# Large-$N$ behavior for independent-value models

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Highly specialized, $N$-component, scalar model quantum field theories invariant under O($N$) transformations are studied in the limit of large $N$. The models are expressed in $n$-dimensional Euclidean space-time and differ from conventional covariant quantum models by the absence of all space-time gradients, a modification that leads to nonrenormalizable O($N$)-invariant interactions for each $N \geq 1$. These models are solved by nonperturbative techniques, and the solutions exhibit two striking and unfamiliar properties: (1) For finite (or infinite) $N$, the solutions of any interacting theory do not reduce to those of the free theory in the limit where the coupling of the nonlinear interaction vanishes; and (2) the relevant (asymptotic) dependence of the parameters of the interacting theories on $N$ differs from the conventional choice, and the limit $N \rightarrow \infty$ does not lead to a Hartree-type solution. It is proposed that similar unconventional behavior may characterize certain O($N$)-invariant, covariant nonrenormalizable quantum field theories, and in particular that the limit $N \rightarrow \infty$ may not lead to a Hartree (or Hartree-Fock) type of solution.

## I. INTRODUCTION

Fields with a large number of components have led to $1/N$ expansion techniques that are finding many applications in the study of quantum field models.[1] Invariably, Hartree (or Hartree-Fock) type solutions are used for $N \rightarrow \infty$ and deviations from such solutions are studied in the $1/N$ expansion. In view of the rather wide interest in these techniques, we propose to survey briefly and to investigate the large-$N$ behavior of a model that we have previously studied. This model is a soluble nonrenormalizable model of an $N$-component boson field that, as we shall show, possesses a meaningful limit as $N \rightarrow \infty$, but a limit that is *not* given by a Hartree-type solution contrary to popular belief.

The model we study is that of an $N$-component boson field $\vec{\Phi}(x) \equiv \{\Phi_k(x),\ k=1,2,\ldots,N\}$, defined in an $n$-dimensional Euclidean space-time, and where $x \in R^n$, $n \geqslant 2$. The Euclidean-space action functional for the model has O($N$) symmetry and is generically of the form

$$I = -\int \{\tfrac{1}{2} m^2 \vec{\Phi}^2(x) + \lambda V[\vec{\Phi}^2(x)]\}\, dx, \tag{1}$$

where $dx \equiv d^n x$ and $\vec{\Phi}^2 \equiv \sum_k \Phi_k^{\,2}$.[2] Evidently, this is a highly artificial model physically since it lacks the conventional $(\nabla\vec{\Phi})^2(x)$ term of a standard covariant theory. Without the gradient term, there is no spread of excitation from one point to another, the field is statistically independent at each space-time point, and as a consequence such models are termed independent-value models.[3] It is also clear that the absence of the gradient term makes such models nonrenormalizable since the propagator is $(m^2)^{-1}$ rather than $(p^2+m^2)^{-1}$ and no high-momentum damping takes place in a conventional perturbation calculation. In spite of this fact we shall present a solution for the independent-value models that not only fails to become the Hartree solution as $N \rightarrow \infty$, as already mentioned, but even fails to reduce to the free-field solution as $\lambda \rightarrow 0$.

Prior to a discussion of these solutions it is worthwhile to present several arguments that may help motivate the study of such models besides the pragmatic one that they can be solved and the idealistic (and probably unrealistic) one that seeks to restore the missing gradient terms by perturbation theory.

One motivational argument stems from the observation that the propagator for a covariant massive vector field has the general form

$$\frac{g_{\mu\nu} + p_\mu p_\nu / m^2}{p^2 + m^2}, \tag{2}$$

and that any nonrenormalizable features associated with Yukawa interactions (when coupled to a nonconserved current) are attributable to the component

$$\frac{p_\mu p_\nu / m^2}{p^2 + m^2}, \tag{3}$$

which looks rather like derivative coupling of a scalar field (Stückelberg decomposition[4]). Although there is important kinematic angular dependence, the large-momentum behavior of the propagator is roughly $(m^2)^{-1}$ and hence more like that in the independent-value model than in a standard covariant scalar theory. While no pretense at equivalence is intended, the similarity in the *order* of the large-momentum dependence of the propagators—so relevant for general perturbation analyses—suggests that the independent-value models may contain useful clues to understand the nonrenormalizable part of interactions that involve massive vector fields.

Another motivation stems from the following argument. For a covariant neutral scalar field $\Phi(x)$ the Euclidean-space functional integral expression for the time-ordered generating functional formally reads

$$S(h)\equiv \mathfrak{N}\int \exp\left(i\int h\Phi\,dx-\int\{\tfrac{1}{2}[(\nabla\Phi)^2+m^2\Phi^2]+\lambda V[\Phi]\}\,dx\right)\mathfrak{D}\Phi. \qquad (4)$$

The independent-value models are related to the covariant ones simply by discarding the $(\nabla\Phi)^2$ term in the exponent, but there is an alternative viewpoint as well. Introduce the real field variable $\Psi\equiv m^{-1}(-\nabla^2+m^2)^{1/2}\Phi$, or in momentum space as $\tilde{\Psi}\equiv m^{-1}(p^2+m^2)^{1/2}\tilde{\Phi}$, and reexpress the functional integral in terms of $\Psi$. Then

$$S(h)\equiv\hat{S}(\hat{h})$$
$$\equiv\mathfrak{N}\int\exp\left(i\int\hat{h}\Psi\,dx-\int\{\tfrac{1}{2}m^2\Psi^2+\lambda V[m(-\nabla^2+m^2)^{-1/2}\Psi]\}\,dx\right)\mathfrak{D}\Psi, \qquad (5)$$

where $\hat{h}\equiv m(-\nabla^2+m^2)^{-1/2}h$. The Jacobian between $\mathfrak{D}\Psi$ and $\mathfrak{D}\Phi$ is purely numerical and is absorbed in the formal factor $\mathfrak{N}$. In this description of a *covariant* problem the free action describes an independent-value model, and thus one which is $\delta$ correlated, and the proper correlations arise both from the definition of $\hat{h}$ and the "nonlocal" nature of the interaction terms. If $\lambda V\equiv 0$ and we deal with the free ($F$) model, then

$$S_F(h)=\hat{S}_F(\hat{h})$$
$$=\mathfrak{N}\int\exp\left(i\int\hat{h}\Psi\,dx-\tfrac{1}{2}m^2\int\Psi^2\,dx\right)\mathfrak{D}\Psi$$
$$=\exp\left[-\tfrac{1}{2}(m^2)^{-1}\int\hat{h}^2(x)dx\right]$$
$$=\exp\left[-\tfrac{1}{2}\int(p^2+m^2)^{-1}|\tilde{h}(p)|^2\,dp\right], \qquad (6)$$

which of course is the correct answer. In studying the independent-value models, therefore, we may say that we drop the $(\nabla\Phi)^2$ term in the *free* action of the expression for $S(h)$ of a covariant theory; or in an alternative viewpoint, we change the *interaction* term of the expression for $\hat{S}(\hat{h})$ of a covariant theory by removing the "ultraviolet damping" represented by the "delocalizing" factor $m(-\nabla^2+m^2)^{-1/2}$ thereby resulting in a "local" interaction term. In the latter view one clearly sees that the interaction term is effectively more singular for independent-value theories than for covariant theories, which ultimately leads to their nonrenormalizability.

Some general remarks regarding such models may be useful. From the structure of the formal functional integral for $N$-component independent-value models one deduces that

$$S(\vec{h})=\exp\left\{-\int L[\vec{h}(x)]dx\right\} \qquad (7)$$

for some even function $L[\vec{h}]$ invariant under $O(N)$. The general form for $L$ follows from the fact that $S(\vec{h})$ is a positive-definite functional. Let $\vec{h}(x)=\vec{s}\chi_\Delta(x)$, where $\chi_\Delta(x)=1$ [$\chi_\Delta(x)=0$] for $x\in\Delta$ [$x\notin\Delta$], and $\vec{s}$ denotes an arbitrary $x$-independent vector. Then

$$\exp(-\Delta L[\vec{s}])\equiv\int\cos(\vec{s}\cdot\vec{u})d\mu_\Delta(\vec{u}) \qquad (8)$$

for some $O(N)$-invariant probability measure on $R^N$. Consequently

$$L[\vec{s}]=\lim_{\Delta\to 0}\Delta^{-1}\int[1-\cos(\vec{s}\cdot\vec{u})]d\mu_\Delta(\vec{u}), \qquad (9)$$

and the most general form for this limit is[5]

$$L[\vec{s}]=a\vec{s}^2+\int_{|u|>0}[1-\cos(\vec{s}\cdot\vec{u})]d\sigma(\vec{u}), \qquad (10)$$

where $a\geq 0$ and $\sigma$ is an $O(N)$-invariant positive measure subject only to the condition

$$\int_{|u|>0}[u^2/(1+u^2)]d\sigma(\vec{u})<\infty. \qquad (11)$$

The case $a\neq 0,\sigma\equiv 0$ corresponds to free theories of different mass, while $a\equiv 0$ and $\sigma\neq 0$ covers all interacting theories. The task then is to find a suitable $\sigma$ for each interacting theory, and this topic is reviewed in Sec. II.

In case we deal with an infinite-component field, we can even say more. The general structure is the same, namely

$$S(\vec{h}) = \exp\left\{-\int L[\vec{h}(x)]dx\right\} \tag{12}$$

for some even $L[\vec{h}]$ invariant under $O(\infty)$. Again let $\vec{h}(x) = \vec{s}\chi_\Delta(x)$ so that

$$\begin{aligned} S'(\vec{s}) &= \exp(-\Delta L[\vec{s}]) \\ &= \int \cos(\vec{s}\cdot\vec{u})d\mu_\Delta(\vec{u}) \\ &\equiv \int_0^\infty e^{-(1/2)b\vec{s}^2}d\mu_\Delta(b), \end{aligned} \tag{13}$$

where in the last relation we have used the fact that every characteristic function with $O(\infty)$ invariance such as $S'(\vec{s})$ is a convex combination of Gaussians.[6] Consequently,

$$L[\vec{s}] = \lim_{\Delta\to 0}\Delta^{-1}\int_0^\infty (1-e^{-(1/2)b\vec{s}^2})d\mu_\Delta(b) \tag{14}$$

and the most general form for this limit is

$$L[\vec{s}] = a\vec{s}^2 + \int_0^\infty (1-e^{-(1/2)b\vec{s}^2})d\sigma(b), \tag{15}$$

where $a \geq 0$ and $\sigma$ is a positive measure on $(0,\infty)$ subject only to the condition

$$\int_0^\infty [b/(1+b)]d\sigma(b) < \infty. \tag{16}$$

Again $a \neq 0$, $\sigma \equiv 0$ corresponds to free theories of different mass (and the Hartree solutions), while $a \equiv 0$, $\sigma \neq 0$ covers all interacting theories. The derivation of suitable $\sigma$ is the subject of Sec. III.

Finally, we note the formal difference in our analysis that permits an $N\to\infty$ limit different from the Hartree solution. Consider as an example the simple interaction $V = (\vec{\Phi}^2)^2$. Customarily, one employs an $N$-dependence of the various terms such that

$$I = -\int\left\{\tfrac{1}{2}m^2\vec{\Phi}^2(x) + \frac{\lambda}{N}[\vec{\Phi}^2(x)]^2\right\}dx, \tag{17}$$

whereas we shall find it necessary to employ an alternative $N$-dependent parametrization so that

$$I = -\frac{1}{N}\int\left\{\tfrac{1}{2}m^2\vec{\Phi}^2(x) + \frac{\lambda}{N}[\vec{\Phi}^2(x)]^2\right\}dx, \tag{18}$$

which can be seen not to affect the formal equations of motion (such as they are). Thus from an equation of motion point of view the standard and nonstandard scalings are indistinguishable and thus equally valid. The difference in the two approaches can be regarded as alternative renormalizations of the model parameters, which, after all, in nonrenormalizable theories are poorly defined to begin with. While it is a common assumption, there is no *a priori* reason that an infinite-component field need be quasifree. More technically, for the models at hand, the standard scaling invariably leads to fields described as Gaussian-distributed random variables, while the nonstandard scaling leads to fields described more accurately as Poisson-distributed random variables that are completely in accord with the character of the required solution of these nonrenormalizable models for finite $N$.[2] Related to this remark is the fact that the nonstandard scaling admits a $1/N$ expansion relevant to the finite-$N$ solutions that the standard scaling simply fails to do. For these reasons we feel the nonstandard scaling is the correct one for the models under discussion. It is interesting to speculate that alternative scalings may be relevant in treating the large-$N$ behavior of some more realistic models.

## II. FINITE-COMPONENT FIELDS

Though the $N$-component fields are treated elsewhere,[2] we think it is convenient to describe the main properties and techniques here. As already mentioned

$$S(\vec{h}) = \exp\left\{-\int L[\vec{h}(x)]dx\right\}, \tag{19}$$

and the most general form that is invariant under $O(N)$ reads

$$S(\vec{h}) = \exp\left(-a\int \vec{h}^2(x)dx - \int dx\int_{|u|>0}\{1-\cos[\vec{u}\cdot\vec{h}(x)]\}d\sigma(\vec{u})\right), \tag{20}$$

where $d\sigma(\vec{u})$ is invariant under $O(N)$. Specializing now we assume that $a = 0$ and $d\sigma(\vec{u}) = C^2(u)d\vec{u}$. Given such a $C(u)$ it is possible to give a useful representation of the field in the following way: Let $A(x,\vec{u})$ and $A^\dagger(x,\vec{u})$ be conventional annihilation and creation operators satisfying

$$[A(x,\vec{u}), A^\dagger(y,\vec{v})] = \delta(x-y)\delta(\vec{u}-\vec{v}) \tag{21}$$

and

$$A(x,\vec{u})|0\rangle = 0, \tag{22}$$

$|0\rangle$ being the vacuum. Define new operators

$$B(x,\vec{u}) = A(x,\vec{u}) + C(u). \tag{23}$$

Then the field can be represented as

$$\Phi_k(x) = \int B^\dagger(x,\vec{u}) u_k B(x,\vec{u}) d\vec{u}, \tag{24}$$

and it follows that

$$\left\langle 0 \left| \exp\left( i \int \vec{h} \cdot \vec{\Phi}\, dx \right) \right| 0 \right\rangle = \exp\left\{ - \int dx \int [1 - \cos(\vec{u} \cdot \vec{h})] C^2(u) d\vec{u} \right\}. \tag{25}$$

We want to find the connection of these relations with the model action functional. In order to do so we need to define an operator product, and it is not difficult to see that the usual Wick product is not satisfactory. Another candidate is $\Phi_{Rij}{}^2(x)$ defined by the decomposition

$$\Phi_i(x)\Phi_j(y) = \delta(x - y)\Phi_{Rij}{}^2(x) + :\Phi_i(x)\Phi_j(y):, \tag{26}$$

where :: denotes normal ordering of $A, A^\dagger$ (or $B, B^\dagger$), or its equivalent. In our representation $\Phi_{Rij}{}^2(x)$ becomes

$$\Phi_{Rij}{}^2(x) = \int B^\dagger(x,\vec{u}) u_i u_j B(x,\vec{u}) d\vec{u}. \tag{27}$$

The generalization to higher-order products is straightforward.

To find a relationship between $C(u)$ and the model problem we proceed as follows. We assume that $C(u)$ is such that

$$S(\vec{h}) = \left\langle 0 \left| \exp\left( i \int \vec{h} \cdot \vec{\Phi}\, dx \right) \right| 0 \right\rangle = \mathfrak{N} \int \exp\left( i \int \vec{h}(x) \cdot \vec{\Phi}(x) dx - \int \{ \tfrac{1}{2} m^2 \vec{\Phi}^2(x) + \lambda V[\vec{\Phi}^2(x)] \} dx \right) \mathfrak{D}\vec{\Phi}. \tag{28}$$

We can change the potential by first adding the term $\int \chi_\Lambda(x) V'[\vec{\Phi}^2(x)]\, dx$, where $\chi_\Lambda$ is the characteristic function of some compact region $\Lambda$; in particular, in the representation introduced above we consider

$$\left\langle 0 \left| \exp\left\{ i \int \vec{h}(x) \cdot \vec{\Phi}(x) dx - \int \chi_\Lambda(x) V'[\vec{\Phi}^2(x)]\, dx \right\} \right| 0 \right\rangle, \tag{29}$$

where the local products are defined through the $R$ prescription. A straightforward evaluation, plus normalization and the limit $\Lambda \to R^n$, leads to the new characteristic functional

$$\lim_{\Lambda \to R^n} N_\Lambda \exp\left\{ - \int dx \int \left[ 1 - e^{i \vec{u} \cdot \vec{h}(x) - \chi_\Lambda(x) V'[u^2]} \right] C^2(u) d\vec{u} \right\} = \exp\left( - \int dx \int \{ 1 - \cos[\vec{u} \cdot \vec{h}(x)] \} e^{-V'[u^2]} C^2(u) d\vec{u} \right). \tag{30}$$

We conclude, therefore, that if the action functional changes according to

$$I \to I - \int V'[\vec{\Phi}^2(x)]\, dx, \tag{31}$$

then the corresponding measure $\sigma(\vec{u})$ changes according to

$$C^2(u) d\vec{u} \to e^{-V'[u^2]} C^2(u) d\vec{u}. \tag{32}$$

If $V' = -\lambda V$ the nonlinear interaction is "canceled" leaving only the term $\frac{1}{2} m^2 \vec{\Phi}^2(x)$, but even the local product $\vec{\Phi}^2(x)$ needs to be defined by the $R$ prescription which does not lead to the free theory but an alternative one termed "pseudofree" (PF). To characterize this special theory we first observe the formal identities so far obtained, namely[7]

$$S_{\rm PF}(\vec{h}) = \mathfrak{N} \int \exp\left[ i \int \vec{h}(x) \cdot \vec{\Phi}(x) dx - \tfrac{1}{2} m^2 \int \vec{\Phi}^2(x) dx \right] \mathfrak{D}\vec{\Phi} \equiv \exp\left( - \int dx \int \{ 1 - \cos[\vec{u} \cdot \vec{h}(x)] \} e^{-(1/2) m^2 u^2} C_0{}^2(u) d\vec{u} \right), \tag{33}$$

where

$$e^{\lambda V[u^2]}C^2(u) \equiv e^{-(1/2)m^2u^2}C_0^2(u)\,, \tag{34}$$

and the form of the dependence on $m$ follows from the general rule for adding $V'$. Invariance of the expression for $S_{PF}(\vec{h})$ under the transformation $\vec{h}(x) \to s\vec{h}(x)$ and $m^2 \to s^2 m^2$ for any $s>0$ leads immediately to $C_0^2(u) = gu^{-N}$. Here $g$ is an undetermined positive constant which incorporates the freedom in normalizing the two-point function (or in choice of $x$ dimensions), and which can therefore be chosen arbitrarily.

In summary, the formal functional integral represented by

$$S(\vec{h}) = \mathfrak{N}\int \exp\left(i\int \vec{h}(x)\cdot\vec{\Phi}(x)dx - \int\{\tfrac{1}{2}m^2\vec{\Phi}^2(x) + \lambda V[\vec{\Phi}^2(x)]\}\,dx\right)\mathfrak{D}\vec{\Phi} \tag{35}$$

is evaluated as

$$S(\vec{h}) = \exp\left(-g\int dx\int\{1-\cos[\vec{u}\cdot\vec{h}(x)]\}e^{-(1/2)m^2u^2-\lambda V[u^2]}|u|^{-N}d\vec{u}\right), \tag{36}$$

where $g$ denotes an arbitrary positive constant without any essential physics. Moreover, every such generating functional reduces as $\lambda \to 0$ to the pseudofree form

$$S_{PF}(\vec{h}) = \exp\left(-g\int dx\int\{1-\cos[\vec{u}\cdot\vec{h}(x)]\}e^{-(1/2)m^2u^2}|u|^{-N}d\vec{u}\right), \tag{37}$$

which is fundamentally different from the free form

$$S_F(\vec{h}) = \exp\left[-\tfrac{1}{2}(m^2)^{-1}\int \vec{h}^2(x)dx\right]. \tag{38}$$

This means that any nonlinear interaction is a discontinuous perturbation of the free theory—but it is a continuous perturbation of the pseudofree theory.

## III. INFINITE-COMPONENT FIELDS

We want the characteristic functional of an infinite-component independent-value field to be obtained as the limit of those with finitely many components. In order that this limit exist we have various parameters to work with, namely $V_N$, $m_N$, and the arbitrary factor $g_N$. We start with the finite-dimensional solution and suppressing the integration over $x$ we consider the relevant expression for the exponent $L[\vec{h}]$ of the expectation functional as given by

$$\begin{aligned}
&\lim_{N\to\infty}\int g_N d\vec{u}\,|u|^{-N}[1-\cos(\vec{u}\cdot\vec{h})]\,e^{-(1/2)m_N^2u^2-V_N(u^2)}\\
&\quad=\lim_{N\to\infty}\int_0^\infty dr\,r^{1/2N-1}\frac{g_N}{\Gamma(\frac{1}{2}N)}\int d\vec{u}[1-\cos(\vec{u}\cdot\vec{h})]\,e^{-[r+(1/2)m_N^2]u^2-V_N(u^2)}\\
&\quad=\lim_{N\to\infty}\int_0^\infty dr\,r^{1/2N-1}\frac{g_N}{\Gamma(\frac{1}{2}N)}\int d\vec{u}[1-\cos(\vec{u}\cdot\vec{h})]\,e^{-[r+(1/2)m_N^2]u^2-i\alpha u^2}f_N(\alpha)\,d\alpha\,,
\end{aligned} \tag{39}$$

where $f_N(\alpha)$ is the Fourier transform of $e^{-V_N(u^2)}$. The integration over $\vec{u}$ is simply the product of Gaussian integrations and leads to

$$\lim_{N\to\infty}\int_0^\infty \frac{dr}{r}\frac{g_N}{\Gamma(\frac{1}{2}N)}\pi^{(1/2)N}\left(1+\frac{m_N^2}{2r}+\frac{i\alpha}{r}\right)^{-(1/2)N}[1-e^{-\vec{h}^2/4[r+1/2m_N^2+i\alpha]}]f_N(\alpha)d\alpha. \tag{40}$$

Finally we have to make our choice of $V_N$, $m_N$, or $g_N$. Evidently $f_N(\alpha)$ should depend on $N$ in a simple way, and we choose $f_N(\alpha) = Nf(N\alpha)$, which simply means that $V_N(u^2) = V((1/N)u^2)$. In addition, we assume $m_N^2 = (1/N)m^2$ and $g_N = \Gamma(N/2)\pi^{-N/2}g$; apart from an arbitrary positive factor $g$, note that $g_N^{-1}$ is one-half the surface of the unit sphere in $N$ dimensions. We are allowed to interchange the order of integration and first take the limit $N\rightarrow\infty$, if, e.g., $\int(1+\alpha^2)|f(\alpha)|^2 d\alpha<\infty$ [which is easily arranged provided that both $e^{-V(b)}$ and $V'(b)e^{-V(b)}$ are square integrable on the interval $(0,\infty)$]. Making the substitution $b=1/2r$ we find for the final result

$$L[\vec{h}]=g\int\frac{db}{b}e^{-[(1/2)m^2+i\alpha]b}(1-e^{-(1/2)b\vec{h}^2})f(\alpha)\,d\alpha$$
$$=g\int_0^\infty\frac{db}{b}e^{-(1/2)m^2b-V(b)}(1-e^{-(1/2)b\vec{h}^2}),\qquad(41)$$

which exactly leads to a characteristic functional of the form stated in Sec. I in which $d\sigma(b)=(g/b)\exp[-\frac{1}{2}m^2b-V(b)]\,db$.

Our choice of $V_N$, $m_N$, and $g_N$ differs from the usual approach, though it occurs to us not unnatural because the free part of the Lagrangian and the potential are treated homogeneously, namely $V_N(\vec{\Phi}^2)=V((1/N)\vec{\Phi}^2)$ and $T_N(\vec{\Phi}^2)=\frac{1}{2}m_N{}^2\vec{\Phi}^2=T((1/N)\vec{\Phi}^2)$.

Besides the expressions we have used, the usual approach can also be examined. Here we fix $m_N{}^2=m^2$, take $V_N(\vec{\Phi}^2)=NV(\vec{\Phi}^2/N)$, for example, and adjust $g_N$ so that the limit exists. With $z=r/N$ and $b=1/2z$, we find for the exponent of interest the expression

$$\lim_{N\rightarrow\infty}\int\frac{dz}{z}\frac{g_N}{\Gamma(\frac{1}{2}N)}\pi^{(1/2)N}\left(1+\frac{m^2}{2Nz}+i\frac{\alpha}{Nz}\right)^{-(1/2)N}[1-e^{-\vec{h}^2/4[Nz+(1/2)m^2+i\alpha]}]f_N(\alpha)d\alpha$$
$$=\lim_{N\rightarrow\infty}\int\frac{db}{b}\frac{g_N}{N\Gamma(\frac{1}{2}N)}\pi^{(1/2)N}e^{-[(1/2)m^2+i\alpha]b}(\tfrac{1}{4}\vec{h}^2b)f_N(\alpha)d\alpha.\qquad(42)$$

With $g_N=N\Gamma(\frac{1}{2}N)\pi^{-(1/2)N}g$ we find the general expression

$$L[\vec{h}]=\tfrac{1}{4}g\vec{h}^2\lim_{N\rightarrow\infty}\int_0^\infty db\,e^{-(1/2)m^2b-V_N(b)}.\qquad(43)$$

The limit that remains can do no more than modify the coefficient of $\vec{h}^2$, i.e., to introduce an effective mass. If $V_N(b)\rightarrow0$ as $N\rightarrow\infty$, e.g., if $V_N(b)=NV(b/N)$ as in the standard expression for polynomial $V$, then any nonlinear potential makes no contribution to the effective mass. If instead $V_N(b)=V(b)$ then the resultant exponent becomes

$$L[\vec{h}]=\tfrac{1}{4}g\vec{h}^2\int_0^\infty db\,e^{-(1/2)m^2b-V(b)}.\qquad(44)$$

In any case the resultant field is invariably free; and since we seek models (and have already exhibited examples) where nonlinear interactions have nontrivial dynamical consequences, the kind of limits that arise when $m_N{}^2=m^2$ are not at all adequate.

It is perhaps of passing interest to point out that a $1/N$ expansion is easy to obtain for our models. We need only consider the basic formula

$$g\int\frac{db}{b}\left(1+\frac{m^2b}{N}+\frac{2i\alpha b}{N}\right)^{-(1/2)N}[1-e^{-(1/2)b\vec{h}^2(1+m^2b/N+i2\alpha b/N)^{-1}}]f(\alpha)d\alpha\qquad(45)$$

that represents the relevant expression for the exponent of the expectation functional for an $N$-component field (with $m_N{}^2=m^2/N$, $b=1/2r$, etc.). An expansion in $1/N$ is straightforward, and we quote the first two terms as

$$g\int\frac{db}{b}f(\alpha)\,d\alpha\,e^{-(1/2)(m^2+2i\alpha)b}\left\{(1-e^{-(1/2)b\vec{h}^2})+\frac{1}{N}\left[\frac{(m^2+2i\alpha)^2b^2}{4}(1-e^{-(1/2)b\vec{h}^2})+\frac{b^2\vec{h}^2(m^2+2i\alpha)}{2}e^{-(1/2)b\vec{h}^2}\right]+\cdots\right\}.\qquad(46)$$

From the identity

$$e^{-V(b)}=\int e^{-i\alpha b}f(\alpha)d\alpha\qquad(47)$$

it follows that

$$(d/db)^p e^{-V(b)} = \int (-i\alpha)^p e^{-i\alpha b} f(\alpha) d\alpha. \tag{48}$$

Consequently the exponent of interest becomes

$$g\int_0^\infty \frac{db}{b}\, e^{-(1/2)m^2 b - V(b)} \Big[ (1 - e^{-(1/2)b\vec{h}^2}) + \frac{1}{N}\big(\{\tfrac{1}{4} m^4 b^2 + m^2 b^2 V'(b) + b^2[V'(b)^2 - V''(b)]\}(1 - e^{-(1/2)b\vec{h}^2}) + b^2 \vec{h}^2[\tfrac{1}{2} m^2 + V'(b)] e^{-(1/2)b\vec{h}^2}\big) + \cdots \Big], \tag{49}$$

which illustrates the dependence on the potential of the two leading $1/N$ terms.

We next seek an operator realization of the infinite-component field. We are guided by the solution in the finite-dimensional case, although we have to be careful in order to make things well defined. In the finite-dimensional case the choice of the measure on $R^N$ was effectively the usual Lebesgue measure $d\vec{u}$. If $N$ becomes infinite, a proper measure has to be defined. In so doing we can conveniently absorb the function $C(u)$ into the measure (which is also possible in finite dimensions) and define a measure $\rho$ first on cylinder-set functions $f(\vec{u})$ that can be written as $\bar{f}(u, \varphi_1, \ldots, \varphi_p)$, depending only on the radius and a finite number of angles by the relation

$$\int d\rho(\vec{u}) f(\vec{u}) = \lim_{N\to\infty} \int_0^\infty du\, C^2(u) g_N \int d\Omega_N \bar{f}(u, \varphi_1, \ldots, \varphi_p), \tag{50}$$

where typically $C^2(u) = (g/u)\exp\{-\frac{1}{2} m^2 u^2 - \lambda V[u^2]\}$. Subsequently, the class of integrable functions may be extended to include functions that are a limit of cylinder-set functions. Note that $g_N = 2/\int d\Omega_N$ and that $f = 1$ is not an integrable function [since $C^2(u) \notin L^1$] but $u_i, u_i u_k$, etc., are integrable functions. Now we start again with creation and annihilation operators $A^\dagger(x, \vec{u}), A(x, \vec{u})$ satisfying

$$[A(x, \vec{u}), A^\dagger(y, \vec{v})] = \delta(x - y)\delta(\vec{u} - \vec{v}), \tag{51}$$

where the distribution $\delta(\vec{u} - \vec{v})$ has to be understood with respect to the measure $d\rho$. The operator $B$ is defined as

$$B(x, \vec{u}) = A(x, \vec{u}) + 1 \tag{52}$$

and our infinite-component field reads

$$\Phi_i(x) = \int B^\dagger(x, \vec{u}) u_i B(x, \vec{u}) d\rho(\vec{u}). \tag{53}$$

Calculating the product of two fields we find

$$\begin{aligned}\Phi_i(x)\Phi_k(y) &= \int B^\dagger(x, \vec{u}) u_i B(x, \vec{u}) B^\dagger(y, \vec{v}) v_k B(y, \vec{v})\, d\rho(\vec{u})\, d\rho(\vec{v}) \\ &= :\Phi_i(x)\Phi_k(y): + \int B^\dagger(x, \vec{u}) u_i v_k B(y, \vec{v}) \delta(x - y)\delta(\vec{u} - \vec{v}) d\rho(\vec{u})\, d\rho(\vec{v}).\end{aligned} \tag{54}$$

Suitable matrix elements (e.g., between general cylinder-set coherent states) give meaning to the last expression with the result that

$$\Phi_i(x)\Phi_k(y) = :\Phi_i(x)\Phi_k(y): + \delta(x - y)\Phi_{Rik}{}^2(x), \tag{55}$$

where

$$\Phi_{Rik}{}^2(x) = \int B^\dagger(x, \vec{u}) u_i u_k B(x, \vec{u}) d\rho(\vec{u}). \tag{56}$$

Higher-order renormalized products are defined analogously.

In finite dimensions we showed how to augment the potential of the theory by starting with some given model, adding a potential with space-time cutoff, renormalizing the over-all expression, and then letting the space-time cutoff go to infinity we obtained the modified expectation functional. In a similar analysis for infinite-component fields we initially have to approximate the additional potential in two ways. We start with a potential with space-time cutoff that involves only a finite number of fields (say $K$), then we remove the cutoff and let the number of fields go to infinity ($K\to\infty$) along with the appropriate scaling. The order of these two limits does not make any difference. The space-time cutoff works exactly as in the finite-dimensional case, so we omit it and assume it has been carried out. We assume without real loss of generality that the unmodified model is the pseudofree model, which means we consider the expression

$$\lim_{K\to\infty}\lim_{N\to\infty} g_N \int d\vec{u}|u|^{-N}[1-\cos(\vec{u}\cdot\vec{h})]e^{-(1/2)m_N{}^2u^2-V_K(u^2)}$$

$$=\lim_{K\to\infty}\lim_{N\to\infty}\int dr\, r^{(1/2)N-1}\frac{g_N}{\Gamma(\frac{1}{2}N)}\int d\vec{u}[1-\cos(\vec{u}\cdot\vec{h})]e^{-[r+(1/2)m_N{}^2]u^2}\exp\left(-i\alpha\sum_{k=1}^{K}u_k{}^2\right)f_K(\alpha)d\alpha\,. \quad (57)$$

We regard $\vec{h}$ as fixed and $K$ big enough so that $\vec{h}$ lies in the $K$-dimensional subspace (in fact, it turns out that this assumption is of no importance). In this case the exponent becomes

$$\lim_{K\to\infty}\lim_{N\to\infty}\int\frac{dr}{r}\frac{g_N}{\Gamma(\frac{1}{2}N)}\pi^{(1/2)N}[1-e^{-\vec{h}^2/4[r+(1/2)m_N{}^2+i\alpha]}]\left(1+\frac{m_N{}^2}{2r}+\frac{i\alpha}{r}\right)^{-(1/2)K}\left(1+\frac{m_N{}^2}{2r}\right)^{-(1/2)(N-K)}f_K(\alpha)d\alpha. \quad (58)$$

Again it is informative to consider the different possibilities discussed earlier. The first choice [$m_N{}^2=m^2/N$, $V_K(u^2)=V(u^2/K)$, and $g_N=\pi^{-(1/2)N}\Gamma(\frac{1}{2}N)g$] leads to

$$g\int_0^\infty\frac{db}{b}(1-e^{-(1/2)b\vec{h}^2})e^{-(1/2)m^2b-V(b)}, \quad (59)$$

which coincides with the previous result obtained when $K\equiv N\to\infty$. The second choice [$m_N{}^2=m^2$, $g_N=N\pi^{-(1/2)N}\Gamma(\frac{1}{2}N)g$, and $V_K(u)\to 0$] leads to

$$\tfrac{1}{4}g\vec{h}^2\int_0^\infty db\, e^{-(1/2)m^2b}=\tfrac{1}{2}g(m^2)^{-1}\vec{h}^2, \quad (60)$$

which is also in agreement with the previous result obtained when $K\equiv N\to\infty$, though again the potential has no effect and therefore the result is unsatisfying. The last approach [as above except that $V_K(u)=V(u)$] also gives $\frac{1}{2}g(m^2)^{-1}\vec{h}^2$, a result completely independent of the potential $V$ and which differs from the conclusion obtained previously when $K\equiv N\to\infty$. Thus not only does the choice $m_N{}^2=m^2$ lead to free fields but, in the latter case, the potential cannot even be added to the model subsequently. Such behavior is certainly unsatisfactory, and is not shared by the alternative solution (based on $m_N{}^2=m^2/N$, etc.) that we advocate.

In addition, the solution we propose for infinite $N$ based on the exponent

$$L[\vec{h}]=g\int_0^\infty\frac{db}{b}e^{-(1/2)m^2b-\lambda V(b)}(1-e^{-(1/2)b\vec{h}^2}) \quad (61)$$

shares the anomalous behavior found for finite $N$, namely, as $\lambda\to 0$ the solution does not converge to the free theory solution (with exponent $L_F[\vec{h}]\equiv\frac{1}{2}(m^2)^{-1}\vec{h}^2$) but to the pseudofree solution based on the exponent

$$L_{PF}[\vec{h}]\equiv g\int_0^\infty\frac{db}{b}e^{-(1/2)m^2b}(1-e^{-(1/2)b\vec{h}^2})$$
$$=g\ln(1+\vec{h}^2/m^2). \quad (62)$$

We remark again that the arbitrary parameter $g$ fixes the scale of the two-point function and can be chosen as desired. Just as in the case of finite $N$, it is the pseudofree theory which is the relevant one with regard to the introduction of interactions, and decidedly not the free theory.

## IV. DISCUSSION

Our purpose in this paper has basically been twofold. On the one hand, we have discussed the solution for the time-ordered generating functional in Euclidean space-time for a model field having $N$ components when $N$ is finite or infinite. No pretense is made that independent-value models are physical, but that does not detract from their possible mathematical interest. After all, artificial fields with identically constant bare propagators have been usefully exploited in studies of $O(N)$ models[1] and in studies of superfields.[8] The initial approach to independent-value models, as with most other models, is to use standard perturbation theory with the necessary introduction of a large-momentum cutoff to render the perturbation theory finite term by term. Indeed, the kinematics is so simple that the theory itself can essentially be solved in closed form with a cutoff that takes the form of a lattice.[9] However, that cutoff cannot be removed no matter how one adjusts the values of various parameters, and this fact simply reflects the conventional nonrenormalizability of such models. Our methods of solution are entirely different, and a high-momentum cutoff *never* enters as it should not since the detailed high-momentum behavior of independent-value models is one of their essential and intrinsic properties. Destroy that property quantum mechanically with a cutoff and a chasm is set up that simply cannot be crossed. Undoubtedly, the non-Hartree-type behavior of independent-value models in the limit $N\rightarrow\infty$ is intimately connected with their high-momentum behavior. For, by introducing a high-momentum cutoff and rendering the theory locally free, the large-$N$ limit does in fact become a Hartree type, and there is no hope of escaping that restriction as the cutoff is removed. Here is *prima facie* evidence that for these models introducing a cutoff is the worst possible mistake to make.

Our second main point is made now and it is the use of the independent-value models and their anomalous behavior as motivation for a larger program directed toward more relevant nonrenormalizable theories. It is our belief that certain essential characteristics of general nonrenormalizable theories are illustrated by these models, to wit, the total disconnection of free and interacting theories, and the replacement of the role of the free theory by that of the pseudofree theory. Such behavior has been demonstrated in problems ranging from harmonic oscillators and singular perturbations, to several soluble model field theories, and up to plausible conjectures covering covariant nonrenormalizable quantum field theories.[10] Why such behavior takes place at all can be heuristically understood in every case by observing that in a functional integral formulation the interaction acts partially as a hard core in the space of field histories projecting out for all coupling-constant values certain field configurations that would be allowed by the free theory, and which remain projected out in the limit that the coupling constant vanishes. The hard-core appearance of the interaction does not need to be manifest (as in the interaction of particles, say) but is effective nonetheless. One need only consider the example treated in this paper where the free action term is $\int\vec{\Phi}^2(x)dx$ (take $N<\infty$) and the interaction term, for example, is $\int[\vec{\Phi}^2(x)]^2dx$. This interaction is not manifestly hard core, but one should recall that $L^2$ functions are generally *not* locally $L^4$ functions, and thus the interaction term effectively acts partially as a hard core *in relation to the specific free theory in question.* This general viewpoint can be persuasively argued for covariant scalar quantum field theories,[11] and for various models involving the gravitational field in interaction with other fields.[12] In line with the discussion in this paper, we would also conjecture that the large-$N$ limit of nonrenormalizable quantum field theories would generally *not* exhibit Hartree (or Hartree-Fock) type behavior.

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[1]See, e.g., K. Wilson, Phys. Rev. D 7, 2911 (1973); L. Dolan and R. Jackiw, *ibid.* 9, 3320 (1974); H. J. Schnitzer, *ibid.* 10, 1800 (1974); S. Coleman, R. Jackiw, and H. D. Politzer, *ibid.* 10, 2491 (1974); D. Gross and A. Neveu, *ibid.* 10, 3235 (1974); R. G. Root, *ibid.* 10, 3322 (1974). In addition see S. Ma, Rev. Mod. Phys. 45, 589 (1973), Sec. IV.

[2]J. R. Klauder and H. Narnhofer, Acta Phys. Austriaca (to be published); J. R. Klauder, *ibid.* 41, 237 (1975).

[3]We should also emphasize that independent-value models exhibit no symmetry-breaking effects, which can be heuristically understood on the basis that there are no free, massless, independent-value models for any number of space-time dimensions.

[4]See, e.g., K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969).

[5]E. Lukacs, *Characteristic Functions* (Hafner, New York, 1970), second edition; I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions Vol. 4: Applications of Harmonic Analysis,* translated by A. Feinstein (Academic, New York, 1964).

[6]I. J. Schoenberg, Ann. Math. 39, 811 (1938).

[7]The next equation in the text is intended to be provocative. Superficially, the formal functional integral appears to be Gaussian but is not; the formal quadratic term $\vec{\Phi}^2(x)$ is to be interpreted (effectively) as $\vec{\Phi}_R{}^2(x)$ and not $:\vec{\Phi}^2(x):$. It is conjectured that analogous anomalies lie at the heart of nonrenormalizable theories more generally [cf. J. R. Klauder, Phys. Lett. 47B, 523 (1973); in *Recent Developments in Mathematical Physics,* Proceedings of the XII Schladming conference on nuclear physics, edited by P. Urban

(Springer, Berlin, 1973) [Acta Phys. Austriaca Suppl. 11 (1973)], p. 341.
[8]See, e.g., J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974); Phys. Lett. 49B, 52 (1974).
[9]W. Kainz, Lett. Nuovo Cimento 12, 217 (1975).
[10]For a brief survey see *International Symposium of Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics, edited by H. Araki (Springer, Berlin, 1975), Vol. 39, p. 160.
[11]See the articles listed in Ref. 7.
[12]J. R. Klauder, Gen. Relativ. Gravit. 6, 13 (1975).