Statistical treatment of annihilation processes

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We present a detailed statistical model for describing annihilation reactions initiated by e^+e^- and $\bar{p}p$ collisions. Charged and total multiplicities, fractional prong cross sections, and the $\pi^+\pi^-\pi^+\pi^-$ production cross section are calculated and compared with experiment. It is argued that the model is valid in the few-GeV energy region and the agreement between theory and experiment is good here.

I. INTRODUCTION

Some aspects of both the inclusive and exclusive data on e^+e^- annihilation into hadrons and $\bar{p}p$ annihilation show similarity.[1] This has led to the use of a statistical model for their description.[1] In the present paper we carry out the statistical approach of Ref. 1 in more detail. One assumes in such a model that the reacting particles collide and form a heavy resonance or fireball, which then decays into all possible final states consistent with quantum-number conservation. Assuming that these resonances add incoherently,[2] the cross section for e^+e^- and $\bar{p}p$ (which will be denoted collectively as $\bar{A}A$) into a particular channel, C, is given by

$$\sigma_C(\bar{A}A) = \sum_R \sigma_R(\bar{A}A) B(R\to C), \qquad (1)$$

where the sum is over all possible resonances. Here $\sigma_R(\bar{A}A)$ is the formation cross section for the resonance, R, and $B(R\to C)$ is the branching ratio of the resonance R into the channel C.

To calculate the branching ratios $B(R\to C)$ for various channels, a model for the resonance, R, is needed. We use the statistical bootstrap model (SBM) of Hagedorn[3] in the particular form derived by Frautschi.[4] In this model, a heavy resonance is composed itself of other hadrons assumed to be contained in a box of volume V of hadronic dimensions. These hadrons can be π, ρ, ω, etc. (i.e., discrete resonances) as well as heavy resonances or fireballs in the continuum. There can be any number of these hadrons in the box consistent with conservation laws, and each state of the system is given an equal *a priori* probability. Under these assumptions[4] the density of hadron states, $\rho(M)$, grows as

$$\rho(M) \propto M^{-3} e^{bM}, \qquad (2)$$

with $b^{-1} \sim 160$ MeV. To calculate a branching ratio one simply counts the number of states for a given channel and divides by the total number of states. Our motivation for using the SBM comes from the various successes of the model in areas such as mass spectra in hadronic collisions,[2] linearity over a range of energy of the $\bar{p}p$ total annihilation multiplicities as a function of $\sqrt{s}$ (Fig. 5) (and over a more limited energy range those of e^+e^-), the exponential falloff of exclusive channel cross section in e^+e^- and $\bar{p}p$ annihilations into pions (Fig. 9), and the exponential falloff, in hadronic collisions, of the single-particle inclusive cross sections with $p_\perp$. It is of interest to note that most of the above considerations including the similarity of e^+e^- and $\bar{p}p$ seem to have a limited energy range in which they are applicable. We have studied the predictions of incoherent resonance formation combined with the SBM to see if it describes well lower-energy data. One expects a breakdown of this picture at higher energies due to the onset of coherence which often takes the form of Regge behavior according to duality principles.[2,4]

The method of calculating branching ratios using the linear bootstrap approximation is discussed in detail in Sec. II. The effect of nonlinear terms is discussed in Sec. III. The results of our calculations are contained in Sec. IV. They are compared with the data and discussed. Our conclusions are summarized in Sec. V.

II. CALCULATION OF BRANCHING RATIOS IN THE LINEAR BOOTSTRAP

A. The linear bootstrap

Frautschi[4] has shown that a fireball has roughly a 70% probability of being made up of two hadrons, about 30% of being composed of three hadrons, and a very small probability for it to contain more than three hadrons. Further, the dominant two-fireball configuration is that in which of the two hadrons in the fireball one is very massive and the other is very light. The decay probability into a particular channel is assumed to be proportional to the available density of states, and we will for the moment describe the decay of a fireball of mass M $[F(M)]$ by the chain

$$F(M)\to \pi + F_1(M_1) \to \pi + \pi + F_2(M_2)\cdots \to n\pi + R\ , \qquad (3)$$

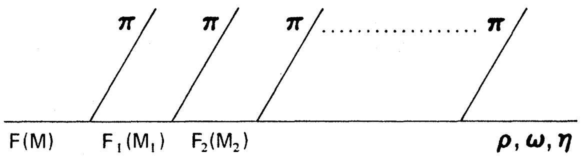

FIG. 1. Decay chain of a fireball $F(M)$ of mass M in the linear bootstrap model in which only pions are emitted except at the end of the chain.

where R is a discrete resonance (e.g. $\rho, \omega, \eta, \ldots$). The decay chain is shown schematically in Fig. 1.

Let $P(M, m)$ be the probability density for a fireball of mass M to decay into a pion plus a fireball of mass m. Then we have for the case of a ρ meson at the end of the chain (this case is taken for definiteness)

$$B(M \to n\pi) = \int \cdots \int P(M, M_1)P(M_1, M_2) \cdots P(M_{n-3}, M_\rho) \times dM_1 \cdots dM_{n-3}. \tag{4}$$

Here $B(M\to n\pi)$ is the branching ratio of a fireball of mass M into n pions. Equation (4) will, in the following, be generalized to calculate branching ratios into charged and neutral pions and we shall allow resonances as well as pions to be emitted throughout the chain. To this end define $P(M_1, M_2, M_3)$ to be the probability density for a fireball of mass M_1 to decay into a particle or discrete resonance of mass M_3 and a fireball of mass M_2.

B. Evaluation of $P(M_1, M_2, M_3)$

Consider the process

$$F(M) \to F(m) + \mu, \tag{5}$$

where $F(r)$ denotes a fireball of mass r and μ is a particle or discrete resonance of mass μ. Let $F(M)$ have isospin and G-parity quantum numbers I and G, respectively. We assume (although this is not an essential limitation) that there exist no exotic resonances, which in practice means that $I < 2$. Let $\rho_I(M)$ be the density of hadron states with isospin I and with a fixed value of I_3. We assume that $\rho_I(M)$ is independent of I_3 and suppress the I_3 label.

We now consider only the more probable configurations of $F(M)$, $F(m)$ plus one of a relatively light set of discrete hadron states $R_1 \cdots R_k$, with masses m_j, isospin I_j, G parity G_j, and spin J_j. Then according to Frautschi[4] and with no exotics present, we have

$$\rho_I(M) = \frac{V}{(2\pi)^3} \int d^3p_1 d^3p_2 d\mu_1 d\mu_2 \delta^3(\vec{p}_1 + \vec{p}_2)\delta(M - E_1 - E_2) \times \sum_{j=1}^{k} \delta(M_2 - m_j)(2J_j + 1)\{\delta_{I0}[\delta_{I_j0}\rho_0(\mu_1) + \delta_{I_j1}\rho_1(\mu_1)] + \delta_{I1}[\delta_{I_j0}\rho_1(\mu_1) + \delta_{I_j1}(\rho_0(\mu_1) + \rho_1(\mu_1))]\}. \tag{6}$$

The factor 1/2! in Ref. 4 is not present since μ_1 and μ_2 are different: μ_1 is in the continuum and μ_2 is a discrete hadron state.

Since the branching ratios are assumed to be the ratios of the phase space available to the total phase space, we find from the foregoing that, generalizing $P(M_1, M_2, M_3)$ to include isospin and G parity,

$$P(M_i, I_i, G_i; M_f, I_f, G_f; m_j, I_j, G_j) = (2J_j + 1)\Delta^{I_j G_j}_{I_i G_i, I_f G_f} A(M_i, M_f, m_j) \frac{\rho_{I_f}(M_f)}{\rho_{I_i}(M_i)}, \tag{7}$$

where

$$A(M_i, M_f, m_j) = \frac{V}{16\pi^2} \frac{M_i^4 - (M_f^2 - m_j^2)^2}{M_i^4} \times \lambda^{1/2}(M_i^2, M_f^2, m_j^2), \tag{8a}$$

with

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \tag{8b}$$

and

$$\Delta^{I_j G_j}_{I_i G_i, I_f G_f} = \delta_{G_i, G_f G_j} \Delta^{I_j}_{I_i I_f}, \tag{9a}$$

$$\Delta^{I_j}_{I_i I_f} = 1 \text{ if } I_i = I_f = 1$$

$$= \delta_{I_j, |I_i + I_f} \text{ otherwise.} \tag{9b}$$

Equation (9) ensures conservation of isospin and G parity.

We now proceed to evaluate $P_n^k(M, I, G)$, which is defined to be the probability that a fireball of mass M, isospin I, and G parity G decays into n pions, k of which are charged. This will be done by means of a recursion relation.

C. The recursion relation for $P_n^k(M, I, G)$

A fireball decays predominantly by emission of a light particle, leaving a lighter fireball as the daughter (at the end of the decay chain the daughter will be a discrete hadron state). Typically, a fireball of mass M_l may decay into n pions of which k are charged [denoted by $(n;k)$] by emitting a π^+ with a fireball of mass M_{l+1} which then decays into $(n-1; k-1)$. At first we will allow only

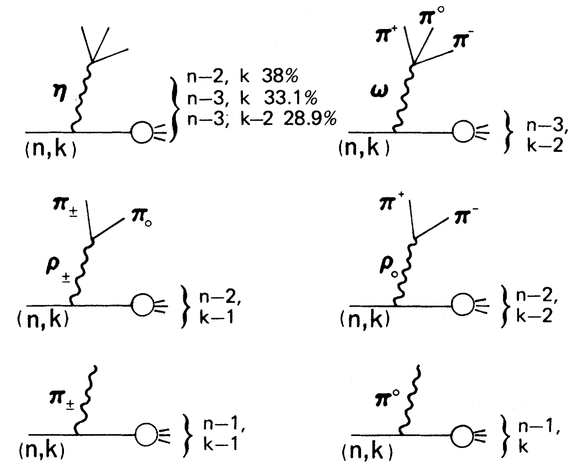

FIG. 2. Schematic representation of the terms in the recursion relation (11) for the decay of a typical fireball.

emission of π, ρ, ω, and η. The effect of kaons and the φ will be considered later. By doing a succession of calculations we can study how the emission of the various resonances affects the results. Including only π, ρ, ω, and η, the relevant diagrams for P_n^k, using the decay modes of ρ, ω, and η which are known, are as shown in Fig. 2. We treat the photon from η decay as a neutral pion for the purpose of calculation. We now define an auxiliary probability $\tilde{P}_n^k$ by

$$\tilde{P}_n^k(M, I, G) = \rho_I(M) P_n^k(M, I, G) . \tag{10}$$

Here we have assumed that $\rho_I(M)$ is independent of the G parity, G, and denotes the density of states for a particular G with G parity label suppressed. Then making use of the appropriate Clebsch-Gordan coefficients, it follows from the diagrams of Fig. 2 that

(a) $\tilde{P}_n^k(M, 0, G) = 0$ if k is odd, (11a)

(b) otherwise:

$$\begin{aligned}\tilde{P}_n^k(M, I, G) = \int_{m_\eta+(n-4)\pi}^{M-m_\pi} A(M, \mu, m_\pi)\{\delta_{I0}(\tfrac{1}{3}\tilde{P}_{n-1}^k(\mu, 1, -G) + \tfrac{2}{3}\tilde{P}_{n-1}^{k-1}(\mu, 1, -G)) \\ + \delta_{I1}(\delta_{k,\text{even}}[\tilde{P}_{n-1}^{k-1}(\mu, 1, -G) + \tilde{P}_{n-1}^k(\mu, 0, -G)] \\ + \delta_{k,\text{odd}}[\tfrac{1}{2}\tilde{P}_{n-1}^{k-1}(\mu, 1, -G) + \tfrac{1}{2}\tilde{P}_{n-1}^k(\mu, 1, -G) + \tilde{P}_{n-1}^{k-1}(\mu, 0, -G)])\} d\mu \\ + 3\int_{m_\eta+(n-6)m_\pi}^{M-m_\omega} d\mu A(M, \mu, m_\omega)\tilde{P}_{n-3}^{k-2}(\mu, I, -G) \\ + \int_{m_\eta+(n-6)m_\pi}^{M-m_\eta} A(M, \mu, m_\eta)[0.38\tilde{P}_{n-2}^k(\mu, I, G) + 0.331\tilde{P}_{n-3}^k(\mu, I, G) + 0.289\tilde{P}_{n-3}^{k-2}(\mu, I, G)] d\mu \\ + 3\int_{m_\eta+(n-5)m_\pi}^{M-m_\rho} A(M, \mu, m_\rho)\{\delta_{I0}(\tfrac{1}{3}\tilde{P}_{n-2}^{k-2}(\mu, 1, G) + \tfrac{2}{3}\tilde{P}_{n-2}^{k-1}(\mu, 1, G)) \\ + \delta_{I1}[\delta_{k,\text{even}}(\tilde{P}_{n-2}^{k-1}(\mu, I, G) + \tilde{P}_{n-2}^{k-2}(\mu, 0, G) \\ + \delta_{k\,\text{odd}}(\tfrac{1}{2}\tilde{P}_{n-2}^{k-2}(\mu, I, G) + \tilde{P}_{n-2}^{k-1}(\mu, 0, G))]\} d\mu .\end{aligned} \tag{11b}$$

Note that the lower limit of integration is $m_\eta + (n-r)m_\pi$, where $r = 4$, 5, or 6, since only ρ, ω, and η are allowed at the end of the chain.

To complete the recursion relation we need initial conditions. These are given by the diagrams of Fig. 3. As an example of the expressions obtained from these diagrams we take $n = 3$ in which case we get

$$P_3^0(M, I, G) = 0.38\delta_{I1}\delta_{G,-}\int A(M, \mu, m_\pi)\frac{\rho_f(\mu)}{\rho_i(M)} d\mu . \tag{12}$$

Now

$$\rho_f(\mu) = \delta(\mu - m_\eta)$$

and so

$$\tilde{P}_3^0(M, I, G) = 0.38\delta_{I1}\delta_{G,-}A(M, m_\eta, m_\pi) . \tag{13}$$

The other conditions can be derived in a similar way. The density of states, $\rho(M)$, has completely disappeared both from the recursion relation and the initial conditions if one works with $\tilde{P}_n^k$. We can calculate, as we shall see later, the density of states for each version of the bootstrap model which we shall use, from knowledge of the $\tilde{P}_n^k$.

Using the method outlined above, the $\tilde{P}_n^k(M, I, G)$ have been evaluated by computer. Results will be presented in Sec. IV.

D. Normalization, cross sections, and multiplicities

The probability $P_n(M, I, G)$ for a neutral fireball to decay into n pions is given by

$$P_n(M, I, G) = \sum_{k \text{ even}} P_n^k(M, I, G) . \quad (14)$$

The normalization condition is clearly

$$\sum_n P_n(M, I, G) = 1 . \quad (15)$$

It follows directly that

$$\sum_{\substack{n,k \\ k \text{ even}}} \tilde{P}_n^k(M, I, G) = \rho_I(M) . \quad (16)$$

Therefore, by evaluating the $\tilde{P}_n^k$ for all n and k we can determine the density of states $\rho_I(M)$.

For the reaction $e^+e^- \rightarrow$ hadrons we assume dominance of the one-photon mechanism for which $C=-1$. Therefore, $I^G = 0^-$ or 1^+. Let p_0 and p_1 be the respective probabilities for these two isospins. These probabilities may be energy-dependent. Let $\sigma_h(s)$ be the total hadronic cross section for e^+e^-. Then for k even

$$\sigma_n^k(s) = \sigma_h(s) \left[p_0 \frac{\tilde{P}_n^k(\sqrt{s}, 0, -1)}{\rho_0(\sqrt{s})} + p_1 \frac{\tilde{P}_n^k(\sqrt{s}, 1, 1)}{\rho_1(\sqrt{s})} \right], \quad (17)$$

where ρ_0 and ρ_1 are evaluated using Eq. (16). We find little sensitivity to the values of p_0 and p_1 in our calculations. We assume that when the photon becomes a fireball it seeks out all possible states with equal *a priori* probability so that

$$p_l = \frac{\rho_l}{\rho_0 + \rho_1}, \quad l = 0, 1 . \quad (18)$$

Equation (17) then simplifies to

$$\sigma_n^k(s) = \sigma_h(s) \frac{\tilde{P}_n^k(\sqrt{s}, 0, -1) + \tilde{P}_n^k(\sqrt{s}, 1, 1)}{\sum_{\substack{l,m \\ m \text{ even}}} [\tilde{P}_l^m(\sqrt{s}, 0, -1) + \tilde{P}_l^m(\sqrt{s}, 1, 1)]} . \quad (19)$$

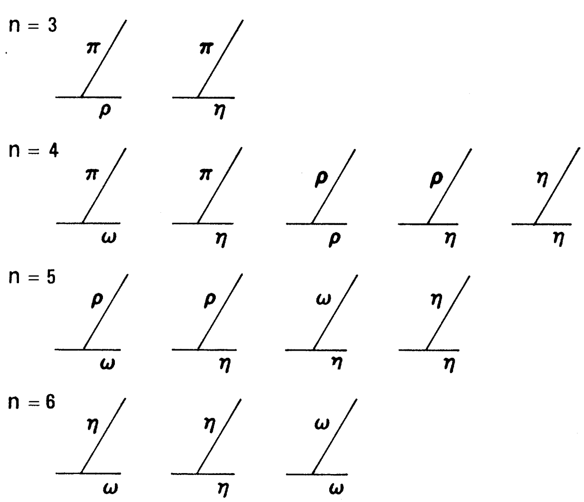

FIG. 3. Schematic representation of the initial conditions of which Eq. (13) is an example. Here n is the number of pions emitted by the fireball.

The prong cross section is given by

$$\sigma_{\text{pr}}^k(s) = \sum_{n \geq k} \sigma_n^k(s) . \quad (20)$$

The cross section $\sigma(n)$ for $e^+e^- \rightarrow n$ pions is

$$\sigma(n) = \sum_{k \text{ even}} \sigma_n^k(s) . \quad (21)$$

We can now evaluate the average total and charged multiplicities. They are

$$\langle n \rangle = \frac{1}{\sigma_h(s)} \sum_n n\sigma(n) ,$$
$$\langle n_{\text{ch}} \rangle = \frac{1}{\sigma_h(s)} \sum_k k\sigma_{\text{pr}}^k(s) . \quad (22)$$

In $\bar{p}p$ annihilation, the $\bar{p}p$ system has equal $I=0$ and $I=1$ components. The G-parity situation is not as straightforward. This does not concern us much due to the lack of sensitivity of most of our results to the isospin and G parity.

E. The emission of ϕ, K, and K^* mesons

The calculations above can be extended to allow for ϕ emission at every stage. We have done this to see whether it is sufficient to include only ρ, ω, and η mesons or whether the ϕ and perhaps heavier resonances should be included. We shall see in Sec. IV that the φ meson has only a small effect on the results of the calculation. The $\phi(1019)$ has $I^G = 0^-$ and $J^P = 1^-$. Using its decay modes which are known (dominantly $K\bar{K}$) one simply adds the necessary terms to the recursion relation (11) and to the initial conditions. The e^+e^- data that we will compare with include K mesons as well as pions. We will in this case treat the $\phi \rightarrow 2K$ decay mode as a $\phi \rightarrow 2\pi$ mode.

We also allow for direct emission of kaons. Since there are four kaons with masses of the same order as that of the η they should, statistically, contribute a significant amount. [We do not consider SU(3) breaking which might cause the K coupling to be smaller than that of other particles.] Secondly, the fact that the kaons have strangeness leads to two charged and two neutral kaons as opposed to two charged and one neutral pion. Therefore, including kaons and counting them as pions could increase the ratio $\langle n \rangle / \langle n_{\text{ch}} \rangle$ and possibly help explain the large amount of neutral energy.[5] We give up G-parity considerations when we include the kaons keeping only isospin and strangeness conservation. We now must allow for strange fireballs, of course. We allow only $I = 0, \frac{1}{2}, 1$ fireballs in keeping with the assumption of no exotic fireballs. Therefore, instead of calculating quantities $P_n^k(M, I, G)$ we consider $P_n^k(M, I)$. When $I = \frac{1}{2}$, if k is even, the fireball has the quantum numbers of K^0 or $\bar{K}^0$ and if k is odd, those of K^+ or K^-. One can derive a recursion relation similar to (11) with similar initial conditions. Since the mass of the $K^*(892)$ is not much larger than that of the ρ, we include it for completeness and we find little difference in the quantities calculated.

F. Invariant and noninvariant phase space

Although the formulas we have used are fully relativistic, the phase space which we have used is noninvariant phase space,

$$\prod_i \frac{V}{h^3} d^3p_i \ .$$

This is the correct phase space to use for noninteracting relativistic particles in a box.[6] However, the branching ratios need not be given by a simple phase-space ratio. In general there could be a matrix element involved. At this point there is some ambiguity. One could, for example, use invariant phase space

$$\prod_i d^3p_i/2E_i \ .$$

However, there is no natural way to normalize this phase space to be dimensionally correct. However, we have looked at the effect of using invariant phase space normalized by the condition that in the limit of nonrelativistic particles in the box, the invariant phase space should reduce to the noninvariant phase space. This leads to the phase space

$$\prod_i \frac{V}{h^3} \frac{m_i}{E_i} d^3p_i \ .$$

Equation (8) is now replaced by

$$A'(M,m,\mu)=\frac{V}{4\pi^2}\frac{m\mu}{M^2}\lambda^{1/2}(M^2,m^2,\mu^2). \qquad (23)$$

This leads to an appreciable change in our results which we shall discuss in Sec. IV.

We have also considered using the matrix element $KM/E_1E_2 = v_1 + v_2$ for the two-body decay of a fireball of mass M. Here E_1, v_1 and E_2, v_2 are the energies and velocities of the two products and K is their center-of-mass (c.m.) momentum. The motivation for this matrix element is discussed in Appendix B. Instead of Eq. (23) we now have

$$A''(M,m,\mu)=\frac{V}{8\pi^2}\frac{\lambda(M^2,m^2,\mu^2)}{M^2} \ . \qquad (24)$$

We shall discuss the results of this calculation as well in Sec. IV.

III. THE NONLINEAR BOOTSTRAP

We discuss in this section the effect of allowing for the possibility that the fireball contains three as well as two hadrons.

A. The three-body phase space

Although we should evaluate the vertices corresponding to $F(M) \to R_1 + R_2 + F^1(m)$ where R_1, R_2 are π, ρ, ω, η, K, K^*, etc., we restrict ourselves for simplicity to the case $R_1 = R_2 = \pi$. This, at any rate, will be an important contribution to the total three-particle effect. Let $\rho^{(3)}(M)$ be the contribution of $\pi\pi F^1$ to the density of states. Then

$$\rho^{(3)}(M)=\frac{9}{2!}\left[\frac{V}{(2\pi)^3}\right]^2\int d^3p_1d^3p_2d^3p_3dm_1dm_2dm_3\,\delta(m_1-m_\pi)\delta(m_2-m_\pi)\rho(m_3)\delta^{(3)}\Big(\sum_{i=1}^{3}\vec{p}_i\Big)\,\delta\Big(M-\sum_{i=1}^{3}E_i\Big). \qquad (25)$$

The factor 9 is the isospin factor for the two pions which will be dropped when specific isospin states are considered and the 1/2! factor results since the two pions are identical. Equation (25) can be rewritten as

$$\rho^{(3)}(M)=\frac{9}{2!}\frac{V^2}{(2\pi)^6}\int \mathcal{V}'(M,m_\pi,m_3)\rho(m_3)\,dm_3 \ , \qquad (26a)$$

where

$$\mathcal{V}'(M,m_\pi,m_3)=\int d^3p_1d^3p_2d^3p_3\,\delta^{(3)}\Big(\sum_i\vec{p}_i\Big)\,\delta\Big(M-\sum_i E_i\Big). \qquad (26b)$$

$\mathcal{V}'$ is just the usual three-body phase space in the c.m. system and has been evaluated by Milburn.[7] We take his result, integrate by parts, and get after doing some algebra

$$\mathcal{V}'(M,m_\pi,m_3)=\frac{\Omega_0}{2\mu^2}\int_1^K d\xi\left[\frac{(K-\xi)(\xi-1)}{\xi(\delta-\xi)}\right]^{1/2}\left[P'(\xi)(1-\xi)+P(\xi)\left(1+\frac{1}{2\xi}+\frac{1}{2}\frac{\xi-1}{\delta-\xi}\right)\right], \qquad (27a)$$

where

$$K=\frac{(1-r)^2}{4\mu^2},\quad \delta=\frac{(1+r)^2}{4\mu^2},\quad \mu=\frac{m_\pi}{M},\quad r=\frac{m_3}{M},\quad \Omega_0=\tfrac{4}{3}\pi^2 M^5\mu^4\,, \tag{27b}$$

$$P(\xi)=(1-r^2+r^4-2r^6+r^8)-4\mu^2\xi(1+2r^2-2r^4+4r^6) + 16\mu^4\xi^2(1+2r^2+6r^4)-64\mu^6\xi^3(2+4r^2)+256\mu^8\xi^4\,, \tag{27c}$$

and

$$P'(\xi)=\frac{dP(\xi)}{d\xi}$$

$\mathcal{V}'$ can now be evaluated numerically.

B. The recursion relation

Defining the quantity

$$B(M,\mu)=\frac{1}{2}\frac{V^2}{(2\pi)^6}\mathcal{V}'(M,m_\pi,\mu) \tag{28}$$

and considering all isospin states of the π-π system and using the relevant Clebsch-Gordan coefficients we get

$$\begin{aligned}\tilde{P}_n^k(M,I,G)=&[\text{the terms from Eq. (11)}]\\ &+\int_{M_\eta+(n-5)m_\pi}^{M-2m_\pi} d\mu B(M,\mu)\{\delta_{I0}[\tfrac{1}{3}\tilde{P}_{n-2}^{k}(\mu,0,G)+\tfrac{2}{3}\tilde{P}_{n-2}^{k-2}(\mu,0,G)+\tfrac{2}{3}\tilde{P}_{n-2}^{k-1}(\mu,1,G)+\tfrac{1}{3}\tilde{P}_{n-2}^{k-2}(\mu,1,G)]\\ &+\delta_{k,\text{even}}\delta_{I,1}[\tilde{P}_{n-2}^{k-2}(\mu,0,G)+\tfrac{3}{5}\tilde{P}_{n-2}^{k}(\mu,1,G)+\tfrac{8}{5}\tilde{P}_{n-2}^{k-1}(\mu,1,G)+\tfrac{4}{5}\tilde{P}_{n-2}^{k-2}(\mu,1,G)]\\ &+\delta_{k,\text{odd}}\delta_{I,1}[\tilde{P}_{n-2}^{k-1}(\mu,0,G)+\tfrac{2}{5}\tilde{P}_{n-2}^{k}(\mu,1,G)+\tfrac{4}{5}\tilde{P}_{n-2}^{k-1}(\mu,1,G)+\tfrac{9}{5}\tilde{P}_{n-2}^{k-2}(\mu,1,G)]\}\end{aligned} \tag{29a}$$

and

$$\tilde{P}_n^k(I=0)=0 \text{ if } k \text{ is odd}\,. \tag{29b}$$

The initial conditions are handled with the same method as in the preceding section.

IV. THE EFFECT OF RESONANCES

The predictions of the version of the linear bootstrap model in which only pions are emitted throughout the decay and with emission of a resonance at the end of the decay have been discussed in a previous paper.[1] The multiplicities rise linearly with $\sqrt{s}$ as expected in the SBM due to the essentially constant maximum temperature which is effectively achieved at these energies and the resulting constant average pion energy. The branching ratios into four and six charged pions fall exponentially with energy and the ratio of total multiplicity to charged multiplicity is approximately 3/2.

In Figs. 4–9 we show the predictions of the SBM with resonances and the three-particle vertex included and we show how these predictions vary with the version of the model chosen. The lighter resonances contribute significantly to all quantities calculated, whereas the ϕ contribution is small. Heavier discrete resonances will contribute even less. Unless otherwise indicated, we have taken, for the volume, a sphere of radius 1.1 F and we have used the mixture of $I=0$ and $I=1$ as in Eq. (18).

We show the results of the calculations of the density of states when all resonances are included in Fig. 4. For $m>2$ GeV, $\rho(m)\sim cm^{-3}e^{bm}$ with $b^{-1}\simeq 163$ MeV, as in Eq. (2). In the cases when fewer resonances are included, the form, (2), is still found with the values of c and b depending on the case considered. In Table I the values of $T_{\max}=b^{-1}$ are summarized for the cases considered.

The total and charged multiplicities for various approximations to the SBM are shown in Figs. 5–7. All versions give both charged and total multiplicities which rise linearly with $\sqrt{s}$, and the total multiplicity has a slope which is equal to the inverse mean pion energy $(\langle E_\pi\rangle)$ as $\sqrt{s}\to\infty$. Table I also summarizes the values of $\langle E_\pi\rangle$ as $\sqrt{s}\to\infty$ for all the cases. With some notable exceptions, $\langle E_\pi\rangle$ is essentially a monotonic function of $T_{\max}$ as expected from thermodynamic considerations. The ratio $\langle n\rangle/\langle n_{ch}\rangle$ of the total to charged multiplicity is also given in Table I in the limit of

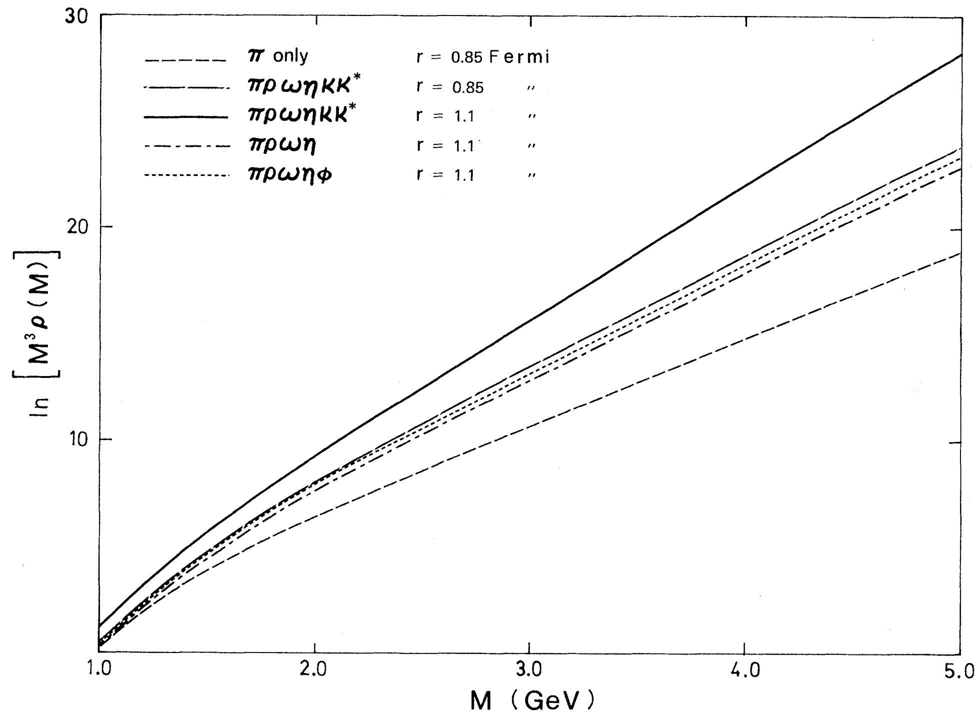

FIG. 4. $\ln[M^3\rho(M)]$ versus M for various versions of the SBM. $\rho(M)$ is the sum of the density of states for $I=0$ and $I=1$, which we have calculated. These results are for a hadronic volume which is a sphere of radius 0.85 or 1.1 F.

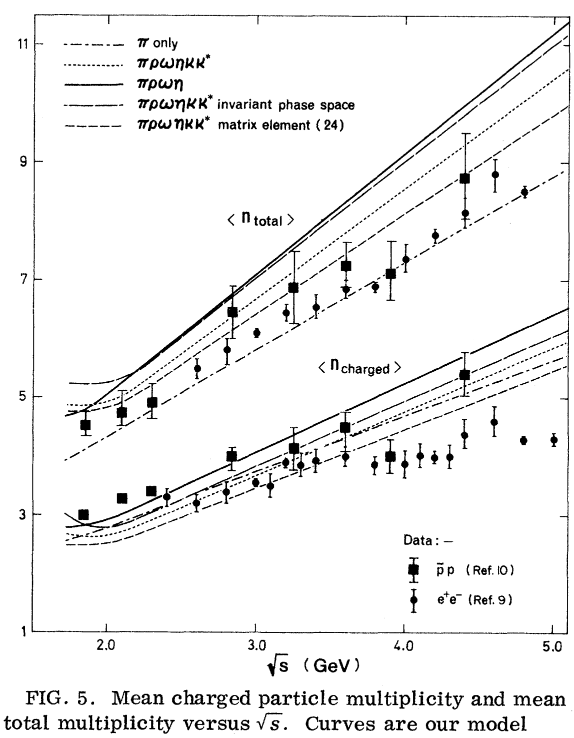

FIG. 5. Mean charged particle multiplicity and mean total multiplicity versus $\sqrt{s}$. Curves are our model predictions. The mean total multiplicity is evaluated by dividing the mean charged multiplicity by the ratio of charged energy to total energy. Unless otherwise indicated the volume used is that of a sphere of radius 1.1 F. Data are from Refs. 9 and 10.

$\sqrt{s}\to\infty$. It is proved in Appendix A that the decay of an $I=0$ system into any number of pions, conserving isospin, is symmetric in isospin space so that

$$\langle n_+\rangle = \langle n_-\rangle = \langle n_0\rangle , \quad (30)$$

where $\langle n_{+,\,-,\,0}\rangle$ is the average number of $\pi_{+,\,-,\,0}$ in the decay. The same theorem is true if the initial system is $I=1$ with equal probability of having $I_3=0,\pm 1$. In both e^+e^- and $\bar{p}p$ the initial fireball is a linear combination of $I=0$ and $I=1$. Therefore, as the length of the decay chain increases we expect that even the $I=1$ fireball will give approximately the same number of π_+, π_-, and π_0 if isospin is conserved throughout. This leads us to expect that

$$\langle n\rangle/\langle n_{\rm ch}\rangle \to \tfrac{3}{2} \text{ as } s\to\infty . \quad (31)$$

Deviations from this ratio are due to η decay and K meson production.

For the linear bootstrap with only pions we see, following Table I, that $\langle n\rangle/\langle n_{\rm ch}\rangle \to \frac{3}{2}$ as $s\to\infty$ as expected from (31). If ρ, ω, and η are included the $\langle n\rangle/\langle n_{\rm ch}\rangle \to 1.67$ as $s\to\infty$. The ratio has increased since the η can now be emitted anywhere along the chain and therefore has an effect even as $\sqrt{s}\to\infty$. We can understand the over-all increase in multiplicity when resonances are included (see Fig. 5) by noting that including ρ, ω, and η in-

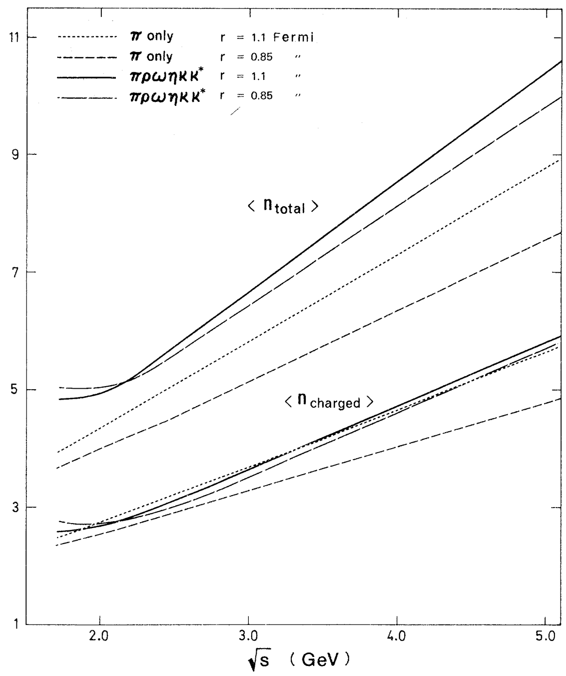

FIG. 6. Caption as for Fig. 5.

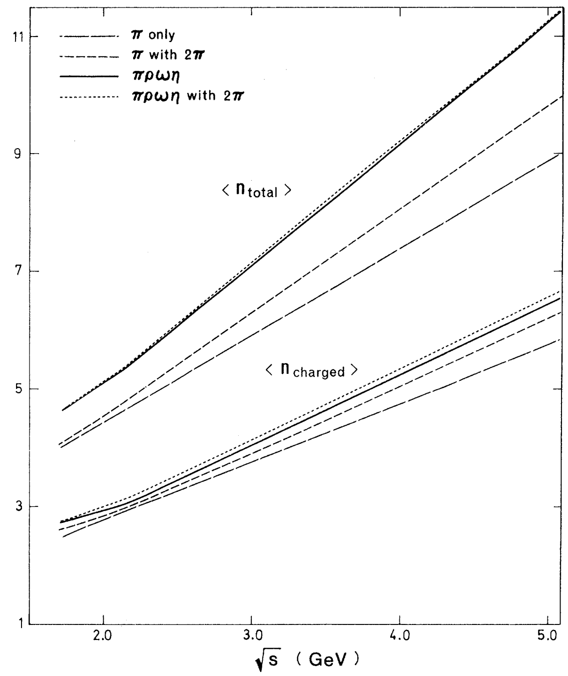

FIG. 7. Caption as for Fig. 5.

creases the density of states. It is therefore not surprising that the explicit calculation shows an increase in the inverse maximum temperature $b = T_0^{-1}$ ($\rho \sim \text{cm}^{-3} e^{bm}$). The resulting decrease in T_0 (from 200 MeV to 170 MeV in our calculations) leads to a decrease in the average pion energy and therefore an increase in the multiplicity.

Including kaons (without ϕ) the temperature lowers to 160 MeV but the multiplicities fall. The lowering of $\langle n \rangle$ and $\langle n_{\text{ch}} \rangle$ is due to the fact that the kaons are much heavier than pions. For a given temperature they will have the same kinetic energy as the pions and therefore a larger total energy which accounts for the reduced multiplicity. The ratio $\langle n \rangle / \langle n_{\text{ch}} \rangle$ increases slightly as expected.

Using invariant phase space, the temperature rises to 180 MeV and the multiplicities rise. The reason for this is that the effective matrix element resulting from the use of invariant phase space for a two-body decay is

$$\frac{m_1 m_2}{E_1 E_2} = \frac{1}{\gamma_1} \frac{1}{\gamma_2}, \tag{32a}$$

with

$$\gamma_i = \frac{1}{(1 - v_i^2)^{1/2}} . \tag{32b}$$

This matrix element is larger when the velocities of the decay products are smaller and thus the average pion energy will decrease which results

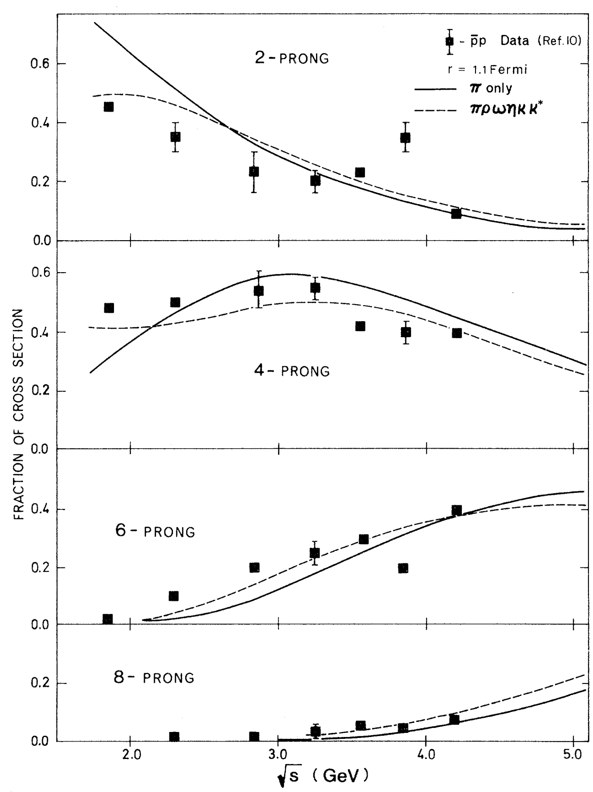

FIG. 8. Fractional prong cross-section predictions for e^+e^- and $\bar{p}p$ annihilation versus $\sqrt{s}$. The squares represent $\bar{p}p$ data (see Ref. 10). The hadronic volume is a sphere of radius 1.1 F.

in an increase of the multiplicity.

We plot in Fig. 6 $\langle n\rangle$ and $\langle n_{ch}\rangle$ for the cases of pions only and all resonances included, for two values of the volume V. There is a significant though not a huge difference between values for spheres of radii 0.85 and 1.1 F. At all but the lowest energies the effect of using an $I=0$ versus an $I=1$ initial fireball is negligible as far as multiplicities are concerned. We show in Fig. 7 the effect of the $\pi\pi F$ vertex on the multiplicities

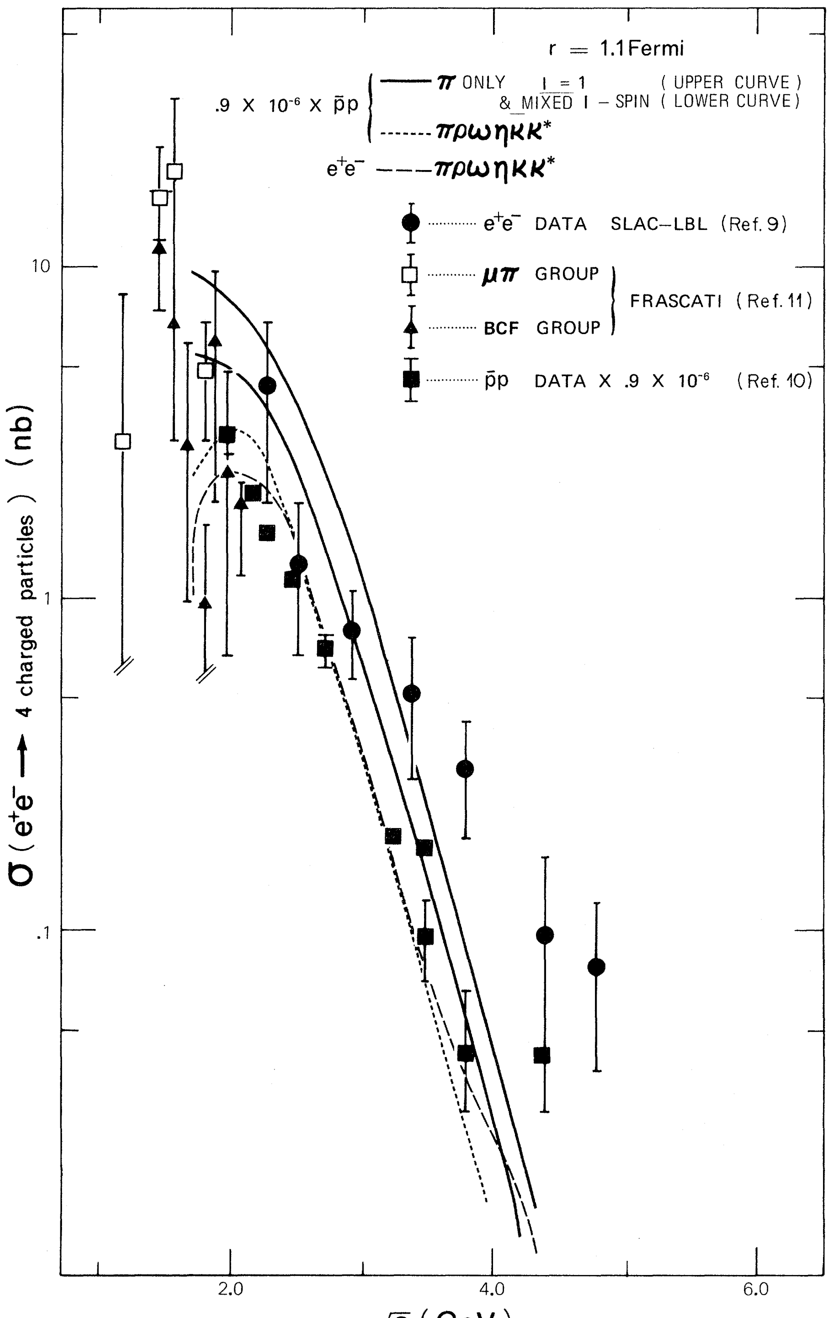

FIG. 9. e^+e^- and $\bar{p}p$ annihilation cross section into four charged particles (pions or kaons). The curves represent the predictions of the SBM with only pions included and with all resonances included. For the e^+e^- hadronic cross section we have used SLAC data (Ref. 9). For the $\bar{p}p$ annihilation cross section we have used an eyeball fit to the data (see Ref. 10).

TABLE I. List of the maximum temperature $T_{max} = b^{-1}$, averaged pion energy $\langle E_\pi \rangle = \sqrt{s}/\langle n \rangle$ as $\sqrt{s} \to \infty$, and the ratio of total to charged multiplicity as $\sqrt{s} \to \infty$, which we have calculated for various cases.

Particles included	Radius of hadron (F)	Maximum temperature (MeV)	Average pion energy $\langle E_\pi \rangle$ (MeV) as $\sqrt{s} \to \infty$	$\langle n \rangle / \langle n_{charged} \rangle$ as $\sqrt{s} \to \infty$
π	0.85	266	890	1.52
	1.1	200	674	1.51
π with 2π	0.85	220	708	1.52
	1.1	173	564	1.50
π, η	0.85	222	615	2.0
	1.1	189	560	2.04
π, η with 2π	0.85	212	610	1.85
	1.1	169	540	1.67
π, ρ, ω, η	0.85	199	535	1.70
	1.1	170	515	1.67
π, ρ, ω, η with 2π	1.1	160	510	1.6
$\pi, \rho, \omega, \eta, \phi$	0.85	197	540	1.70
$\pi, \rho, \omega, \eta, K, K^*$	0.85	186	574	1.75
	1.1	160	545	1.74
$\pi, \rho, \omega, \eta, K, K^*$ with invariant phase space	1.1	180	535	1.76
$\pi, \rho, \omega, \eta, K, K^*$ with matrix element (24)	1.1	165	577	1.73

for the cases of π only and π, ρ, ω, η. It is interesting to note that including the $\pi\pi F$ vertex lowers the temperature in every case by 10—20 MeV. This implies a higher multiplicity and indicates that if the SBM is to be interpreted strictly, the three-body phase space must be included if one desires detailed accuracy. When all resonances are included the $\pi\pi F$ vertex has considerably less of an effect due to competition from the resonances. However, this might change if R_1R_2F vertices were also included, where R_1 and R_2 would be discrete resonances.

We present in Figs. 8 and 9 the results of our calculations of the prong probabilities and exclusive channel $\sigma(\pi^+\pi^-\pi^+\pi^-)$. The results are compared with the data. The agreement is better when heavy resonances (and kaons) are included.

V. CONCLUSIONS

Although the calculations which we have discussed describe the low-energy data on e^+e^- and $\bar{p}p$ annihilations quite well, all of them fail to explain the flattening with energy of the charged multiplicity in e^+e^- collisions in the 3.5–5 GeV center-of-mass energy range. This may be due mainly to new effects coming into play in the region of the ψ resonances. The narrow resonances by themselves should not be much of a problem as long as they are ignored, since they only appear as spikes. However, they could contribute along with other particles as threshold phenomena at high enough energies. Further, there is a possibility of heavy lepton production which must be separated from the hadronic processes in order to compare with a hadronic model such as the one we use. In spite of this, it is possible that when the threshold phenomena subside one could again reach a region of validity for the statistical picture. However, it is expected that at high energies there are new mechanisms which dominate over incoherent resonance formation and decay discussed in this paper. The simplest such mechanisms would be a possible coherence of the various resonances. This is expected according to duality considerations.[2,4] This could affect e^+e^- and $\bar{p}p$ differently as well as affecting various exclusive channels in different ways. The multiplicities, prong probabilities, and exclusive channel cross sections should change character at high enough energy. At high energies one might also expect any constituent structure of the hadrons (such as partons) to be prominent. We see such a possible effect in high $p_\perp$ single-particle inclusive cross sections in hadronic collisions where the cross section starts falling exponentially as predicted by the SBM[3] and eventually a power-law behavior such as is predicted by parton models,[8] for example, takes over.

All of this leads us to believe that the validity of the picture above is limited to the region of a few GeV only.

APPENDIX A

Consider the decay of an $I=0$ system S into n $I=1$ systems $S_1 \cdots S_n$. Pick any S_k. Since S_k has $I=1$ and it must combine with $\{S_1 \cdots S_{k-1} S_{k+1} \cdots S_n\}$ to give $I=0$, the system $\{S_1 \cdots S_{k-1} S_{k+1} \cdots S_n\}$ must have $I=1$. Now

$$|0,0\rangle = \frac{1}{\sqrt{3}}|1,1\rangle|1,-1\rangle - \frac{1}{\sqrt{3}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle|1,1\rangle .$$

Thus S_k has equal probability of having $I_3=0, \pm 1$. Since this is true for all S_k, the isospin symmetry is established. Now the system $\{S_1 \cdots S_{k-1}, S_{k+1} \cdots S_n\}$ is an $I=1$ system with equal probability of having $I_3=0, \pm 1$, and from the last argument any one of its decay products has equal probability of having $I_3=0, \pm 1$. It is easy to establish from this that any system with $I=1$ and equal probability for $I_3=0, \pm 1$, which decays into $I=1$ systems conserving isospin, has a decay which is symmetric in isospin space.

APPENDIX B

Consider the decay of a fireball of mass M into two products (either fireballs, discrete resonances or pions) of masses m_1 and m_2. Then the partial width for this process is given by

$$\Gamma(M; m_1 m_2) = \hat{P}(M; m_1 m_2)\hat{\Gamma}(M; m_1 m_2) , \quad \text{(B1)}$$

where $\hat{P}(M; m_1 m_2)$ is the probability that M is composed of m_1 plus m_2 and $\Gamma(M; m_1 m_2)$ is the decay width of a system containing two masses $m_1 m_2$ with invariant mass M. For $\hat{P}(M; m_1 m_2)$ we take the phase-space ratio discussed in this paper. For $\hat{\Gamma}(M; m_1 m_2)$ assume there is no interaction between m_1 and m_2 and that when they have separated a distance r_0 they decay. Then

$$\hat{\Gamma}(M; m_1 m_2) = \frac{v_1 + v_2}{r_0} , \quad \text{(B2)}$$

where v_i is the velocity of m_i. Calling the total width $\Gamma(M)$ the branching ratio for M going to $m_1 + m_2$ is given by

$$B(M; m_1 m_2) = \frac{1}{\Gamma(M)} \Gamma(M; m_1 m_2) = \hat{P}(M; m_1 m_2) \frac{v_1 + v_2}{r_0} \frac{1}{\Gamma(M)} . \quad \text{(B3)}$$

To calculate $\Gamma(M)$ we consider the most likely configuration for a fireball of mass M, i.e., when M is composed of a large mass particle and a small mass particle. Then $(v_1+v_2)/r_0 \simeq 1/r_0$. We thus take $\Gamma(M) = 1/r_0$. We then get

$$B(M; m_1 m_2) = (v_1 + v_2) \times \text{noninvariant phase space.} \quad \text{(B4)}$$

But

$$v_1 + v_2 = \frac{KM}{E_1 E_2} , \quad \text{(B5)}$$

hence the origin of the matrix element discussed in the text.

Note that if the expression for $\Gamma(M)$ is correct and $1/r_0$ is in fact the true average of $(v_1+v_2)/r_0$ over all configurations, then the density of states calculated using the matrix element (B5) should be the same as that coming from noninvariant phase space. In our calculations we find the difference between the two to be very small relative to the differences encountered when including resonances or using invariant phase space.

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