{+}\tfrac{1}{2}\rangle\sim\left(\frac{\sinh\eta}{\cosh^2\frac{1}{2}\beta}\right)^{n-1}\frac{\sinh\frac{1}{2}\eta}{\cosh^4\frac{1}{2}\beta},\tag{3.7d}$$

although, strictly speaking, they are affected by the term $\sinh^4\frac{1}{2}\eta/\cosh^2\frac{1}{2}\beta$. The dipole behavior is always built into the term $1/\cosh^4\frac{1}{2}\beta$, which actually originates from the $O(2,1)$ representation functions. From Eq. (3.3a)

$$\cosh^2\tfrac{1}{2}\beta=\tfrac{1}{2}(\cosh\theta_1\cosh\theta_{n'}\cosh\eta-\sinh\theta_1\sinh\theta_{n'}+1),$$

where the boost angle η is given by

$$\sinh^2\tfrac{1}{2}\eta=\frac{1}{4M_{n'}M}[(M_{n'}-M)^2+Q^2],\qquad \cosh^2\tfrac{1}{2}\eta=\frac{1}{4M_{n'}M}[(M_{n'}+M)^2+Q^2].\tag{3.7e}$$

Therefore,

$$\cosh^4\tfrac{1}{2}\beta=g_{n'}{}^2\left(1+\frac{Q^2}{t_{n'}}\right)^2,$$

where

$$g_{n'}=1+\frac{1}{2M_{n'}M}(M_{n'}{}^2+M^2)\cosh\theta_1\cosh\theta_{n'}-\sinh\theta_1\sinh\theta_{n'},$$

and the dipole mass [after using Eqs. (2.16)]

$$t_{n'}=M_{n'}{}^2\left(1+\frac{2}{\alpha_1}M\alpha_3\tanh\theta_1\right)-\frac{2M}{\alpha_1\cosh\theta_1}[\alpha_1{}^2M_{n'}{}^2-(b-\alpha_3M_{n'}{}^2)^2]^{1/2}+\left(M^2-\frac{2}{\alpha_1}Mb\tanh\theta_1\right).\tag{3.8}$$

Hence for an increasing mass spectrum such as the one given by Eq. (2.15) the dipole resonance mass $t_{n'}$ increases with n'. This means that the dipole curve of the form factor of any nth resonance should lie above the corresponding one of the $(n-1)$th resonance. The experimental data seem to support this conclusion[18] (at least for the first four prominent nucleon resonances). The maximum value of $t_{n'}$ may be obtained from the largest nucleon resonance mass $M_{n'}$ $(n'\to\infty)$. From the mass spectrum given by Eq. (2.15) we get the largest mass

$$M_{\max}{}^2=\frac{1}{2\alpha_3{}^2}(\alpha_1{}^2+2b\alpha_3)-\frac{\alpha_1}{2\alpha_3{}^2}(\alpha_1{}^2+4b\alpha_3)^{1/2}\sim21.91\ (\text{GeV})^2,$$

$$M_{\max}\sim4.68\ \text{GeV}$$

where we have used the values of the constants given in Ref. 17. They are $\alpha_1=-0.909$, $\alpha_2=1.66$, $\alpha_3=0.166$, $b=-0.618$, $c=0.891$, $\sinh\theta_1=2$, $\cosh\theta_1=\sqrt{5}$, $M=0.94$ GeV. Using this largest mass we get the largest dipole mass $t_{\max}\sim14.92$ (GeV)^2. This means that $t_{n'}$ is bound by the limit $0.71\leq t_{n'}\leq14.92$ (GeV)^2. From Eq. (3.8) one can predict the dipole mass of all the nuclear resonances that fit into the $O(4,2)$ hadron spectrum.[19] Furthermore, if we take $t_{n'}=\text{constant}\times M_{n'}{}^2$ (for simplicity) in the scaling limit ($Q^2\to\infty$, ξ fixed) we find $t_{n'}=\text{constant}\times Q^2(1/\xi)(1-\xi)$. This shows that each and every nth resonance form factor remains finite in the scaling limit. Or, in other words, the structure function νW_2 of Eq. (2.5e) should remain finite in the scaling region. Also, in (3.7d) the presence of the term $[\sinh\eta/\cosh^2\frac{1}{2}\beta]^{n-1}$ is quite harmless, since, with respect to their large Q^2 behavior, $\cosh^2\frac{1}{2}\beta\sim\sinh\eta$. However, the existence of the terms $\sinh\frac{1}{2}\eta$, $\cosh\frac{1}{2}\eta$ in (3.7d) is extremely crucial. If we compare (3.7d) with Eqs. (2.5c), then these terms may be appropriately absorbed by the factors h_0 and h_1. Therefore, from this calculation we suppose, at least for those normal parity transitions (with $j=n-\frac{1}{2}$), $\frac{1}{2}^+\to\frac{1}{2}^+, \frac{5}{2}^+, \frac{9}{2}^+\ldots$, that $h_0=\cosh\frac{1}{2}\eta$ and $h_1=\sinh\frac{1}{2}\eta$. The factors h_0, h_1 connecting the vertex function to the form factors are in principle fixed from the spinor character of the wave function, but involve the generalization of the invariant forms $(\bar\psi\gamma_\mu\psi F_1+\bar\psi\sigma_{\mu\nu}q^\nu\psi F_2)$ to higher spins.

Next, we want to deduce the elastic transition form factors from our general expressions [Eqs. (3.7a)–(3.7c)]. For this we take $n=1, \delta=0$, $\sigma=\theta_1$ and obtain

$$G_E(q^2)=\frac{1}{(N_1')^2\cosh^4\frac{1}{2}\beta}\left\{(N_1')^2-\frac{\sinh^2\frac{1}{2}\eta}{\cosh^2\frac{1}{2}\beta}[B_1-\sinh^2\tfrac{1}{2}\eta\cosh^2\theta_1B_1']\right\}, \tag{3.9a}$$

where

$$\begin{aligned}(N_1')^2&=\tfrac{3}{2}\alpha_1\cosh\theta_1+2\alpha_2M+3\alpha_3M\sinh\theta_1,\\ B_1&=-(N_1')^2-\mu_1(3+4\sinh^2\theta_1)\\ &\quad+4M\alpha_3\sinh\theta_1\cosh^2\theta_1,\\ \mu_1&=-\tfrac{1}{2}\alpha_1\cosh\theta_1+i\alpha_4M,\\ B_1'&=(N_1')^2+3\mu_1-4\alpha_3M\sinh\theta_1,\end{aligned} \tag{3.9b}$$

and

$$G_1^+(q^2)=0,$$

$$G_1^-(q^2)=G_M(q^2)=\frac{\mu_1}{(N_1')^2\cosh^4\frac{1}{2}\beta},$$

where from Eqs. (3.7e) and (3.8)

$$\cosh^2\tfrac{1}{2}\beta=2\left(1+\frac{Q^2}{t_1}\right), t_1=\frac{4M^2}{\cosh^2\theta_1},$$

and

$$\sinh^2\tfrac{1}{2}\eta=\frac{Q^2}{4M^2}.$$

The above results exactly agree with those of the previous calculations,[17] and thus support the validity of our most general expressions [Eqs. (3.6a)–(3.6c)] for the nucleon inelastic excitation form factors.

As a final remark in this section we want to show how the leading $1/\cosh^4\frac{1}{2}\beta$ behavior in Eqs. (3.7a)–(3.7c) arises. For this we have already given an intermediate step before we reached Eq. (3.7c). From Eq. (3.3b) we notice that each Bargmann function $V^k_{m,n}(\beta)$ introduces a term $1/(\cosh\frac{1}{2}\beta)^{2k}$. In Eq. (3.6c) the corresponding function with the smallest k is, say, $V^{3/2}_{n+1/2,3/2}(\beta)$ and this term gives $1/\cosh^3\frac{1}{2}\beta$. Then, when we take the complex conjugate in Eq. (3.6c), because the Bargmann functions are all real, only the angles α and γ get coupled together according to the parity assignment and we get terms such as $\cos[(\alpha+\gamma/2]$, $\cos[(\alpha-\gamma)/2]$, $i\sin[(\alpha+\gamma)/2]$, and $i\sin[(\alpha-\gamma)/2]$. According to Eq. (3.3a) each of the latter terms behaves as $\sinh\frac{1}{2}\eta/\cosh\frac{1}{2}\beta$ or $\cosh\frac{1}{2}\eta/\cosh\frac{1}{2}\beta$. The term $\sinh\frac{1}{2}\eta$ or $\cosh\frac{1}{2}\eta$ comes outside and it is absorbed by h_0 or h_1 of Eq. (2.5c) and $1/\cosh\frac{1}{2}\beta$ gets multiplied with the previous $1/\cosh^3\frac{1}{2}\beta$, giving the leading behavior of $1/\cosh^4\frac{1}{2}\beta$. The important point to note is that the parity superposition is quite essential to get the right behavior.

IV. ELECTROMAGNETIC STRUCTURE FUNCTIONS

In order to calculate the structure functions MW_1 and νW_2, we have to evaluate first the tensor components W_{11} and W_{33} (or W_{00}) from Eq. (2.1), i.e.,

$$W_{11}=\frac{1}{8\pi^2\alpha}\sum_{n,j,m}\delta((p+q)^2-M_{n'}^2)|\langle\tilde n j^\pm m,p|j_1(0)|\tilde 1\tfrac{1}{2}^+\tfrac{1}{2}\rangle|^2 \tag{4.1a}$$

and

$$W_{33}=\frac{1}{8\pi^2\alpha}\sum_{n,j,m}\delta((p+q)^2-M_{n'}^2)|\langle\tilde n j^\pm m,p|j_3(0)|\tilde 1\tfrac{1}{2}^+\tfrac{1}{2}\rangle|^2 \tag{4.1b}$$

We now substitute explicitly the current (2.11) and the physical states (2.12) in the equation given above and obtain in terms of the notation given in (3.1d) [for every cross term it is understood that its complex conjugate should also be added with it, for example, $I_{36}(\frac{1}{2})I^*_{46}(\frac{1}{2})$ means $I_{36}(\frac{1}{2})I^*_{46}(\frac{1}{2})+I^*_{36}(\frac{1}{2})I_{46}(\frac{1}{2})$]

$$W_{11}=\frac{1}{8\pi^2\alpha}\sum_{n,j^\pm}\frac{1}{N_1^2N_{n'}^2}\delta((p+q)^2-M_{n'}^2)\{\alpha_1^2[I_{16}(\tfrac{3}{2})I^*_{16}(\tfrac{3}{2})+I_{16}(-\tfrac{1}{2})I^*_{16}(-\tfrac{1}{2})]$$
$$+\alpha_4^2q_0^2\cosh^2\theta_1[I_{15}(\tfrac{3}{2})I^*_{15}(\tfrac{3}{2})+I_{15}(-\tfrac{1}{2})I^*_{15}(-\tfrac{1}{2})]$$
$$+\alpha_4^2q_0^2\sinh^2\theta_1[I_{14}(\tfrac{3}{2})I^*_{14}(\tfrac{3}{2})+I_{14}(-\tfrac{1}{2})I^*_{14}(-\tfrac{1}{2})]$$
$$+\alpha_4^2q_3^2[I_{13}(\tfrac{3}{2})I^*_{13}(\tfrac{3}{2})+I_{13}(-\tfrac{1}{2})I^*_{13}(-\tfrac{1}{2})]$$
$$+\alpha_1\alpha_4q_0\cosh^2\theta_1[I_{16}(\tfrac{3}{2})I^*_{15}(\tfrac{3}{2})+I_{16}(-\tfrac{1}{2})I^*_{15}(-\tfrac{1}{2})]$$
$$+\alpha_1\alpha_4q_0\sinh\theta_1[I_{16}(\tfrac{3}{2})I^*_{14}(\tfrac{3}{2})+I_{16}(-\tfrac{1}{2})I^*_{14}(-\tfrac{1}{2})]$$
$$-\alpha_1\alpha_4q_3[I_{16}(\tfrac{3}{2})I^*_{13}(\tfrac{3}{2})+I_{16}(-\tfrac{1}{2})I^*_{13}(-\tfrac{1}{2})]$$
$$+\alpha_4^2q_0^2\cosh\theta_1\sinh\theta_1[I_{15}(\tfrac{3}{2})I^*_{14}(\tfrac{3}{2})+I_{15}(-\tfrac{1}{2})I^*_{14}(-\tfrac{1}{2})]$$
$$-\alpha_4^2q_0q_3\cosh\theta_1[I_{15}(\tfrac{3}{2})I^*_{13}(\tfrac{3}{2})+I_{15}(-\tfrac{1}{2})I^*_{13}(-\tfrac{1}{2})]$$
$$-\alpha_4^2q_0q_3\sinh\theta_1[I_{14}(\tfrac{3}{2})I^*_{13}(\tfrac{3}{2})+I_{14}(-\tfrac{1}{2})I^*_{13}(-\tfrac{1}{2})]\},\qquad(4.2\text{a})$$

$$W_{33}=\frac{1}{8\pi^2\alpha}\sum_{n,j^\pm}\frac{1}{N_1^2N_{n'}^2}\delta((p+q)^2-M_{n'}^2)[\alpha_1^2I_{36}(\tfrac{1}{2})I^*_{36}(\tfrac{1}{2})$$
$$+\alpha_3^2P_3^2\cosh^2\theta_1I_{46}(\tfrac{1}{2})I^*_{46}(\tfrac{1}{2})+\alpha_3^2P_3^2\sinh^2\theta_1I_{56}(\tfrac{1}{2})I^*_{56}(\tfrac{1}{2})$$
$$+\alpha_4^2q_0^2\cosh^2\theta_1I_{35}(\tfrac{1}{2})I^*_{35}(\tfrac{1}{2})+\alpha_4^2q_0^2\sinh^2\theta_1I_{34}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})+\alpha_2^2P_3^2I_{00}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})$$
$$+\alpha_1\alpha_3P_3\cosh\theta_1I_{36}(\tfrac{1}{2})I^*_{46}(\tfrac{1}{2})+\alpha_1\alpha_3P_3\sinh\theta_1I_{36}(\tfrac{1}{2})I^*_{56}(\tfrac{1}{2})$$
$$+\alpha_1\alpha_4q_0\cosh\theta_1I_{36}(\tfrac{1}{2})I^*_{35}(\tfrac{1}{2})+\alpha_1\alpha_4q_0\sinh\theta_1I_{36}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})$$
$$+\alpha_1\alpha_2P_3I_{36}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})$$
$$+\alpha_3^2P_3^2\cosh\theta_1\sinh\theta_1I_{46}(\tfrac{1}{2})I^*_{56}(\tfrac{1}{2})+\alpha_3\alpha_4P_3q_0\cosh^2\theta_1I_{46}(\tfrac{1}{2})I^*_{35}(\tfrac{1}{2})$$
$$+\alpha_3\alpha_4P_3q_0\cosh\theta_1\sinh\theta_1I_{46}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})+\alpha_3\alpha_2P_3^2\cosh\theta_1I_{46}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})$$
$$+\alpha_3\alpha_4P_3q_0\cosh\theta_1\sinh\theta_1I_{56}(\tfrac{1}{2})I^*_{35}(\tfrac{1}{2})+\alpha_3\alpha_4P_3q_0\sinh^2\theta_1I_{56}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})$$
$$+\alpha_3\alpha_2P_3^2\sinh\theta_1I_{56}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})+\alpha_4^2q_0^2\cosh\theta_1\sinh\theta_1I_{35}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})$$
$$+\alpha_4\alpha_2P_3q_0\cosh\theta_1I_{35}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})+\alpha_4\alpha_2P_3q_0\sinh\theta_1I_{34}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})].\qquad(4.2\text{b})$$

In the above expressions for W_{11} and W_{33}, the summation over $j^\pm$ of the product of two matrix elements can be easily done using Eqs. (3.5) and the following orthonormality condition of the SO(4) rotation functions, because each matrix element can be, as detailed in the preceding section, expressed in terms of Bargmann and SO(4) functions and the final-state spin-parity $j^\pm$ are involved only with the latter, i.e.,

$$\sum_j D^{[j_+,j_-]}_{j,j',m}(\alpha)\,D^{[j_+,j_-]}_{j,j'',m}(-\alpha)=\delta_{j'j''}.\qquad(4.3)$$

We below give all the relevant final results:

$$\sum_{j^\pm}I_{36}(\tfrac{1}{2})I^*_{36}(\tfrac{1}{2})=\tfrac{1}{12}V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\sin^2\gamma\,V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\cos^2\gamma\,V^{5/2}_{n+1/2,5/2}(\beta)V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j^\pm}I_{46}(\tfrac{1}{2})I^*_{46}(\tfrac{1}{2})=\tfrac{1}{12}V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\cos^2\gamma\,V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\sin^2\gamma\,V^{5/2}_{n+1/2,5/2}(\beta)V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j^\pm}I_{56}(\tfrac{1}{2})I^*_{56}(\tfrac{1}{2})=\tfrac{9}{4}V^{3/2}_{n+1/2,3/2}(\beta)V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j^\pm}I_{35}(\tfrac{1}{2})I^*_{35}(\tfrac{1}{2})=\tfrac{1}{12}V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\sin^2\gamma\,V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)+\tfrac{2}{3}\cos^2\gamma\,V^{5/2}_{n+1/2,5/2}(\beta)V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j^\pm}I_{34}(\tfrac{1}{2})I^*_{34}(\tfrac{1}{2})=\tfrac{1}{4}V^{3/2}_{n+1/2,3/2}(\beta)V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j^\pm}I_{00}(\tfrac{1}{2})I^*_{00}(\tfrac{1}{2})=V^{3/2}_{n+1/2,3/2}(\beta)V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j^\pm}I_{36}(\tfrac{1}{2})I^*_{46}(\tfrac{1}{2})=\tfrac{2}{3}\sin(2\gamma)V^{3/2}_{n+1/2,5/2}(\beta)V^{3/2}_{n+1/2,5/2}(\beta)-\tfrac{2}{3}\sin(2\gamma)V^{5/2}_{n+1/2,5/2}(\beta)V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j\pm} I_{36}(\tfrac{1}{2}) I^*_{56}(\tfrac{1}{2}) = -\frac{3\sqrt{3}}{2}\sin\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{36}(\tfrac{1}{2}) I^*_{00}(\tfrac{1}{2}) = -\sqrt{3}\sin\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{46}(\tfrac{1}{2}) I^*_{56}(\tfrac{1}{2}) = -\frac{3\sqrt{3}}{2}\cos\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{46}(\tfrac{1}{2}) I^*_{00}(\tfrac{1}{2}) = -\sqrt{3}\cos\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta), \tag{4.4}$$

$$\sum_{j\pm} I_{56}(\tfrac{1}{2}) I^*_{00}(\tfrac{1}{2}) = 3 V^{3/2}_{n+1/2,3/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{35}(\tfrac{1}{2}) I^*_{34}(\tfrac{1}{2}) = -\frac{1}{2\sqrt{3}}\cos\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{16}(\tfrac{3}{2}) I^*_{16}(\tfrac{3}{2}) = \tfrac{1}{2} V^{5/2}_{n+1/2,5/2}(\beta) V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j\pm} I_{15}(\tfrac{3}{2}) I^*_{15}(\tfrac{3}{2}) = \tfrac{1}{2} V^{5/2}_{n+1/2,5/2}(\beta) V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j\pm} I_{16}(-\tfrac{1}{2}) I^*_{16}(-\tfrac{1}{2}) = \tfrac{1}{12} V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,5/2}(\beta) + \tfrac{1}{6} V^{5/2}_{n+1/2,5/2}(\beta) V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j\pm} I_{15}(-\tfrac{1}{2}) I^*_{15}(-\tfrac{1}{2}) = \tfrac{1}{12} V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,5/2}(\beta) + \tfrac{1}{6} V^{5/2}_{n+1/2,5/2}(\beta) V^{5/2}_{n+1/2,5/2}(\beta),$$

$$\sum_{j\pm} I_{14}(-\tfrac{1}{2}) I^*_{14}(-\tfrac{1}{2}) = \tfrac{1}{4} V^{3/2}_{n+1/2,3/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{13}(-\tfrac{1}{2}) I^*_{13}(-\tfrac{1}{2}) = \tfrac{1}{4} V^{3/2}_{n+1/2,3/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{15}(-\tfrac{1}{2}) I^*_{14}(-\tfrac{1}{2}) = -\frac{1}{2\sqrt{3}}\cos\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta),$$

$$\sum_{j\pm} I_{15}(-\tfrac{1}{2}) I^*_{13}(-\tfrac{1}{2}) = \frac{1}{2\sqrt{3}}\sin\gamma\, V^{3/2}_{n+1/2,5/2}(\beta) V^{3/2}_{n+1/2,3/2}(\beta).$$

The angles γ and β can be expressed in terms of invariant quantities ν, q^2, and $M_{n'}{}^2$ using Eqs. (3.3a) and Eqs. (2.16) as follows:

$$\cos\gamma = \frac{n'}{\sinh\beta}\left[\cosh\theta_1\left(\frac{b-\alpha_3 M_{n'}{}^2}{c-\alpha_2 M_{n'}{}^2}\right) - \alpha_1 \sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2 M_{n'}{}^2}\right)\right],$$
$$\sin\gamma = \frac{-n'}{\sinh\beta}\left[\frac{\alpha_1(\nu^2-q^2)^{1/2}}{c-\alpha_2 M_{n'}{}^2}\right]. \tag{4.5}$$

and

$$\cosh^2\tfrac{1}{2}\beta = \tfrac{1}{2}(xn'+1), \quad \sinh^2\tfrac{1}{2}\beta = \tfrac{1}{2}(xn'-1),$$
$$\tanh^2\tfrac{1}{2}\beta = \left(\frac{xn'-1}{xn'+1}\right),$$

where

$$x = \alpha_1 \cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2 M_{n'}{}^2}\right) - \sinh\theta_1\left(\frac{b-\alpha_3 M_{n'}{}^2}{c-\alpha_2 M_{n'}{}^2}\right). \tag{4.6}$$

It is important to emphasize that the present theory does not take into account the widths of the intermediate states $|n\rangle$. These are treated as one-particle states. However, they decay strongly, and a further vertex representing the decay of the state $|n\rangle$ into all possible decay products must be included in order to calculate more accurately the experimentally measured cross sections. However, we shall evaluate first the structure functions in the zero-width approximation. Note that in most of the phenomenological direct-channel-resonance models[8,18] mass formulas and appropriate factors taking into account the widths of the resonances are introduced phenomenologically in order to obtain scaling [equivalently, the requirement of the finite-energy sum rule

$$\frac{1}{t}\int_t^{(t/a-M/2)} \nu W_2(\nu, Q^2)d\nu = \int_a^1 F_2(\xi)d\xi, \quad t = \frac{Q^2}{2M}$$

a = fixed number, can be used]. Clearly the widths of the resonances will be important, especially in the region of overlapping resonances, i.e., for large n' (and for continuum), and we assume a factor $1/M_{n'}^2$ as in the phenomenological models.

With this remark in mind we substitute Eqs. (4.4)–(4.6) and (3.6d) into (4.2a) and (4.2b) and after some simplications obtain

$$W_{11}=\frac{1}{8\pi^2\alpha}\sum_n\frac{1}{N_1^2N_{n'}^2}\delta((p+q)^2-M_{n'}^2)\left(\frac{xn'-1}{xn'+1}\right)^n$$
$$\times\left\{\tfrac{1}{3}(\alpha_1^2+\alpha_4^2\cosh^2\theta_1\nu^2)\frac{n(n+1)}{(xn'+1)^3}\left[3-4\frac{(n-1)}{(xn'-1)}+4\frac{(n-1)(n+1)}{(xn'-1)^2}\right]\right.$$
$$+\tfrac{2}{3}\alpha_4^2\left[\alpha_1\cosh\theta_1\left(\frac{\nu}{c-\alpha_2M_{n'}^2}\right)(\nu^2-q^2)+\alpha_1\sinh^2\theta_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)\nu^2\right.$$
$$\left.-\cosh^2\theta_1\sinh\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)\nu^2\right]\frac{n'n(n+1)}{(xn'+1)^3(xn'-1)}\left[-3+2\frac{(n-1)}{(xn'-1)}\right]$$
$$\left.+\alpha_4^2[\sinh^2\theta_1\nu^2+(\nu^2-q^2)]\frac{n(n+1)}{(xn'+1)^2(xn'-1)}\right\}, \tag{4.7}$$

$$W_{33}=\frac{1}{8\pi^2\alpha}\sum_n\frac{1}{N_1^2N_{n'}^2}\delta((p+q)^2-M_{n'}^2)\left(\frac{xn'-1}{xn'+1}\right)^n$$
$$\times\left\{\tfrac{1}{9}[\alpha_1^2+\alpha_3^2\cosh^2\theta_1(\nu^2-q^2)+\alpha_4^2\cosh^2\theta_1\nu^2]\frac{n(n+1)}{(xn'+1)^3}\left[9-12\frac{(n-1)}{(xn'-1)}+4\frac{(n-1)^2}{(xn'-1)^2}\right]\right.$$
$$+\tfrac{8}{9}\left(\left[\alpha_1^2\left(\frac{1}{c-\alpha_2M_{n'}^2}\right)-\alpha_3\cosh^2\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)+\alpha_1\alpha_3\sinh\theta_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)\right]^2(\nu^2-q^2)\right.$$
$$\left.+\alpha_1^2\alpha_4^2\cosh^2\theta_1\left(\frac{\nu^2-q^2}{(c-\alpha_2M_{n'}^2)^2}\right)\nu^2\right)\frac{n'^2(n+1)}{(xn'+1)^4(xn'-1)}\left[9-12\frac{(n-1)}{(xn'-1)}+4\frac{(n-1)^2}{(xn'-1)^2}\right]$$
$$+\tfrac{8}{9}\left(\left[\alpha_1\cosh\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)-\alpha_1^2\sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)+\alpha_1\alpha_3\cosh\theta_1\left(\frac{\nu^2-q^2}{c-\alpha_2M_{n'}^2}\right)\right]^2\right.$$
$$\left.+\left[\alpha_4\cosh^2\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)-\alpha_1\alpha_4\cosh\theta_1\sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)\right]^2\nu^2\right)\frac{n'^2n(n-1)(n+1)(n+2)}{(xn'+1)^4(xn'-1)^3}$$
$$+\tfrac{2}{3}\left(6(\alpha_2+\tfrac{3}{2}\alpha_3\sinh\theta_1)\left[\alpha_1^2\left(\frac{1}{c-\alpha_2M_{n'}^2}\right)-\alpha_3\cosh^2\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)\right.\right.$$
$$\left.+\alpha_1\alpha_3\cosh\theta_1\sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)\right](\nu^2-q^2)$$
$$\left.-\left[\alpha_4^2\cosh^2\theta_1\sinh\theta_1\left(\frac{b-\alpha_3M_{n'}^2}{c-\alpha_2M_{n'}^2}\right)-\alpha_1\alpha_4^2\cosh\theta_1\sinh^2\theta_1\left(\frac{\nu+M}{c-\alpha_2M_{n'}^2}\right)\right]\nu^2\right)$$
$$\times\frac{n'n(n+1)}{(xn'+1)^3(xn'-1)}\left[-3+2\frac{(n-1)}{(xn'-1)}\right]$$
$$\left.+[\alpha_4^2\sinh^2\theta_1\nu^2+4(\alpha_2+\tfrac{3}{2}\alpha_3\sinh\theta_1)^2(\nu^2-q^2)]\frac{n(n+1)}{(xn'+1)^2(xn'-1)}\right\}. \tag{4.8}$$

Equations (4.7) and (4.8) are the exact expressions for the tensor components W_{11} and W_{33}, respectively. Since the mass $M_{n'}$ is a function of n' we explicitly substitute $n=n'-\frac{1}{2}$ and sum over n'. The quantum number n' takes only discrete positive values. If $g(n')=W-M_{n'}^2$, $W=(p+q)^2$, then

$$\delta(g(n'))=\frac{1}{|g'(n')|_{n'=N}}\delta(n'-N), \tag{4.9}$$

where

$$|g'(n')|_{n'=N}=\left|\frac{2[\alpha_1^2W-(b-\alpha_3W)^2]^{3/2}}{W[\alpha_2(\alpha_1^2+2\alpha_3b)-2\alpha_3^2c]+[c(\alpha_1^2+2\alpha_3b)-2b^2\alpha_2]}\right|,$$

$$N=\frac{(\alpha_2W-c)}{[\alpha_1^2W-(b-\alpha_3W)^2]^{1/2}}.$$

Using Eq. (4.9) the summation over n' can be trivially done. We finally obtain from Eqs. (2.4a), (2.4b), and (2.8) and the remarks given below Eq. (4.6)

$$MW_1(\nu, q^2) = \frac{1}{8\pi^2\alpha}\,\frac{M}{N_1^2N_N^2}\,\frac{1}{W}\,\frac{1}{|g'(N)|}\sum_{i=1}^{3} b_{11}^{(i)}(\nu, Q^2, W)g_{11}^{(i)}(N, W), \tag{4.10a}$$

$$\nu W_2(\nu, q^2) = \frac{1}{8\pi^2\alpha}\,\frac{1}{N_1^2N_N^2}\,\frac{1}{W}\,\frac{1}{|g'(N)|}\left[\frac{\nu}{(1+\nu^2/Q^2)}\sum_{i=1}^{3} b_{11}^{(i)}(\nu, Q^2, W)g_{11}^{(i)}(N, W) + \frac{Q^2}{\nu(1+\nu^2/Q^2)}\sum_{j=1}^{5} b_{33}^{(j)}(\nu, Q^2, W)g_{33}^{(j)}(N, W)\right], \tag{4.10b}$$

where

$$N = \frac{(\alpha_2W - c)}{[\alpha_1^2W-(b-\alpha_3W)^2]^{1/2}},$$

$$N_N^2 = \frac{1}{2}\left\{\frac{[\alpha_1^2+2\alpha_3(b-\alpha_3W)](c-\alpha_2W)}{[\alpha_1^2W-(b-\alpha_3W)^2]} + 2\alpha_2\right\},$$

$$N_1^2 = \frac{1}{2M}\left(\tfrac{3}{2}\alpha_1\cosh\theta_1 + 2\alpha_2M + 3\alpha_3M\sinh\theta_1\right),$$

$$|g'(N)| = \left|\frac{2[\alpha_1^2W-(b-\alpha_3W)^2]^{3/2}}{W[\alpha_2(\alpha_1^2+2\alpha_3b)-2\alpha_3^2c]+[c(\alpha_1^2+2\alpha_3b)-2b^2\alpha_2]}\right|,$$

$$b_{11}^{(1)}(\nu, Q^2) = \tfrac{1}{3}(\alpha_1^2+\alpha_4^2\cosh^2\theta_1\,\nu^2),$$

$$b_{11}^{(2)}(\nu, Q^2) = \tfrac{2}{3}\alpha_4^2\left[\alpha_1\sinh^2\theta_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right)\nu^2 - \sinh\theta_1\cosh^2\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right)\nu^2 + \alpha_1\cosh\theta_1\left(\frac{\nu}{c-\alpha_2W}\right)(\nu^2+Q^2)\right],$$

$$b_{11}^{(3)}(\nu, Q^2) = \alpha_4^2[\sinh^2\theta_1\,\nu^2+(\nu^2+Q^2)],$$

$$b_{33}^{(1)}(\nu, Q^2) = \tfrac{1}{9}[\alpha_1^2+\alpha_3^2\cosh^2\theta_1(\nu^2+Q^2)+\alpha_4^2\cosh^2\theta_1\,\nu^2],$$

$$b_{33}^{(2)}(\nu, Q^2) = \frac{8}{9}\left\{\left[\alpha_1^2\left(\frac{1}{c-\alpha_2W}\right) - \alpha_3\cosh^2\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right) + \alpha_1\alpha_3\sinh\theta_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right)\right]^2(\nu^2+Q^2) + \alpha_1^2\alpha_4^2\cosh^2\theta_1\left(\frac{\nu^2+Q^2}{(c-\alpha_2W)^2}\right)\nu^2\right\},$$

$$b_{33}^{(3)}(\nu, Q^2) = \frac{8}{9}\left\{\left[\alpha_1\cosh\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right) - \alpha_1^2\sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right) + \alpha_1\alpha_3\cosh\theta_1\left(\frac{\nu^2+Q^2}{c-\alpha_2W}\right)\right]^2 + \left[\alpha_4\cosh^2\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right) - \alpha_1\alpha_4\sinh\theta_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right)\right]^2\nu^2\right\},$$

$$b_{33}^{(4)}(\nu, Q^2) = \frac{2}{3}\left\{6(\alpha_2+\tfrac{3}{2}\alpha_3\sinh\theta_1)\left[\alpha_1^2\left(\frac{1}{c-\alpha_2W}\right) - \alpha_3\cosh^2\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right) + \alpha_1\alpha_3\cosh\theta_1\sinh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right)\right](\nu^2+Q^2) + \left[\alpha_1\alpha_4^2\cosh\theta_1\sinh^2\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right) - \alpha_4^2\cosh^2\theta_1\sinh\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right)\nu^2\right\},$$

$$b_{33}^{(5)}(\nu, Q^2) = \alpha_4^2\sinh^2\theta_1\,\nu^2 + 4(\alpha_2+\tfrac{3}{2}\alpha_3\sinh\theta_1)^2(\nu^2+Q^2),$$

$$g_{11}^{(1)}(N, W) = \left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{(N^2-\frac{1}{4})}{(xN+1)^3}\left[3 - 4\,\frac{(N-\frac{3}{2})}{(xN-1)} + 4\,\frac{(N-\frac{3}{2})(N+\frac{1}{2})}{(xN-1)^2}\right],$$

$$g_{11}^{(2)}(N, W) = \left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{N(N^2-\frac{1}{4})}{(xN+1)^3(xN-1)}\left[-3+2\,\frac{(N-\frac{3}{2})}{(xN-1)}\right],$$

$$g_{11}^{(3)}(\nu, Q^2) = \left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{(N^2-\frac{1}{4})}{(xN+1)^2(xN-1)},$$

$$g_{33}^{(1)}(\nu,Q^2)=\left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{(N^2-\frac{1}{4})}{(xN+1)^3}\left[9-12\,\frac{(N-\frac{3}{2})}{(xN-1)}+4\,\frac{(N-\frac{3}{2})^2}{(xN-1)^2}\right],$$

$$g_{33}^{(2)}(\nu,Q^2)=\left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{N^2(N^2-\frac{1}{4})}{(xN+1)^4(xN-1)}\left[9-12\,\frac{(N-\frac{3}{2})}{(xN-1)}+4\,\frac{(N-\frac{3}{2})^3}{(xN-1)^2}\right],$$

$$g_{33}^{(3)}(\nu,Q^2)=\left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{N^2(N^2-\frac{1}{4})(N^2-\frac{9}{4})}{(xN+1)^4(xN-1)^3},$$

$$g_{33}^{(4)}(\nu,Q^2)=\left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{N(N^2-\frac{1}{4})}{(xN+1)^3(xN-1)}\left[-3+2\,\frac{(N-\frac{3}{2})}{(xN-1)}\right],$$

$$g_{33}^{(5)}(\nu,Q^2)=\left(\frac{xN-1}{xN+1}\right)^{N-1/2}\frac{(N^2-\frac{1}{4})}{(xN+1)^2(xN-1)},$$

$$x=\alpha_1\cosh\theta_1\left(\frac{\nu+M}{c-\alpha_2W}\right)-\sinh\theta_1\left(\frac{b-\alpha_3W}{c-\alpha_2W}\right),$$

and the kinematical quantities

$$\nu=\frac{Q^2}{2M\xi}-\tfrac{1}{2}M,\quad 0<\xi\le 1,\quad W=Q^2\,\frac{1}{\xi}\,(1-\xi).$$

Equations (4.10a) and (4.10b) give the exact final results for the structure functions MW_1 and νW_2. In the next section we will consider a special case and will evaluate the structure functions in the scaling limit ($\nu, Q^2\to\infty$, ξ fixed).

V. SPECIAL CASE

In this section we consider a special case in which the resonances satisfy an indefinitely increasing mass trajectory. We achieve this by taking explicitly the "saturation" constant $\alpha_3=0$ in Eq. (2.11). We obtain the mass formula

$$n'^2=\frac{(\alpha_2M_{n'}^2-c)^2}{(\alpha_1^2M_{n'}^2-b^2)} \tag{5.1}$$

In the scaling limit, $n'^2\sim(\alpha_2/\alpha_1)^2M_{n'}^2$ or $n'\sim(\alpha_2/\alpha_1)M_{n'}$. We give below all the relevant expressions in the scaling limit:

$$N^2\sim W\frac{\alpha_2^2}{\alpha_1^2}+(\text{const}),$$

$$N_N^2\sim\tfrac{1}{2}\alpha_2+\left(\frac{1}{W}\right),$$

$$|g'(N)|\sim\sqrt{W}\,\frac{2\alpha_1}{\alpha_2}+\left(\frac{1}{\sqrt{W}}\right),$$

$$x\sim-\frac{\alpha_1}{\alpha_2}\frac{\cosh\theta_1}{2M(1-\xi)},$$

$$xN\sim-\sqrt{W}\,\frac{\cosh\theta_1}{2M(1-\xi)},$$

$$b_{11}^{(1)}(\nu,Q^2)\sim\nu^2\tfrac{1}{3}\alpha_4^2\cosh^2\theta_1,$$

$$b_{11}^{(2)}(\nu,Q^2)\sim\nu^2\left[-\frac{2}{3}\frac{\alpha_1}{\alpha_2}\alpha_4^2\frac{\cosh^3\theta_1}{2M(1-\xi)}\right],$$

$$b_{11}^{(3)}(\nu,Q^2)\sim\nu^2\alpha_4^2\cosh^2\theta_1,$$

$$b_{33}^{(1)}(\nu,Q^2)\sim\nu^2\tfrac{1}{9}\alpha_4^2\cosh^2\theta_1,$$

$$b_{33}^{(2)}(\nu,Q^2)\sim\nu^2\frac{8}{9}\left[\frac{\alpha_1\alpha_4}{\alpha_2}\frac{\cosh\theta_1}{2M(1-\xi)}\right]^2,$$

$$b_{33}^{(3)}(\nu,Q^2)\sim\nu^2\frac{8}{9}\left[\frac{\alpha_1\alpha_4}{\alpha_2}\frac{\sinh\theta_1\cosh\theta_1}{2M(1-\xi)}\right]^2,$$

$$b_{33}^{(4)}(\nu,Q^2)\sim\nu^2\left[-\frac{2}{3}\frac{\alpha_1\alpha_4^2}{\alpha_2}\frac{\cosh\theta_1\sinh^2\theta_1}{2M(1-\xi)}\right],$$

$$b_{33}^{(5)}(\nu,Q^2)\sim\nu^2(\alpha_4^2\sinh^2\theta_1+4\alpha_2^2),$$

$$g_{11}^{(1)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^3(3+4B+4B^2),$$

$$B=\frac{\alpha_2}{\alpha_1}\frac{2M(1-\xi)}{\cosh\theta_1}$$

$$g_{11}^{(2)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^4(3+2B),$$

$$g_{11}^{(3)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^3,$$

$$g_{33}^{(1)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^3(9+12B+4B^2),$$

$$g_{33}^{(2)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^5(9+12B+4B^2),$$

$$g_{33}^{(3)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^7,$$

$$g_{33}^{(4)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^4(3+2B),$$

$$g_{33}^{(5)}(N,W)\sim\frac{-1}{\sqrt{W}}\frac{\alpha_1}{\alpha_2}B^3.$$

Substituting the above limits explicitly in Eqs. (4.10a) and (4.10b) we obtain after some trivial simplifications

$$\begin{aligned}F_1(\xi)&=\frac{1}{12\pi^2\alpha}\frac{1}{N_1^2}\frac{\alpha_4^2}{|\alpha_1|}\left(\frac{2M\alpha_2}{\alpha_1}\right)^4\frac{1}{\cosh^3\theta_1}(1-\xi)^3\\&\quad+O\left(\frac{1}{\nu}\right),\\F_2(\xi)&=2\xi F_1(\xi).\end{aligned} \tag{5.2}$$

Thus we find that the structure functions $MW_1(\nu, Q^2)$ and $\nu W_2(\nu, Q^2)$ satisfy Bjorken scaling, the Callan-Gross relation, and the Drell-Yan-West condition. We also find that in the scaling limit the contribution from W_{33} [or the second term of Eq. (4.10b)] vanishes always like $1/\nu$. This means that, as explained in Sec. II, the magnetic part of the inelastic excitation form factors $G(Q^2)$ [Eq. (2.5d)] dominates, in the scaling limit, over its electric counterpart. This situation is true in many direct-channel-resonance models.[8,9,13,18–20] Also a close scrutiny of the calculation reveals that any resonance model based on this O(4,2) infinite-component field framework, if it scales, will scale like $\nu^2(1/W^2)(1-\xi)^5$. Here ν^2 comes from the kinematics and $(1/W^2)(1-\xi)^5$ comes from the dynamics, and in the scaling limit one is related to the other by $W = 2M\nu(1-\xi)$.

VI. CONCLUSIONS

We have succeeded in evaluating the most general expressions in closed form for proton inelastic transition form factors [Eqs. (3.6a)–(3.6c)] and structure functions [Eqs. (4.10a) and (4.10b)] within the framework of O(4,2) infinite-component fields. The transition form factors all have leading dipole behavior in Q^2 (although there are small deviations for the electric form factors) as conjectured in various dynamical and phenomenological resonance models. The elastic magnetic form factor $[G_M(Q^2)]$ has exact dipole form in Q^2 and there is no immediate way that this behavior would be broken for large Q^2. If the experimental evidence on the deviation of G_M from the dipole behavior is indeed accurately established, then perhaps, to exlain such deviation, one may have to incorporate new features into the theory such as inclusion of a higher-order term in the electromagnetic current, etc. The calculation also gives an expression for the dipole mass $(t_{n'})$ of the nth resonance and its direct variation with $M_{n'}{}^2$ seems to be in agreement with the experimental data (at least for prominent nucleon resonances). Our final expressions for structure functions are indeed quite general. We have used them to a special case with an indefinitely increasing mass (Regge) trajectory. The model does indeed scale and satisfies the Callan-Gross relation and the Drell-Yan-West condition exactly. From the general expressions for νW_2 [Eq. (4.10b)] one could determine the value of Q^2 at which the scaling might start (precocious scaling) by making various plots of νW_2 versus the scaling parameter ξ $(0 \leq \xi \leq 1)$ at definite values of Q^2.

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