Calculation of the partial width for the decay $\eta \to \pi^+\pi^-\pi^0$ *

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We calculate the partial width for the decay $\eta \to \pi^+\pi^-\pi^0$ in a linear $U(3)\otimes U(3)_{\rm chiral}$ Lagrangian, using a model discussed previously in the literature by Haymaker and Carruthers. We find $\Gamma(\eta\to\pi^+\pi^-\pi^0) = 58$ eV which is consistent with a recent calculation of Weinberg using a nonlinear $U(3)\otimes U(3)_{\rm chiral}$ Lagrangian modified so as to incorporate the new mechanism proposed by Kogut and Susskind. However, the result is quite below the latest experimental result of 204 ± 22 eV. In addition, we briefly discuss the history behind the decay $\eta\to\pi^+\pi^-\pi^0$, emphasizing those points leading to the present understanding of the problem.

INTRODUCTION

The second-order electromagnetic decay of $\eta\to\pi^+\pi^-\pi^0$ is a long-standing problem of current algebra, partial conservation of axial-vector current (PCAC), and the "standard"[1] model of strong interactions. From 1966 when Sutherland[2] proved a theorem forbidding the decay until recently when Kogut and Susskind[3] (KS) proposed a way out of this dilemma the problem had reached several plateaus of partial understanding.

Most recently Weinberg[4] has recalculated the decay using a nonlinear $U(3)\otimes U(3)_{\rm chiral}$ Lagrangian, modified so as to incorporate the new mechanism proposed by KS.[5] He obtains a decay rate $\Gamma(\eta\to\pi^+\pi^-\pi^0)\simeq 54$ eV,[6] much smaller than the latest experimental result of 204 ± 22 eV.[7] Weinberg then suggests improving the calculation by including the leading edge of the ϵ(700 MeV) enhancement of π-π scattering. This paper takes the first step in that direction. We calculate the decay rate $\Gamma(\eta\to\pi^+\pi^-\pi^0)$ in the tree approximation in a linear $U(3)\otimes U(3)_{\rm chiral}$ Lagrangian, using a model discussed previously in the literature by Carruthers and Haymaker.[8] We find the amplitude is dominated by the ϵ pole (which lies outside the physical region). However, this affects a subsequent increase in the decay rate to only 58 eV, thus suggesting the model independence of the result. We also find that the usually assumed linear behavior of the amplitude on the π^0 energy (E_0) is only valid within the physical region. The amplitude rises sharply as $E_0\to 0$. However, this is clearly due to the close proximity of the ϵ to the point $E_0=0$ (only ~56 MeV away) and the failure of the tree approximation to properly account for its very large experimental and theoretical decay width ~579 MeV. We thus expect that the over-all picture of the amplitude would be significantly improved by going to the next order in the loop expansion, i.e., the one-loop approximation. However, the value of the amplitude in the physical region which is ~194 MeV from the ϵ pole may not be significantly changed. If the result is indeed model-independent then the puzzle of the decay $\eta\to\pi^+\pi^-\pi^0$ is still a puzzle since the theoretical prediction is clearly quite below the latest experimental value.

This paper is organized as follows: In Sec. I we briefly review the history behind the decay $\eta\to\pi^+\pi^-\pi^0$. We try to give the main points leading up to the present understanding of the problem. In Sec. II we present the model and our results. Sec. II is divided into four sections. In subsections A and B we discuss the strong-interaction and electromagnetic-interaction parts respectively of the Lagrangian, explaining how the parameters in the theory were fitted. In subsection C the results of the calculation are given and also checks for possible calculational errors. Finally in subsection D we discuss our results and present our conclusions.

I. REVIEW

The amplitude for the decay is experimentally known to have the form[9]

$$T(\eta\to\pi^+\pi^-\pi^0)=\alpha+\beta E_0, \tag{1}$$

where E_0 is the energy of the neutral pion and $\beta/\alpha\simeq -2/m_\eta$. Sutherland[2] showed that the amplitude should vanish when any one of the three pions' 4-momentum is taken to zero. In particular when either charged pions' 4-momentum vanishes $E_0=m_\eta/2$ and the extrapolated experimental result upholds this prediction. However, if one extrapolates the experimental amplitude, assuming it behaves linearly in E_0 even to the point $(E_0,\vec{q}_{\pi^0})=0$, then one evidently predicts $T(\eta\to\pi^+\pi^-\pi^0)\equiv 0$.

Soon after this result Bell and Sutherland[10] removed the restriction of the linear dependence of the amplitude, allowing for a quadratic dependence on the pion momenta as well. As a result the decay was no longer forbidden but still required to be exceedingly suppressed.

There is evidently a problem with the Sutherland theorem at the neutral-soft-pion point. In order to facilitate discussion of the problem we will calculate the amplitude explicitly in this limit. The invariant amplitude for $\eta \to 3\pi$ to second order in electromagnetism is given by[11]

$$T(\eta \to 3\pi) = \langle \pi^+\pi^-\pi^0 | \mathcal{L}_{\rm em}(0) | \eta \rangle, \tag{2}$$

where

$$\mathcal{L}_{\rm em}(0) = -e^2 \int d^4y\, T(j^\alpha_{\rm em}(y) j^\alpha_{\rm em}(0)) D_{\alpha\beta}(y). \tag{3}$$

$j^\alpha_{\rm em}$ is the electromagnetic current and $D_{\alpha\beta}$ is the free photon propagator. Using the standard reduction technique and the PCAC relation

$$\partial_\mu j_3^{\mu 5}(x) = m_\pi^2 f_\pi \phi_{\pi 0}(x), \tag{4}$$

we obtain

$$T(\eta \to 3\pi) = -e^2 \frac{(-K^2 + m_\pi^2)}{m_\pi^2 f_\pi} \int d^4x\, d^4y\, e^{iK\cdot x} D_{\alpha\beta}(y) \langle \pi^+\pi^- | T(\partial_\mu j_3^{\mu 5}(x) j^\alpha_{\rm em}(y) j^\beta_{\rm em}(0)) | \eta \rangle. \tag{5}$$

Pulling out the derivative from inside the time-ordering we have

$$\begin{aligned} T(\eta \to 3\pi) = {} & ie^2 \frac{(-K^2 + m_\pi^2) K_\mu}{m_\pi^2 f_\pi} \int d^4x\, d^4y\, e^{iK\cdot x} \langle \pi^+\pi^- | T(j_3^{\mu 5}(x) j^\alpha_{\rm em}(y) j^\beta_{\rm em}(0)) | \eta \rangle D_{\alpha\beta}(y) \\ & + e^2 \frac{(-K^2 + m_\pi^2)}{m_0^2 f_\pi} \int d^4x\, d^4y\, e^{iK\cdot x} \langle \pi^+\pi^- | F^{\alpha\beta}(x, y) | \eta \rangle D_{\alpha\beta}(y), \end{aligned}$$

where

$$\begin{aligned} F_{\alpha\beta}(x, y) \equiv {} & \{\delta(x_0)\theta(y_0) j^\alpha_{\rm em}(y) [j_3^{05}(x), j^\beta_{\rm em}(0)] + \delta(x_0)\theta(-y_0) [j_3^{05}(x), j^\beta_{\rm em}(0)] j^\alpha_{\rm em}(y) \\ & + \delta(x_0 - y_0)\theta(y_0) [j_3^{05}(x), j^\alpha_{\rm em}(y)] j^\beta_{\rm em}(0) + \delta(x_0 - y_0)\theta(-y_0) j^\beta_{\rm em}(0) [j_3^{05}(x), j^\alpha_{\rm em}(y)] \\ & + \delta(x_0)\theta(x_0 - y_0)\theta(y_0) j_3^{05}(x) j^\alpha_{\rm em}(y) j^\beta_{\rm em}(0) + \delta(x_0 - y_0)\theta(x_0)\theta(-y_0) j_3^{05}(x) j^\beta_{\rm em}(0) j^\alpha_{\rm em}(y) \\ & - \delta(x_0 - y_0)\theta(y_0)\theta(-x_0) j^\alpha_{\rm em}(y) j^\beta_{\rm em}(0) j_3^{05}(x) - \delta(x_0)\theta(-y_0)\theta(y_0 - x_0) j^\beta_{\rm em}(0) j^\alpha_{\rm em}(y) j_3^{05}(x)\}. \end{aligned} \tag{6}$$

At the neutral-soft-pion point $K_\mu \to 0$ the amplitude naively speaking vanishes. The first part vanishes since it is proportional to K_μ and there are no massless poles in the theory to cancel this zero.[12] In the second part $F^{\alpha\beta}$ contains eight terms. The first four terms vanish identically as a result of current algebra.[13] They are all proportional to the matrix elements of the operator

$$[Q_3^5(y_0), j^\alpha_{\rm em}(y)] \equiv 0. \tag{7}$$

It is the last four terms which have been a source of confusion for many years. It was originally thought that they also vanish [because of the product of θ functions $\theta(x_0)\theta(-x_0)$], thus obtaining Sutherland's neutral-soft-pion zero and the consequent suppression of $\eta \to 3\pi$.[2]

It was at this point that some people suggested another electromagnetic interaction, i.e., the so-called "u_3 tadpole" of Coleman and Glashow.[14] As is well known the K^+-K^0 and p-n mass differences are not explained by a second-order electromagnetic mass shift. The correction obtained in this way always has the wrong sign.[15] Coleman and Glashow showed that the u_3 tadpole, belonging to a nonet of scalar densities u_a, $a = 0, \ldots, 8$, in addition to the second-order electromagnetic interaction, was sufficient to fit the measured hadronic SU(2)-breaking mass differences. In addition, the u_8 member of the multiplet fits the SU(3)-breaking mass differences. Thus the effective phenomenological interaction Lagrangian was assumed to have the form

$$\mathcal{L}_{\rm eff}(0) = \mathcal{L}_{\rm em}(0) - m_3 u_3(0). \tag{8}$$

With this extra term it was shown in some models that the amplitude for the decay $\eta \to 3\pi$ no longer vanishes at the neutral-soft-pion point.[16]

Later Wilson[17] realized that the u_3 tadpole arises naturally in the "standard" model of strong interactions. The "standard" model is a model in which quarks and gluons are the only elementary strongly interacting fields. The quarks transform as the fundamental representation of an $SU(3) \otimes SU(3)_{\rm color}$ and the gluons form a multiplet of Yang-Mills gauge bosons coupled to the color degrees of freedom. The "standard" model has many desirable features. It explains the quark-model spectrum of low-lying states,[18] and is a basis for current algebra.[13] It has the proper short-distance behavior, given by asymptotic freedom,[19] to explain deep-inelastic electron-proton and neutron scattering.[20] Combined with the necessary large-distance behavior (infrared slavery) it can explain quark confinement.[21]

Finally, theories of this type are the natural "effective field theories" which arise in unified gauge theories of the weak, electromagnetic, and strong interactions.[22]

Wilson noted that included in the short-distance expansion of the operator product $j^{\alpha}_{\rm em}(y)j^{\beta}_{\rm em}(0)$ is the SU(2)-symmetry-breaking operator $C^{\alpha\beta}(y)u_3(0)$, where $u_3 \equiv \bar\psi\lambda_3\psi/2$ is the tadpole of Coleman and Glashow and $C^{\alpha\beta}$ is a singular c-number function of y with canonical dimensions of (mass)^3. This term is responsible for the electromagnetic mass splitting of the p and n quarks.

Let us now reanalyze the last four terms of Eq. (6) which were previously thought to vanish. The first term is proportional to the matrix element of the operator expression

$$\lim_{\epsilon\to 0}\left[e^2\int d^4y\ \theta(-y_0+\epsilon)\theta(y_0)Q^5_3(\epsilon)j^{\alpha}_{\rm em}(y)j^{\beta}_{\rm em}(0)D_{\alpha\beta}(y)\right], \tag{9}$$

where we have taken the limit $K_\mu \to 0$ and used the identity

$$\delta(x_0)=\lim_{\epsilon\to 0}\delta(x_0-\epsilon).$$

Now replacing $j^{\alpha}_{\rm em}j^{\beta}_{\rm em}$ at short distances by the term $C^{\alpha\beta}u_3$ we have

$$\lim_{\epsilon\to 0}\int d^4y\ \theta(-y_0+\epsilon)\theta(y_0)C^{\alpha\beta}(y)D_{\alpha\beta}(y)Q^5_3(\epsilon)u_3(0). \tag{10}$$

Suppose the produce CD has the form

$$C^{\alpha\beta}(y)D_{\alpha\beta}(y)\,\alpha\,\frac{m}{y^4}f(m^2y^2), \tag{11}$$

which is dimensionally correct and reflects the fact that a mass renormalization is always proportional to the bare mass, m. Integrating over space we obtain

$$\lim_{\lambda\to\infty}\lim_{\epsilon\to 0} m\int_{\epsilon/\lambda}^{\epsilon}\frac{dy_0}{y_0}g(my_0)Q^5_3(\epsilon)u_3(0), \tag{12}$$

where we have cut off the divergent integral at short distances and

$$mg(my_0)/y_0 \equiv e^2\int d^3y\ C^{\alpha\beta}(y)D_{\alpha\beta}(y). \tag{13}$$

Integrating once more we have

$$\lim_{\lambda\to\infty}(mg(0)\ln\lambda)Q^5_3(0)u_3(0). \tag{14}$$

Finally combining all four terms which are evaluated similarly, we obtain

$$\lim_{\lambda\to\infty}(mg'(0)\ln\lambda)[Q^5_3(0),u_3(0)], \tag{15}$$

where g' is the sum of g plus a similar term obtained from the operator product $j^{\beta}_{\rm em}(0)j^{\alpha}_{\rm em}(y)\theta(-y_0)$.[23] This expression is equivalent to what we would have obtained as a result of the effective Lagrangian of Eq. (8) with $m_3=+\lim_{\lambda\to\infty}(mg'(0)\ln\lambda)$. In unified theories of weak and electromagnetic interactions there are additional terms of order α which contribute to the p-n quark mass difference and to the decay $\eta\to 3\pi$. These terms effectively cutoff the $\ln\lambda$ divergence at the masses of the heavy bosons in the theory, resulting in a finite value of m_3.[24]

To summarize our results of the preceding discussion we write the amplitude for the decay $\eta\to 3\pi$ as it now stands in the neutral-soft-pion limit:

$$T(\eta\to 3\pi)\xrightarrow[K_\mu\to 0]{}\frac{m_3}{f_\pi}\langle\pi^+\pi^-|[Q^5_3(0),u_3(0)]|\eta\rangle. \tag{16}$$

The chiral charge Q^5_3 is just one generator of the group $U(3)\otimes U(3)_{\rm chiral}$. The vector charges

$$Q_a\equiv\int d^3x j^0_a,\quad a=0,\ldots,8$$

where $j^{\mu}_a\equiv\psi\gamma^{\mu}\lambda_a\psi/2$ together with the chiral charges

$$Q^5_a\equiv\int d^3x\, j^{05},\quad a=0,\ldots,8$$

where $j^{\mu 5}_a=\bar\psi\gamma^{\mu}\gamma^5\lambda_a\psi/2$ satisfy the well-known equal-time commutation relations of current algebra (see Ref. 13, p.28). Similarly u_3 is a member of the $(3,\bar 3)\oplus(\bar 3,3)$ representation of $U(3)\otimes U(3)_{\rm chiral}$. The nonet of scalar densities $u_a=\bar\psi\lambda_a\psi/2$, $a=0,\ldots,8$, along with the nonet of pseudoscalar densities $v_a=-i\bar\psi\gamma 5\lambda a\psi/2$, $a=0,\ldots,8$, satisfy the equal-time commutation relations

$$\begin{aligned}
&[Q_a,u_b]=if_{abc}u_c,\\
&[Q_a,v_b]=if_{abc}v_c,\\
&[Q^5_a,u_b]=-id_{abc}v_c,\\
&[Q^5_a,v_b]=id_{abc}u_c,
\end{aligned} \tag{17}$$

with $d_{0ab}\equiv(\frac{2}{3})^{1/2}\delta_{ab}$, $a,b=0,\ldots,8$ and $\lambda_0\equiv(\frac{2}{3})^{1/2}I$. Thus we can evaluate the commutator in Eq. (16) and we obtain

$$[Q^5_3(0),u_3(0)]=-i\left[(\tfrac{2}{3})^{1/2}v_0(0)+\frac{1}{\sqrt 3}v_8(0)\right]. \tag{18}$$

It is only now that we confront the last problem for the decay $\eta\to 3\pi$. It turns out that the right-hand side of Eq. (18) is proportional to the total divergence of the 0 and 8 axial-vector currents. Thus if we recall Eq. (16) we see that the amplitude for the decay $\eta\to 3\pi$ at the neutral-soft-pion point vanishes anyway (unless for some reason the 0 and/or 8 axial-current creates a massless particle). Therefore, to verify the above statement let us calculate the current divergence.

We write the strong-interaction Lagrangian in the form suggested by Gell-Mann, Oakes, and Renner,[25]

$$\mathcal{L}_{st} = \mathcal{L}_{sym} - m_0 u_0 - m_8 u_8, \tag{19}$$

where $\mathcal{L}_{sym}$ is $U(3)\otimes U(3)_{chiral}$-symmetric and the terms m_0 and m_8 break this symmetry. We recall that in the limit m_0, $m_8 \to 0$, the symmetry is realized by the Nambu-Goldstone[26] mechanism and a nonet of massless pseudoscalar bosons. However, the real world is closer to the special case $m_8/m_0 = -\sqrt{2}$ with $U(3)\otimes U(3)_{chiral}$ broken leaving $SU(2)\otimes SU(2)_{chiral}$ invariant and a triplet of massless bosons. Moreover, we should emphasize that although an octet of low-mass pseudoscalar states is found in nature—the π, K, and η mesons—the ninth pseudoscalar meson which should by symmetry arguments have a mass comparable to or lower than that of the pions (see e.g. Weinberg, Ref. 4) can only be identified with the $\eta'(958)$. The problem of the ninth Goldstone boson is as old as that of $\eta \to 3\pi$ and as we shall see is solved by the same mechanism. Now using the expression for $\mathcal{L}_{st}$ [Eq. (19)], the standard relation

$$\partial_\mu j_a^{\mu 5}(0) = i[Q_a^5(0), \mathcal{L}_{st}(0)], \tag{20}$$

and the transformation relations of the u's and v's [Eq. (17)], we find

$$\begin{aligned}\partial_\mu j_0^{\mu 5} &= -(\tfrac{2}{3})^{1/2} m_0 v_0 - (\tfrac{2}{3})^{1/2} m_8 v_8,\\ \partial_\mu j_8^{\mu 5} &= -(\tfrac{2}{3})^{1/2} m_0\left(1 - \frac{m_8}{\sqrt{2}m_0}\right) v_8 - (\tfrac{2}{3})^{1/2} m_8 v_0,\end{aligned} \tag{21}$$

or

$$(\tfrac{2}{3})^{1/2} v_0 + \frac{1}{\sqrt{3}} v_8 = \frac{-1}{m_0(1+m_8/\sqrt{2}m_0)} \times\left(\frac{1}{\sqrt{2}}\partial_\mu j_8^{\mu 5} + \partial_\mu j_0^{\mu 5}\right). \tag{22}$$

Thus

$$\begin{aligned}T(\eta\to 3\pi) &\xrightarrow[K_\mu\to 0]{} + \frac{i m_3/m_0}{(1+m_8/\sqrt{2}m_0)f_\pi} \times\left\langle \pi^+\pi^- \left| \left(\frac{1}{\sqrt{2}}\partial_\mu j_8^{\mu 5}(0) + \partial_\mu j_0^{\mu 5}(0)\right)\right| \eta\right\rangle\\ &\xrightarrow[K_\mu\to 0]{} \frac{m_3/m_0}{(1+m_8/\sqrt{2}m_0)f_\pi} \times K_\mu\left\langle \pi^+\pi^- \left| \left(\frac{1}{\sqrt{2}} j_8^{\mu 5}(0) + j_0^{\mu 5}(0)\right)\right| \eta\right\rangle\end{aligned} \tag{23}$$

vanishes unless there is a pole at $K^2 = 0$ in the amplitude $\langle \pi^+\pi^- | (1/\sqrt{2}) j_8^{\mu 5} + j_0^{\mu 5} | \eta\rangle$. As there are no known massless particles in the physical space of states with these quantum numbers the amplitude once again apparently vanishes at the neutral soft-pion point.

Recently KS proposed a solution to this puzzle. The solution focuses on the following point. All the operators $j_a^{\mu 5}$, $a = 1, \ldots, 8$ are invariant under the color gauge group; however $j_0^{\mu 5} \equiv \bar{\psi}\gamma^\mu\gamma^5\lambda_0\psi/2$ is not. In order to construct a gauge invariant operator one must take the limit $\epsilon \to 0$ of the point split and explicitly gauge-invariant operator

$$\hat{j}_0^{\mu 5}(x,\epsilon) \equiv \bar{\psi}(x+\epsilon)\gamma^\mu\gamma^5 \frac{\lambda_0}{2}\, \mathrm{L}\left[\exp\left(i\int_x^{x+\epsilon} d1^\mu B_\mu(x')\right)\right]\psi_{(x)},$$

where $B_\mu \equiv \frac{1}{2}\rho^\alpha B_\mu^\alpha$.

$$\rho^\alpha, \quad \alpha = 1, \ldots, 8,$$

are the generators of the group $SU(3)_{color}$ in the fundamental representation, B_μ^α is the octet of colored gluons, and L symbolizes a line-ordered product. (We use the Greek indices α, β, γ to denote the color degrees of freedom and the caret over an operator denotes a gauge-invariant operator.) In the limit $\epsilon \to 0$ Adler, Bell, and Jackiw[27] have shown that the operator $\hat{j}_0^{\mu 5}(x,\epsilon)$ has the limiting form

$$\begin{aligned}\hat{j}_0^{\mu 5}(x) &\equiv \lim_{\epsilon\to 0}\hat{j}_0^{\mu 5}(x,\epsilon)\\ &= j_0^{\mu 5}(x) + \frac{g^2}{4\pi} B_\nu^\alpha(x) G_\alpha^{*\mu\nu}(x),\end{aligned} \tag{24}$$

where $G_\alpha^{*\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^\alpha$ is the dual of the gluon field strength $G_{\rho\sigma}^\alpha$ and

$$G_{\rho\sigma}^\alpha \equiv \partial_\rho B_\sigma^\alpha - \partial_\sigma B_\rho^\alpha + g f^{\alpha\beta\gamma} B_\rho^\beta B_\sigma^\gamma.$$

Since $j_0^{\mu 5}$ is not gauge-invariant it can create states in the nonphysical negative metric sector of the Hilbert space.[28] Thus Goldstone's theorem (which implies that $j_0^{\mu 5}$ creates a massless particle in the symmetry limit $m_0, m_8 \to 0$) can be satisfied by an unphysical, negative metric, ghost state.

KS demonstrate in a relativistic quantum field theory—two-dimensional, massless, Abelian QED—that the same long-range forces which confine quarks create a "dipole" pair of massless pseudoscalar mesons, one with positive metric and the other with negative metric. The dipole pair cancel identically in all matrix elements of gauge-invariant operators so that as required there are no physical states of zero mass, which by analogy would correspond to a ninth Goldstone boson. However, in matrix elements of nongauge-invariant operators only the ghost state contributes. They also show that when the symmetry is broken by a small bare mass term the dipole pair remains massless, and thus again by analogy there is a pole at $K^2=0$ to save $\eta\to 3\pi$. Finally, they propose this mechanism to operate in the "standard" model of strong interactions where similar long-range forces have been been shown to confine color.[21]

Thus for our purposes the proposed mechanism has the following two features.

The massless ghost state contributes to the matrix element $\langle\pi^+\pi^-|j_0^{\mu 5}|\eta\rangle$ so that at the neutral soft-pion point

$$T(\eta\to 3\pi)\xrightarrow[K_\mu\to 0]{}\frac{m_3/m_0}{(1+m_8/\sqrt{2}m_0)f_\pi}\times K_\mu\langle\pi^+\pi^-|j_0^{\mu 5}(0)|\eta\rangle \tag{25}$$

is not zero. Equivalently

$$T(\eta\to 3\pi)\xrightarrow[K_\mu\to 0]{}-i\frac{m_3/m_0}{(1+m_8/\sqrt{2}m_0)f_\pi}\times\left\langle\pi^+\pi^-\left|\frac{g^2}{4\pi}G^\alpha_{\mu\nu}(0)G^{*\mu\nu}_\alpha(0)\right|\eta\right\rangle \tag{26}$$

is proportional to the matrix element of the Adler-Bell-Jackiw anomaly which likewise by virtue of the ghost does not vanish.[29]

Secondly, the symmetry limit $m_0, m_8\to 0$ is realized by an octet of massless pseudoscalar mesons identified with the π, K, η octet and the massless dipole pair. The dipole pair cancels in all matrix elements of gauge-invariant operators, so that there is no ninth Goldstone boson in the gauge-invariant space of states. Equivalently and self-consistently, the ghost contributes in matrix elements of nongauge-invariant operators so that in the symmetry limit $m_0, m_8\to 0$ the $U(3)\otimes U(3)_{\text{chiral}}$ symmetry is effectively broken down to $SU(3)\otimes SU(3)_{\text{chiral}}$. We have

$$\partial_\mu \hat{j}_0^{\mu 5}=\frac{g^2}{4\pi}G^\alpha_{\mu\nu}G^{*\mu\nu}_\alpha,$$

where the anomaly behaves as if it were not a total divergence, thus invalidating Goldstone's theorem.

Weinberg[4] in a recent paper incorporated the dipole pair in a nonlinear $U(3)\otimes U(3)_{\text{chiral}}$ Lagrangian. The dipole pair transformed as the ninth component of a nonet of pseudoscalar mesons.

In this paper we work with a linear $U(3)\otimes U(3)_{\text{chiral}}$ Lagrangian with a nonet of scalar and pseudoscalar mesons. In the symmetry limit $U(3)\otimes U(3)$ is broken down to $SU(3)\otimes SU(3)$ by an explicit symmetry-breaking interaction. Thus the ninth axial-vector current is not conserved. We have

$$\partial_\mu \hat{j}_0^{\mu 5}=3g\delta_5,$$

where the operator δ_5 (to be discussed in the next section) plays the role of the Adler-Bell-Jackiw anomaly with its accompanying ghost pole contribution. The model contains no explicit ghost states. We work in the physical gauge-invariant space of states with the ninth pseudoscalar meson identified with the $\eta'(958)$.[30]

II. MODEL AND RESULTS

A. The strong-interaction Lagrangian

The model consists of a nonet of pseudoscalar mesons ϕ_a, $a=0,\ldots,8$ and a nonet of scalar mesons σ_a, $a=0,\ldots,8$ which together transform as the $(3,\overline{3})\oplus(\overline{3},3)$ representation of the group $U(3)\otimes U(3)_{\text{chiral}}$. Explicitly we have

$$\begin{aligned}
&[Q_a,\phi_b]=if_{abc}\phi_c,\\
&[Q_a,\sigma_b]=if_{abc}\sigma_c,\\
&[Q_a^5,\phi_b]=id_{abc}\sigma_c,\\
&[Q_a^5,\sigma_b]=-id_{abc}\phi_c,
\end{aligned}\tag{27}$$

with $d_{0ab}\equiv(\frac{2}{3})^{1/2}\delta ab$, $a,b,=0,\ldots,8$ and Q_a, Q_a^5, $a,=0,\ldots,8$ the ordinary U(3) and chiral generators respectively of the group. We also define the 3×3 matrices

$$\phi\equiv\phi_a\frac{\lambda_a}{\sqrt{2}},\quad \sigma=\sigma_a\frac{\lambda_a}{\sqrt{2}},\ \text{and}\ M=\sigma+i\phi,$$

with $\lambda_0\equiv(\frac{2}{3})^{1/2}I$.

Using this notation the strong-interaction Lagrangian has the form[31,32]

$$\begin{aligned}
\mathcal{L}_{st}=&\tfrac{1}{2}\mathrm{Tr}\partial_\mu M^+\partial^\mu M+f_1(\mathrm{Tr}M^+M)^2\\
&+f_2\mathrm{Tr}M^+MM^+M+g(\det M+\det M^+)\\
&-\epsilon_0\sigma_0-\epsilon_8\sigma_8.
\end{aligned}\tag{28}$$

In the symmetry limit $\epsilon_0,\epsilon_8\to 0$ the Lagrangian is invariant under the group $SU(3)\otimes SU(3)_{\text{chiral}}$ with the $U(3)\otimes U(3)_{\text{chiral}}$ symmetry broken only by the term $g\delta$ ($\delta\equiv\det M+\det M^+$). The operator δ along with $\delta_5\equiv -i(\det M-\det M^+)$ transform into each other under the action of the ninth axial charge. We have

$$\begin{aligned}
&[Q_0^5,\delta]=-3id_{000}\delta_5,\\
&[Q_0^5,\delta]=3id_{000}\delta.
\end{aligned}\tag{29}$$

Together δ and δ_5 effectively do the work of the massless ghost and the Adler, Bell, Jackiw anomaly discussed in Sec. I.

The symmetry-breaking part of $\mathcal{L}_{st}$, on the other hand, transforms in the way suggested by Gell-Mann, Oakes, and Renner.[25] The tadpoles σ_0 and σ_8 explicitly break the $SU(3)\otimes SU(3)_{\text{chiral}}$

symmetry down to SU(3) and SU(2), respectively, with the special case $a \equiv \epsilon_8/\sqrt{2}\,\epsilon_0 = -\sqrt{2}$ leaving $\mathcal{L}_{st}$ invariant under the group $SU(2)\otimes SU(2)_{chiral}$. In fact, after fitting all the parameters of the theory in a way we will soon specify, we find $a = -0.919$ as compared to a (Gell-Mann, Oakes, and Renner) $= -0.89$. The difference between our result and that of Gell-Mann, Oakes, and Renner lies in the fact that we calculate in the tree approximation whereas they calculate to first order in ϵ_8. The two approximations are not equivalent as will increasingly become evident as we continue.

When calculating the effective potential[33] for this theory, which in the tree approximation is the negative sum of all nonderivative terms in the Lagrangian, we find the minimum, in the symmetry limit $\epsilon_0, \epsilon_8 \to 0$, to be at the point $\xi_0' \equiv \langle\sigma_0\rangle \neq 0$. Thus although the Lagrangian is symmetric under $SU(3)\otimes SU(3)_{chiral}$, the vacuum is spontaneously broken down to SU(3) resulting via Goldstone's theorem (see Goldstone, Ref. 26) in an octet of massless pseudoscalar mesons. When the symmetry-breaking tadpoles are then turned on ($\epsilon_0, \epsilon_8 \neq 0$) the minimum of the potential moves still further to the point $\xi_a \equiv \langle\sigma_a\rangle_0 \neq 0$, $a = 0, 8$. In order to expand in a perturbation series of small oscillations about the true vacuum we define the shifted fields

$$\sigma \to \sigma + \Sigma,$$

where

$$\Sigma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \alpha \equiv \frac{\xi_0}{\sqrt{3}}(1+b),$$

$$\beta \equiv \frac{\xi_0}{\sqrt{3}}(1-2b), \text{ and } b \equiv \xi_8/\sqrt{2}\,\xi_0.$$

ξ_0 and ξ_8 are defined by the condition that the coefficients of the linear terms in $\mathcal{L}_{st}$ vanish. We have as a result of this condition the relations

$$\begin{aligned} \epsilon_0 &= \xi_0^2[\tfrac{4}{3}\xi_0 G(b) + \gamma(1-b^2)], \\ \epsilon_8/\sqrt{2} &= \xi_0^2 b[4\xi_0 H(b) - \gamma(1+b)], \end{aligned} \tag{30}$$

where

$$G(b) = 3f_1(1+2b^2) + f_2(1+6b^2-2b^3),$$

$$H(b) = f_1(1+2b^2) + f_2(1-b+b^2),$$

and

$$\gamma \equiv 2g/\sqrt{3}.$$

Fitting the parameters in the shifted strong-interaction Lagrangian proceeds as follows. The terms quadratic in the fields determine the masses, in the tree approximation, of the 18 scalar and pseudoscalar particles. In Table I, the masses are listed in terms of the parameters in the Lagrangian. For the unmixed fields ϕ_a, σ_a, $a = 1, \ldots, 7$ the mass is identified by the relation e.g. $\mathcal{L}_{st} = -\frac{1}{2}m_a^2\phi_a^2$. However, for the mixed fields, ϕ_0, ϕ_8 and σ_0, σ_8 which appear in $\mathcal{L}_{st}$ as e.g.

$$\mathcal{L}_{st} = -\tfrac{1}{2}(m_{00}^2\phi_0^2 + m_{88}^2\phi_8^2 + 2m_{08}^2\phi_0\phi_8),$$

the mass matrix is diagonalized by the rotated fields, e.g.,

$$\eta = \phi_8\cos\theta_P - \phi_0\sin\theta_P,$$

$$\eta' = \phi_8\sin\theta_P + \phi_0\cos\theta_P,$$

with the corresponding eigenvalues identified as the masses. As apparent from Table I, there are four independent parameters which are chosen to be $f_1\xi_0^2$, $f_2\xi_0^2$, $\gamma\xi_0$, and b. These are fitted using the well-known masses for the pseudoscalar particles π, K, η, and η' (or X_0) as input.[34] Thus we take as input $m_\pi^2 = 0.01906$ GeV^2, $m_K^2 = 0.2450$ GeV^2, $m_\eta^2 = 0.3003$ GeV^2, and $m_{\eta'}^2 = 0.9178$ GeV^2. Determining the four independent parameters then fixes the values of the eight dependent parameters

$$\epsilon_0/\xi_0, \quad a \equiv \epsilon_8/\sqrt{2}\epsilon_0, \quad m_\delta, \quad m_\kappa, \quad m_\epsilon, \quad m_{\epsilon'}, \quad \tan\theta_P,$$

and $\tan\theta_S$.

In Table II is the solution obtained to these fits by Haymaker and Carruthers (Ref. 8). The values of the masses obtained for the scalar particles should be compared with the experimental results listed in Ref. 34. The δ is identified with the

TABLE I. Masses in terms of parameters in the Lagrangian. Mass is identified by the relation $\mathcal{L} = -\frac{1}{2}m^2\phi^2$ for the unmixed fields, and is given by $m^2 = -2f_1\xi_0^2A_1 - 2f_2\xi_0^2A_2 - \gamma\xi_0A_3$. For the mixed fields, $\mathcal{L} = -\frac{1}{2}(m_{00}^2\phi_0^2 + m_{88}^2\phi_8^2 + 2m_{08}^2\phi_0\phi_8)$.

m^2	A_1	A_2	A_3
π	$2(1+2b^2)$	$\frac{2}{3}(b+1)^2$	$1-2b$
K	$2(1+2b^2)$	$\frac{2}{3}(7b^2-b+1)$	$1+b$
δ	$2(1+2b^2)$	$2(b+1)^2$	$-(1-2b)$
κ	$2(1+2b^2)$	$2(b^2-b+1)$	$-(1+b)$
η_{00}	$2(1+2b^2)$	$\frac{2}{3}(1+2b^2)$	-2
η_{88}	$2(1+2b^2)$	$\frac{2}{3}(3b^2-2b+1)$	$1+2b$
η_{08}	0	$\frac{2}{3}\sqrt{2}b(b-2)$	$\sqrt{2}b$
σ_{00}	$2(3+2b^2)$	$2(1+2b^2)$	2
σ_{88}	$2(1+6b^2)$	$2(3b^2-2b+1)$	$-(1+2b)$
σ_{08}	$4b\sqrt{2}$	$2\sqrt{2}b(2-b)$	$-\sqrt{2}b$

$\delta(970)$, $I^G = 1^-$ with a dominant decay mode to $\eta\pi$. The fitted mass is 911 MeV. The κ is the broad bump in the $K\pi$ channel $I=\frac{1}{2}$ with the phase shift slowly going through 90° near 1300 MeV. The fitted mass is ~903 MeV. The ϵ is the broad enhancement in π-π phase shifts with $I^G = 0^+$, a mass $\leq$700 MeV, and a width $\geq$600 MeV. The fitted mass is ~604 MeV. Lastly, the ϵ' is identified with the $S^*(993)$, $I^G = 0^+$ with a fitted mass 1094 MeV. All the fitted masses, except for the κ, are within approximately 10% of the experimental results. However, improved results can be obtained for the masses of the scalar mesons if one includes a mass term in $\mathcal{L}_{st}$.[35] We will come back to this point again when we discuss the results of the calculation. Two other notable features of the fit are the mixing angles θ_P and θ_S which agree well with quark-model expectations. The η is found to be almost pure octet (ϕ_8) with $\sin\theta_P \sim 0.04$. The ϵ, on the other hand, is very nearly the nonstrange combination of σ_0 and σ_8 with

$$-\epsilon \sim (\tfrac{2}{3})^{1/2}\sigma_0 + \frac{1}{\sqrt{3}}\sigma_8, \text{ i.e., } -\cos\theta_S = 0.61 \sim 1/\sqrt{3}$$

TABLE II. Solution I of Haymaker and Carruthers (Ref. 8). (Dimensional quantities are in GeV units.)

$\tan\theta_P$	0.040 66	
m_δ	0.9110	
m_K	0.9025	
$m_{\epsilon'}$	1.0940	
m_ϵ	0.6035	
$\tan\theta_S$	−1.283	
$f_1\xi_0^2$	−0.075 75	
$f_2\xi_0^2$	−0.056 92	
$\gamma\xi_0$	0.2522	
b	−0.2102	$b \equiv \xi_8/\sqrt{2}\xi_0$
ϵ_0/ξ_0	−0.1861	
a	−0.9189	$a \equiv \epsilon_8/\sqrt{2}\epsilon_0$

The mixing angles are defined by

$$\eta = \phi_8\cos\theta_P - \phi_0\sin\theta_P$$
$$\eta' = \phi_8\sin\theta_P + \phi_0\cos\theta_P$$
$$\epsilon = \sigma_8\cos\theta_S - \sigma_0\sin\theta_S$$
$$\epsilon' = \sigma_8\sin\theta_S + \sigma_0\cos\theta_S$$

$\xi_0 = 0.1470$ $\quad f_\pi/f_K \equiv \dfrac{1+b}{1-b/2} = 0.7147$

$\Gamma_\epsilon = 0.5794$

and

$$\sin\theta_S = 0.79 \sim (\tfrac{2}{3})^{1/2}.$$

One parameter of $\mathcal{L}_{st}$ which has not been fitted thus far is the value of ξ_0. As we will see the overall amplitude for $\eta \to 3\pi$ is proportional to $1/\xi_0^2$. We determine ξ_0 using operator PCAC satisfied to each order in the loop expansion.[36]

As a result we have $f_\pi \equiv (\frac{2}{3})^{1/2}\xi_0(1+b)$. Using the experimental value $f_\pi = 0.095$ Gev we obtain $\xi_0 = 0.147$ Gev.

Additional tests of the accuracy of the parameters in $\mathcal{L}_{st}$ other than those already discussed include decay rates, scattering lengths, and other experimental parameters uniquely predicted by $\mathcal{L}_{st}$. Two such parameters which we have calculated and present now for later reference are the decay rate $\Gamma_\epsilon \to 2\pi$ and the ratio f_π/f_K. The calculation of Γ_ϵ makes use of the fitted values of $f_1\xi_0^2$, $f_2\xi_0^2$, b, and ξ_0 found in Table II. We find $\Gamma_\epsilon \sim 579$ MeV, which is consistent with the very large width of the ϵ. The ratio f_π/f_K is determined by operator PCAC and the fitted value of b. We have $f_\pi/f_K \equiv (1+b)/(1-b/2) \sim 0.72$, which agrees well with the experimental value 0.78.[37]

B. The electromagnetic tadpole

The effective second-order electromagnetic interaction Lagrangian has the form

$$\mathcal{L}_{em}(0) = -e^2 \int d^4y\, T(j^{\alpha}_{em}(y) j^{\beta}_{em}(0)) D_{\alpha\beta}(y) - \epsilon_3\sigma_3(0), \tag{31}$$

where σ_3 is the Coleman-Glashow tadpole.[14] As discussed in Sec. I, only the tadpole contributes directly to the decay $\eta \to 3\pi$. However, the first term does contribute in the usual way to electromagnetic mass corrections of the scalar and pseudoscalar mesons. In particular there is a correction to the charged pion and kaon masses. We denote by

$$(\delta\mu^2_{\pi^+-\pi^0})_{em} \equiv (m_{\pi^+}{}^2 - m_{\pi^0}{}^2)_{em}$$

and

$$(\delta\mu^2_{K^+-K^0})_{em} \equiv (m_{K^+}{}^2 - m_{K^0}{}^2)_{em}$$

the contribution to the mass differences resulting from the first term in $\mathcal{L}_{em}$.

The tadpole also contributes to electromagnetic mass differences. To obtain the mass difference resulting from the tadpole and also to evaluate ϵ_3 we shift the field $\sigma_3 \to \sigma_3 + \xi_3$, where ξ_3 is defined by the condition that the term in $\mathcal{L} \equiv \mathcal{L}_{st} + \mathcal{L}_{em}$ linear in σ_3 vanishes. This condition implies the relation

$$\epsilon_3 = c\xi_0[4f_1\xi_0^2(1+2b^2) + 4f_2\xi_0^2(1+b)^2 - \gamma\xi_0(1-2b)] + O(c^2), \tag{32}$$

where $c \equiv \xi_3/\xi_0$ and since we wish to calculate the decay $\eta \to 3\pi$ only to first order in ϵ_3 (and *not* in the tree approximation) we have neglected all terms of order ${\xi_3}^2$ which appear in the shifted Lagrangian.[38] The contribution of the tadpole to the K^+-K^0 mass difference is then denoted by $(\delta\mu^2_{K^+-K^0})_{\rm tad}$. It is identified in the shifted Lagrangian by the relation

$$\mathcal{L} = -\delta\mu^2_{45}\phi^+_{45}\phi^-_{45} - \delta\mu^2_{67}\phi^+_{67}\phi^-_{67}\,,$$

where

$$\delta\mu^2_{45} \equiv -\delta\mu^2_{67}\,, \quad \phi^\pm_{ab} \equiv \frac{\phi_a \pm i\phi_b}{\sqrt{2}}\,,$$

and

$$\begin{aligned}(\delta\mu^2_{K^+-K^0})_{\rm tad} &\equiv 2\delta\mu^2_{45}\\ &= c[-4(\tfrac{2}{3})^{1/2} f_2 {\xi_0}^2(1+4b) + \sqrt{6}\gamma\xi_0]. \end{aligned} \tag{33}$$

Finally we have the identity

$${m_{K^+}}^2 - {m_{K^0}}^2 \equiv (\delta\mu^2_{K^+-K^0})_{\rm tad} + (\delta\mu^2_{K^+-K^0})_{\rm em}. \tag{34}$$

Thus in order to evaluate c and consequently ϵ_3 we need to know the value of $(\delta\mu^2_{K^+-K^0})_{\rm em}$. This we obtain in the following manner. First we know from the model that $(\delta\mu^2_{\pi^+-\pi^0})_{\rm tad} \equiv 0$. Therefore,

$$(\delta\mu^2_{\pi^+-\pi^0})_{\rm em} \equiv {m_{\pi^+}}^2 - {m_{\pi^0}}^2. \tag{35}$$

It has been known since the work of Das *et al.* in 1967 (see Ref. 39) that the first term of $\mathcal{L}_{\rm em}$ is sufficient to describe the entire π^+-π^0 mass difference. The second step uses the modified Dashen sum rule[40]

$$(\delta\mu^2_{K^+-K^0})_{\rm em} = (f_\pi/f_K)^2(\delta\mu^2_{\pi^+-\pi^0})_{\rm em}. \tag{36}$$

We note that Dashen takes $f_\pi/f_K = 1$ in the original sum rule. However, since the derivation is to first order in $\mathcal{L}_{\rm em}$ (without the tadpole) and to all orders in the strong interactions (or to any given order in the loop expansion) we must take f_π/f_K as given in the tree approximation (see Table II) for the calculation to be self-consistent.

To summarize we then have

$$(\delta\mu^2_{K^+-K^0})_{\rm em} = (f_\pi/f_K)^2({m_{\pi^+}}^2 - {m_{\pi^0}}^2)$$

and thus

$$\begin{aligned}(\delta\mu^2_{K^+-K^0})_{\rm tad} = &({m_{K^+}}^2 - {m_{K^0}}^2)\\ &-(f_\pi/f_K)^2({m_{\pi^+}}^2 - {m_{\pi^0}}^2).\end{aligned} \tag{37}$$

Solving for c, we find

$$c = -0.007133 \tag{38}$$

and thus

$$\epsilon_3/\epsilon_0 = -0.031829. \tag{39}$$

C. Results and checks

We can now proceed to calculate the amplitude for the decay $\eta \to \pi^+\pi^-\pi^0$. We calculate to first order in ϵ_3 and in the tree approximation for the strong interactions. The amplitude includes the 20 Feynman diagrams depicted in Fig. 1. In Tables III and IV we list the strong-interaction and electromagnetic vertices, respectively. The vertices are listed in terms of the unmixed fields, i.e., ϕ_a and σ_a. For the case of the η, η' and ϵ, ϵ' we use the mixing angles given in Table II to calculate the vertices entering in the Feynman diagrams of Fig. 1. In Fig. 2 we present our results. We plot the amplitude $T(\eta \to 3\pi)$ as a function of the neutral-pion energy E_0. The charged-pion energies are fixed at the value $E_{\pi^+} = E_{\pi^-} = (m_\eta - E_0)/2$. Thus we plot the value of the amplitude which lies on a line running down the center of the Dalitz plot. We note that the amplitude is symmetric about this line and relatively constant as a function of the distance away from it.

As we see the amplitude rises sharply as $E_0 \to 0$, passing through the value $T(\eta \to 3\pi) = 0.6912$ at $E_0 = 0$. In the physical region which runs from $E_0 = m_\pi$ to $({m_\eta}^2 - 3{m_\pi}^2)/2m_\eta$ the amplitude is to a good approximation a linear function of E_0 of the form

$$T(\eta \to 3\pi) = \alpha + \beta E_0\,, \tag{40}$$

with $\beta/\alpha = -1/0.255 \simeq -2/m_\eta$ and $\alpha = 0.4053$ [see Eq. (1)]. We note that $\alpha \ll T(\eta \to 3\pi)(E_0 = 0)$. It is clear that the amplitude calculated in the tree approximation with a linear chiral Lagrangian does not extrapolate linearly in E_0 from the physical region to the neutral-soft-pion point at $K_\mu = 0$. The amplitude is instead dominated by the ϵ pole which lies only ~56 MeV to the left of the neutral-soft-pion point.

To calculate the partial width of $\eta \to \pi^+\pi^-\pi^0$ we need to integrate $|T(\eta \to 3\pi)|^2$ [given in Eq. (40)] over the area of the Dalitz plot. Using the relativistic phase-space formula calculated previously in the literature[41] we have

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) = 489\,|x|^2(1 + 0.02y + 0.02y^2)\ \text{eV}, \tag{41}$$

where

$$x \equiv \alpha\left(3 + \frac{\beta}{\alpha} m_\eta\right)$$

and

$$y \equiv \frac{m_\eta - 3m_\pi}{m_\eta + 3(\alpha/\beta)}\,.$$

As a result we obtain

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) = 58\ \text{eV}\,. \tag{42}$$

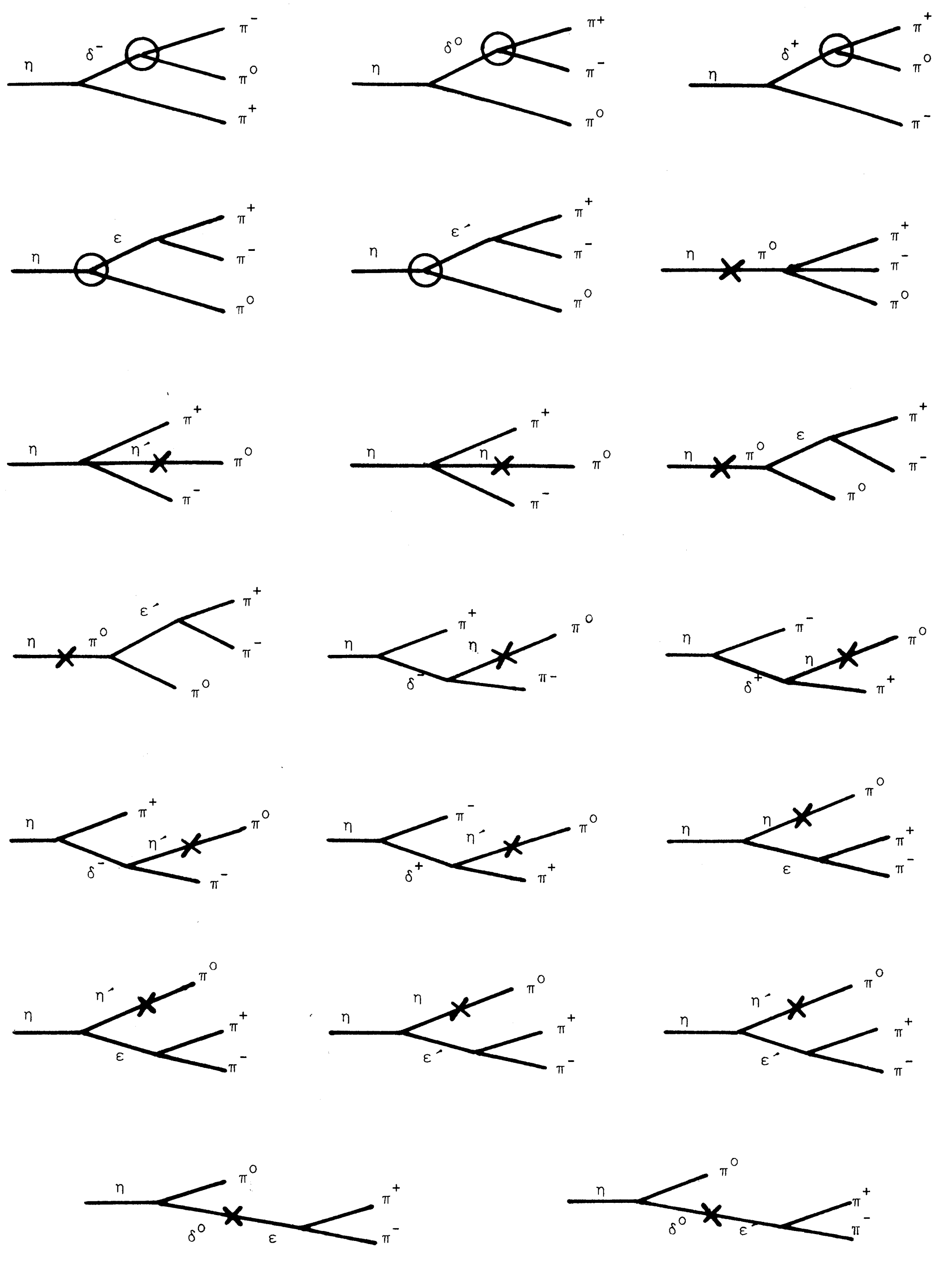

FIG. 1. The 20 Feynman diagrams included in the amplitude for the decay $\eta\rightarrow\pi^{+}\pi^{-}\pi^{0}$. The circles and crosses denote electromagnetic vertices.

TABLE III. Strong-interaction vertices. The vertices are identified by the relation e.g. $\mathcal{L}_{st} = h\phi_a\phi_b\sigma_c$, where $h = (1/\xi_0 n)(Af_1\xi_0^2 + Bf_2\xi_0^2 + C\gamma\xi_0)$.

vertex	n	A	B	C
$\phi_8\phi_3\sigma_3$	1	0	$\frac{4}{3}\sqrt{2}(1+b)$	$-\sqrt{2}$
$\phi_0\phi_3\sigma_3$	1	0	$\frac{8}{3}(1+b)$	1
$\phi_8(\phi_{12}^+\sigma_{12}^- + \text{H.c.})$	1	0	$\frac{4}{3}\sqrt{2}(1+b)$	$-\sqrt{2}$
$\phi_0(\phi_{12}^+\sigma_{12}^- + \text{H.c.})$	1	0	$\frac{8}{3}(1+b)$	1
$\phi_8^2\sigma_0$	1	4	$\frac{4}{3}(1-b)$	$\frac{1}{2}$
$\phi_8^2\sigma_8$	1	$4\sqrt{2}b$	$-\frac{2}{3}\sqrt{2}(1-3b)$	$1/\sqrt{2}$
$\phi_8\phi_0\sigma_0$	1	0	$\frac{8}{3}\sqrt{2}b$	0
$\phi_8\phi_0\sigma_8$	1	0	$\frac{8}{3}(1-b)$	1
$\phi_0^2\sigma_0$	1	4	$\frac{4}{3}$	-1
$\phi_0^2\sigma_8$	1	$4\sqrt{2}b$	$\frac{4}{3}\sqrt{2}b$	0
$\phi_8^2\phi_{12}^+\phi_{12}^-$	2	4	2	0
$\phi_8\phi_0\phi_{12}^+\phi_{12}^-$	2	0	$4\sqrt{2}$	0
$\phi_0^2\phi_{12}^+\phi_{12}^-$	2	4	4	0
$\phi_3^2\phi_{12}^+\phi_{12}^-$	2	4	2	0
$\sigma_8\phi_{12}^+\phi_{12}^-$	1	$8\sqrt{2}b$	$\frac{4}{3}\sqrt{2}(1+b)$	$-\sqrt{2}$
$\sigma_0\phi_{12}^+\phi_{12}^-$	1	8	$\frac{8}{3}(1+b)$	1
$\sigma_8\phi_3^2$	1	$4\sqrt{2}b$	$\frac{2}{3}\sqrt{2}(1+b)$	$-1/\sqrt{2}$
$\sigma_0\phi_3^2$	1	4	$\frac{4}{3}(1+b)$	$\frac{1}{2}$
$\phi_{12}^{\pm} \equiv \dfrac{\phi_1 \pm i\phi_2}{\sqrt{2}}$, etc.				

TABLE IV. Electromagnetic vertices. The vertices are identified by the relation e.g. $\mathcal{L}_{em} = h\phi_a\phi_b\sigma_c$, where $h = (c/\xi_0 n)(Af_1\xi_0^2 + Bf_2\xi_0^2 + C\gamma\xi_0)$, and $c \equiv \xi_3/\xi_0$.

vertex	n	A	B	C
$\phi_8\sigma_8\phi_3$	1	0	$\frac{4}{3}$	0
$\phi_8\sigma_0\phi_3$	1	0	$\frac{4}{3}\sqrt{2}$	0
$\phi_0\sigma_8\phi_3$	1	0	$\frac{4}{3}\sqrt{2}$	0
$\phi_0\sigma_0\phi_3$	1	0	$\frac{8}{3}$	0
$(\sigma_{12}^+\phi_{12}^- + \text{H.c.})\phi_3$	1	0	-4	0
$\sigma_3\phi_{12}^+\phi_{12}^-$	1	8	12	0
$\phi_8\phi_3$	0	0	$\frac{4}{3}\sqrt{2}(1+b)$	$-\sqrt{2}$
$\phi_0\phi_3$	0	0	$\frac{8}{3}(1+b)$	1
$\sigma_0\sigma_3$	0	8	$8(1+b)$	-1
$\sigma_8\sigma_3$	0	$8\sqrt{2}b$	$4\sqrt{2}(1+b)$	$\sqrt{2}$

To see how sensitive the partial width is to the slope of the amplitude we fix α at the value obtained and change β/α to $-2/m_\eta$. We find $\Gamma(\eta\to\pi^+\pi^-\pi^0) = 80$ eV. However, in order to obtain the experimentally measured partial width of ~ 204 eV with an amplitude linear in E_0 of the form $T(\eta\to 3\pi) = \alpha + \beta E_0$ and $\beta/\alpha = -2/m_\eta$ we need a value of $\alpha = 0.648$ or about 1.6 times greater than the value we have obtained.

In order to check our results for possible calculational errors we first note that the amplitude satisfies the Sutherland charged-pion zeros. This is apparent in Fig. 2, where the amplitude vanishes at $E_0 \sim m_\eta/2$. In addition we can calculate the amplitude at the neutral-soft-pion point using PCAC and compare the value obtained with Fig. 2. We have

$$
\begin{aligned}
T(\eta\to 3\pi)((E_0,\vec{K})=0) &= \lim_{K_\mu\to 0} \langle \pi^+\pi^-\pi^0(K)|\mathcal{L}_{em}^{(0)}|\eta\rangle \\
&= \lim_{K_\mu\to 0} -\epsilon_3\langle \pi^+\pi^-\pi^0(K)|\sigma_3(0)|\eta\rangle \\
&= \epsilon_3/f_\pi\langle \pi^+\pi^-|[Q_3^5(0),\sigma_3(0)]|\eta\rangle \\
&= -i\frac{\epsilon_3}{f_\pi}\left\langle \pi^+\pi^-\left|\left((\tfrac{2}{3})^{1/2}\phi_0(0) + \frac{1}{\sqrt{3}}\phi_8(0)\right)\right|\eta\right\rangle .
\end{aligned} \tag{43}
$$

Using the identities

$$
\begin{aligned}
\partial_\mu j_8^{\mu 5} &= -(\tfrac{2}{3})^{1/2}\epsilon_0(1-a)\phi_8 - (\tfrac{2}{3})^{1/2}\epsilon_8\phi_0 , \\
\partial_\mu j_0^{\mu 5} &= -(\tfrac{2}{3})^{1/2}\epsilon_0\phi_0 - (\tfrac{2}{3})^{1/2}\epsilon_8\phi_8 + (\tfrac{2}{3})^{1/2}3g\delta_5 , \\
a &\equiv \epsilon_8/\sqrt{2}\,\epsilon_0 ,
\end{aligned} \tag{44}
$$

and

$$
\delta_5 \equiv \frac{2}{\sqrt{3}}\xi_0^2(1-b^2)\phi_0 - 2(\tfrac{2}{3})^{1/2}\xi_0^2 b(1+b)\phi_8 + \delta_5' , \tag{45}
$$

where δ_5' is defined not to have any linear terms in the fields, we obtain

$$T(\eta\to 3\pi)((E_0,\vec{K})=0)=\frac{-i\epsilon_3/\epsilon_0}{[(1+a)-\iota(b)]f_\pi}\left\langle \pi^+\pi^-\left|\frac{3\gamma\delta_5'(0)}{\sqrt{2}}\right|\eta\right\rangle , \tag{46}$$

where

$$\iota(b)\equiv\left(\frac{\xi_0}{\epsilon_0}\right)3\gamma\xi_0(1-b^2)\left[1-\frac{(1-3b)a}{(1-b)}\right]\Big/(1-2a) .$$

The piece of δ_5' which contributes to the amplitude is as follows:

$$\begin{aligned}\delta_5'=&-2(\tfrac{2}{3})^{1/2}\phi_8\phi_{12}^+\phi_{12}^-+\frac{2}{\sqrt{3}}\phi_0\phi_{12}^+\phi_{12}^--\frac{2}{\sqrt{3}}\xi_0(1+2b)\phi_8\sigma_8-2(\tfrac{2}{3})^{1/2}\xi_0 b\,\phi_8\sigma_0\\&-2(\tfrac{2}{3})^{1/2}\xi_0 b\,\phi_0\sigma_8+\frac{4}{\sqrt{3}}\xi_0\phi_0\sigma_0-\frac{2}{\sqrt{3}}\xi_0(1-2b)(\phi_{12}^+\sigma_{12}^-+\phi_{12}^-\sigma_{12}^+) .\end{aligned} \tag{47}$$

The Feynman diagrams included in the calculation are given in Fig. 3. As a result of the calculation we have $T(\eta\to 3\pi)((E_0,\vec{K})=0)=0.69$, which equals the value of the amplitude at the neutral-soft-pion point obtained previously.

D. Discussion and conclusion

We have calculated the amplitude for the decay $\eta\to\pi^+\pi^-\pi^0$ in the tree approximation for the strong interactions and to first order in the electromagnetic interaction with a linear $U(3)\otimes U(3)_{\text{chiral}}$ Lagrangian. We have obtained a result for the partial width $\Gamma(\eta\to\pi^+\pi^-\pi^0)=58$ eV. This is much smaller than the experimentally observed width of 204 ± 22 eV.[7] The strong-interaction part of the chiral Lagrangian has fitted the masses of the scalar mesons with an accuracy of about 10%. We noted previously that some improvement can be obtained if we introduce an additional mass term in the Lagrangian. However, it is apparent that such a correction is irrelevant at present.

By far the greatest source of uncertainty in the present calculation is the close proximity of the ϵ pole to the physical region. Its large width as calculated in the tree approximation indicates that in one-loop approximation the ϵ pole would move far off the real axis. As a result it is likely that the amplitude at the neutral-soft-pion point would be greatly decreased. However, it is difficult to say *a priori* what would be the effect in the physical region. Perhaps, since the physical region is ~194 MeV from the ϵ pole, there would be no significant change. It is reasonable to believe though that in the one-loop approximation the amplitude would now extrapolate linearly from the physical region to the neutral-soft-pion point. In this case one would only need to calculate the amplitude at the point $(E_0,\vec{K})=0$ to evaluate $\Gamma(\eta\to\pi^+\pi^-\pi^0)$. This then seems like the first step toward a better calculation within the present framework.

In conclusion, we recall that our result is con-

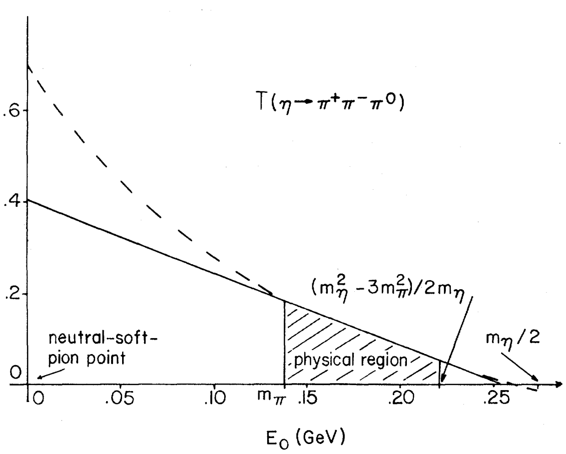

FIG. 2. A plot of the amplitude $T(\eta\to 3\pi)$ as seen down the center of the Dalitz plot, i.e., $E_{\pi^+}=E_{\pi^-}=(m_\eta-E_0)/2$, as a function of the neutral pion energy E_0.

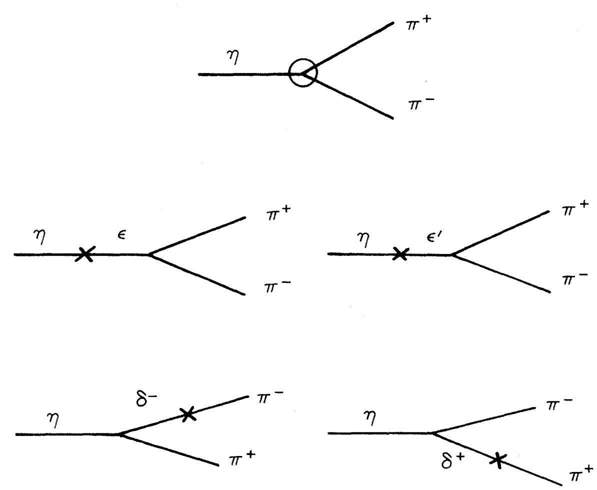

FIG. 3. The Feynman diagrams included in the amplitude $T(\eta\to\pi^+\pi^-\pi^0)$ evaluated at the neutral soft-pion point $K_\mu\to 0$. The circles and crosses denote δ_5' vertices.

sistent with the result obtained by Weinberg[4] using a nonlinear $U(3)\otimes U(3)_{\text{chiral}}$ Lagrangian, thus suggesting the model independence of the calculation. However, a reliable confirmation of this hypothesis, we feel, would require a similar calculation in the one-loop approximation. Finally, if the result is indeed model-independent and consequently also true in the "standard" model of strong interactions, then we are once again at an impasse for the decay $\eta\to\pi^+\pi^-\pi^0$ since the result is clearly still quite below the experimental value.

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[1]By the standard model we mean a Lagrangian field theory of nine elementary quark fields transforming as the fundamental representation of $SU(3)\otimes SU(3)_{\text{color}}$ coupled only via color to Yang-Mills multiplet of gluons.

[2]D. G. Sutherland, Phys. Lett. 23, 384 (1966).

[3]J. Kogut and L. Susskind, Phys. Rev. D 11, 3594 (1975); 10, 3468 (1974).

[4]S. Weinberg, Phys. Rev. D 11, 3583 (1975).

[5]The first calculation of $\eta\to 3\pi$ using a nonlinear $SU(3)\otimes SU(3)_{\text{chiral}}$ Lagrangian was probably performed by A. J. Cantor, Harvard University, Ph.D. thesis, 1969 (unpublished).

[6]Weinberg quotes a partial width of 65 eV in the text. He has apparently used nonrelativistic phase space to obtain this result. We prefer to use the relativistic phase-space formula, which results in a significant difference.

[7]For several years the actual value has been quoted as 630 ± 140 eV [see e.g. Particle Data Group, Phys. Lett. 50B, 1 (1974)]. Recently, a remeasurement of the Primakoff effect has led to a new value for the $\eta\to 2\gamma$ width, with a consequent reduction in the value of the $\eta\to\pi^+\pi^-\pi^0$ width to 204 ± 22 eV [see A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974).]

[8]P. Carruthers, Phys. Rev. D 3, 959 (1971); P. Carruthers and R. Haymaker, *ibid.* 4, 406 (1971); 4, 1808 (1971). The original linear $SU(3)\otimes SU(3)_{\text{chiral}}$ Lagrangian was by M. Lévy, Nuovo Cimento 52, 23 (1967). There is a fine review article on chiral Lagrangians by S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969), from which we have also taken some of our notation. Finally, J. Schechter and Y. Ueda, Phys. Rev. D 4, 733 (1971) and references therein, have discussed the decay $\eta\to 3\pi$ using the most general linear $SU(3)\otimes SU(3)_{\text{chiral}}$ Lagrangian.

[9]Columbia-Berkeley-Purdue-Wisconsin-Yale Collaboration, Phys. Rev. 149, 1044 (1966).

[10]J. S. Bell and D. G. Sutherland, Nucl. Phys. B4, 315 (1968).

[11]We have

$$S(\eta\to 3\pi)=i(2\pi)^4\delta^4(P_\eta-P_{\pi^+}-P_{\pi^-}-K)\times\prod_{i=1}^{4}\frac{1}{[(2\pi)^3 2E i]^{1/2}}\,T(\eta\to 3\pi),$$

where $i=1,2,3,4$ corresponds to η, π^+, π^-, π^0, respectively.

[12]All physical particles are massive. Later we will come back to this point when we discuss the massless dipole pair of KS which, as we will see then, does not contribut to gauge-invariant amplitudes. Thus though there *are* massless particles in the theory they are unphysical and do not contribute to this gauge-invariant amplitude.

[13]See e.g. S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968), p. 35.

[14]S. Coleman and S. L. Glashow, Phys. Rev. Lett. 6, 423 (1961); Phys. Rev. 134, B671 (1964).

[15]See e.g. J. H. Wojtaszek, R. E. Marshak, and Riazuddin, Phys. Rev. 136, B1053 (1964); M. Elitzur and H. Harari, Ann. Phys. (N.Y.) 56, 81 (1970).

[16]See e.g. R. N. Mohapatra, Nuovo Cimento 2A, 707 (1971); J. Schechter and Y. Ueda, Ref. 8; P. Dittner, P. H. Dondi, and S. Eliezer, Phys. Rev. D 8, 2253 (1973).

[17]K. Wilson, Phys. Rev. 179, 1499 (1969).

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[20]H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics,* edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

[21]K. Wilson, Phys. Rev. D 10, 2445 (1974); J. Kogut and L. Susskind, 11, 395 (1975); 9, 3501 (1974).

[22]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).

[23]If $T(j^\alpha(y)j^\beta(0))\propto C'^{\alpha\beta}(y)u_3(0)$ at short distances then $mg'(my_0)/y_0\equiv\int d^3\vec{y}\,C'^{\alpha\beta}(y)D_{\alpha\beta}(y)$ and $g'(y_0>0)\equiv g(y_0)$.

[24]See e.g. S. Weinberg, Phys. Rev. Lett. 31, 494 (1973); Phys. Rev. D 8, 605 (1973); 8, 4482 (1973).

[25]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[26]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961); J. Goldstone, Nuovo Cimento 19, 15 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[27]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969).

[28]For a reference on the S. N. Gupta and K. Bleuler formalism, see S. S. Schweber, *Introduction to Relativ-*

istic Quantum Field Theory (Harper and Row, New York, 1961), p. 242.

[29]Since $\partial_\mu j_0^{\mu 5}$ and $(g^2/4\pi)G^\alpha_{\mu\nu}G^{*\mu\nu}_\alpha$ are both gauge-invariant operators, the residue of the ghost pole at $K^2=0$ is also a gauge-invariant quantity, as it must be since the amplitude for the decay $\eta \rightarrow 3\pi$ is gauge-invariant.

[30]We remark that Carruthers and Haymaker, Ref. 8, had previously noted the necessity of the U(3)$\otimes$U(3) symmetry-breaking interaction (the determinental interaction) to account for the large η' mass.

[31]This model was previously discussed and the parameters fitted by Carruthers and Haymaker (Ref. 8).

[32]The symmetric part of $\mathcal{L}_{st}$, i.e., ϵ_0, $\epsilon_8=0$ is not the most general SU(3)$\otimes$SU(3)$_{\text{chiral}}$-invariant Lagrangian that can be constructed with the 18 scalar and pseudoscalar fields. However, with the addition of the term $(\mu^2/2)\,\mathrm{Tr}M^\dagger M$ it would be the most general renormalizable SU(3)$\otimes$SU(3) symmetric Lagrangian; see Lévy (Ref. 8).

[33]The effective potential was introduced by Goldstone, Salam, and Weinberg (Ref. 26), and by J. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964). For a review see Sydney Coleman, lectures given at the 1973 International Summer School of Physics "Ettore Majorana" (unpublished).

[34]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[35]See L.-H. Chan and R. W. Haymaker, Phys. Rev. D 10, 4143 (1974).

[36]We have $(2K_0)^{1/2}\langle 0|j_1^{\mu 5}+ij_2^{\mu 5}|\pi\rangle = -\mathrm{i}f_\pi K^\mu$ and similarly for f_K with $j_a^{\mu 5}\equiv d_{abc}\phi_b \overleftrightarrow{\partial}^\mu \sigma_c$.

[37]We note that Gell-Mann, Oakes, and Renner (Ref. 25), calculating to first order in SU(3) symmetry breaking, find $f_\pi/f_K=1$. As we see the tree approximation gives a marked improvement $\sim 20\%$ in this result. A relation which is exact in the tree approximation is $(4f_K-f_\pi)/3=f_{8\eta}\cos\theta_P+f'_{8\eta}\sin\theta_P$, where $(2K_0)^{1/2}\langle 0|j_8^{\mu 5}|\eta\rangle \equiv -\mathrm{i}f_{8\eta}K^\mu$, $(2K_0)^{1/2}\langle 0|j_8^{\mu 5}|\eta'\rangle \equiv -\mathrm{i}f_{8\eta'}K^\mu$, and $f_{8\eta}=-(\frac{2}{3})^{1/2}\xi_0[(1-b)\cos\theta_P-\sqrt{2b}\,\sin\theta_P]$. If we set $\sin\theta_P\equiv 0$ we find the approximate mass formula $4(f_K m_K{}^2-f_\pi m_\pi{}^2)/3\sim f_\eta m_\eta{}^2$ or using the above relation in the limit $\sin\theta_P\rightarrow 0$, $4(f_K m_K{}^2-f_\pi m_\pi{}^2)/(4f_K-f_\pi)\sim m_\eta{}^2$. This is to be compared with the SU(3) mass formula $(4m_K{}^2-m_\pi{}^2)/3=m_\eta{}^2$. Both relations are well satisfied by experiment.

[38]To show that a calculation to first order in ϵ_3 is indeed the same as calculating to first order in ξ_3 we simply have to calculate $\xi_3\equiv\langle\sigma_3\rangle_0$ to first order in ϵ_3. We have

$$\begin{aligned}\xi_3=\langle\sigma_3\rangle_0 &\equiv -i\epsilon_3\int d^4x\,\langle\, T(\sigma_3(0)\sigma_3(x))\,\rangle_0\\ &=\lim_{p\rightarrow 0}\frac{\epsilon_3}{p^2-m_\delta{}^2}\\ &=-\epsilon_3/m_\delta{}^2\end{aligned}$$

or $\epsilon_3=-\xi_3 m_\delta{}^2=-c\xi_0 m_\delta{}^2$. It is easy to check that this relation is equivalent to Eq. (32).

[39]T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Lett. 18, 759 (1967). Using PCAC and Weinberg's spectral-function sum rules [S. Weinberg, Phys. Rev. Lett. 18, 507 (1967)], they calculate $(m_{\pi^+}-m_{\pi^0})$ starting from the relation

$$\Delta E=\int d^4x\,\langle\pi^+|\,T(j_\mu^{\text{em}}(x)j_\nu^{\text{em}}(0))\,|\pi^+\rangle_{\text{conn}}D^{\mu\nu}(x).$$

[40]See R. Dashen, Phys. Rev. 183, 1245 (1969).

[41]See P. Dittner, P. H. Dondi, and S. Eliezer, Ref. 16, and references therein.