Action principle for spin-$\frac{1}{2}$ wave equations

Henry F. Ahner
Adelphi University, Garden City, New York 11530
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An action principle is presented whose variation yields *both* Dirac's spin-1/2 equation and Staunton's positive-energy spin-1/2 equation. The Lagrangian used is a function not only of the matter, electromagnetic, and gravitational fields, but also of the structure constants of the internal group. Variation with respect to the matter field gives the dynamical equations of motion. Variation with respect to the electromagnetic and gravitational fields yields Maxwell's and Einstein's equations, respectively. These equations include source terms that are the electromagnetic current and the stress-energy tensors for the matter and electromagnetic fields. Variation with respect to the structure constants determines the internal group, thereby projecting out either Dirac's or Staunton's equation. The matter stress-energy tensor herein determined is used to construct the energy operator for the second-quantized free fields. The results obtained in the case of Dirac's equation are the standard ones. In the case of Staunton's equation, the matter stress-energy tensor and energy operator are found for the first time.

I. INTRODUCTION

The recent discovery by Staunton[1] of a spin-$\frac{1}{2}$ positive-energy wave equation raises many questions concerning its interpretation, its physical significance, and its relationship to Dirac's spin-$\frac{1}{2}$ equation. Already resolved are the forms of the electromagnetic[1] and gravitational[2] interactions in Staunton's equation. In addition, the classical limit has been determined[3] and a perturbative solution for scattering from a static Coulomb potential has been found.[1] Questions of interpretation in terms of a quantum-front subdynamics, as well as generalization of the equation for other spins using higher powers of the momentum operator, have been treated by Biedenharn, Han, and van Dam.[4] Many questions concerning the second quantization of Staunton's field have yet to be considered. If a correspondence could be set up between the two known spin-$\frac{1}{2}$ equations, then the well-known methods used to answer these questions for Dirac's equation might be transferred to the new problems involving Staunton's equation.

The major purpose of this paper is to show how one may set up this correspondence and simultaneously obtain both Dirac's and Staunton's equations from a single variational principle. This is achieved by requiring that the variation of the Lagrangian not only yields the dynamical equations of motion for the fields but also determines the theories' internal group structure. After the formalism has been developed, it is used to obtain the stress-energy tensors for the matter fields, and to determine the energy operators for the second-quantized free-field equations. In the Dirac case the expected results are obtained, while new results are found for the Staunton case.

In Sec. II, the Dirac and Staunton equations are listed and briefly discussed. Section III contains the action principle and all the relations obtained upon its variation. These include some group structure constants, the Dirac and Staunton equations, the conserved electromagnetic currents, and the stress-energy tensors. The free-field equations are second-quantized to obtain an energy operator in Sec. IV. Finally, in Sec. V, a discussion of some areas of future research as well as a summary of the paper is presented.

II. SPIN-$\frac{1}{2}$ WAVE EQUATIONS

Consider a geometrical object ψ that transforms like a scalar under the space-time manifold mapping group. In addition, let ψ form the basis of a representation of the SO(3,2) group. The generators of this internal Lie group, V_μ and $S_{\mu\nu}$, satisfy the algebra

$$\begin{aligned}
[V_\mu, V_\nu] &= C(1,\mu,\nu,\rho)V^\rho + C(2,\mu,\nu,\rho\sigma)S^{\rho\sigma}\,,\\
[V_\mu, S_{\rho\sigma}] &= C(3,\mu,\rho\sigma,\nu)V^\nu + C(4,\mu,\rho\sigma,\tau\kappa)S^{\tau\kappa}\,,\\
[S_{\mu\nu}, S_{\rho\sigma}] &= C(5,\mu\nu,\rho\sigma,\tau)V^\tau + C(6,\mu\nu,\rho\sigma,\tau\kappa)S^{\tau\kappa}\,.
\end{aligned} \tag{1}$$

The structure constants, C (indices deleted), are labeled by a number from 1 to 6 to avoid confusion (even though their indicial structure permits identification). They are

$$\begin{aligned}
&C(1) = C(4) = C(5) = 0\,,\\
&C(2,\mu,\nu,\rho\sigma) = \tfrac{1}{2}i(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,,\\
&C(3,\mu,\rho\sigma,\nu) = i(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\rho}g_{\nu\sigma})\,,\\
&C(6,\mu\nu,\rho\sigma,\tau\kappa) = i(g_{\mu\rho}g_{\nu\tau}g_{\sigma\kappa} - g_{\mu\sigma}g_{\nu\tau}g_{\rho\kappa}\\
&\qquad + g_{\nu\sigma}g_{\tau\mu}g_{\rho\kappa} - g_{\nu\rho}g_{\mu\tau}g_{\sigma\kappa})\,.
\end{aligned} \tag{2}$$

The $S_{\mu\nu}$ satisfy the algebra of the Lorentz group when $g_{\mu\nu} = \eta_{\mu\nu}$. Therefore, in special relativity, the internal group and the group of space-time mappings may be related and spinors can be introduced onto the space-time manifold. However, in the case of general relativity, the internal group and the manifold mapping group are no longer related.[5]

It can be shown[2] that the generalized covariant derivative D_μ is given by

$$D_\mu \psi = \psi_{;\mu} - i\lambda_\mu{}^{ab} S_{ab}\psi \,, \tag{3}$$

where[6]

$$\lambda_\mu{}^{ab} = \tfrac{1}{4}(h^{\nu b} h_\nu{}^a{}_{;\mu} - h^{\nu a} h_\nu{}^b{}_{;\mu}) \,. \tag{4}$$

Accordingly, we may take

$$\Pi_\mu \equiv iD_\mu + eA_\mu \tag{5}$$

as a generalized momentum operator, where A_μ is the electromagnetic potential vector and e the particle's charge. Note that the covariant derivative is defined and determined solely by using the internal Lorentz group generators S_{ab}. Any V_μ selected that, together with $S_{\mu\nu}$, satisfies Eq. (1) will fit into this pattern regardless of the values of $C(1)$ to $C(4)$.

Defining the Lorentz Casimir operators F and G by

$$F \equiv \tfrac{1}{4} S_{\mu\nu} S^{\mu\nu} \,,$$

$$G \equiv \tfrac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} \,,$$

we may write in a formal manner

$$[V_\mu, F] \equiv C(7, \mu, \nu) V^\nu + C(8, \mu, \rho\sigma) S^{\rho\sigma} \,. \tag{6}$$

It has been shown by Staunton and Browne[3] that F can take on only three values if the quantities $C(7)$ and $C(8)$ defined by Eq. (6) are zero. These values are $F=0$ (trivial representation), $F=\frac{3}{4}$ (Dirac representation), and $F=-\frac{3}{8}$ (Majorana representation). In these cases, with $\alpha = -\frac{8}{3}F$ for convenience below, Staunton and Browne[3] showed that

$$\tfrac{4}{3}F = D \equiv V_\mu V^\mu \,,$$

$$-S_{\mu\nu} V^\mu = V^\mu S_{\mu\nu} = i\lambda V_\nu \,,$$

with $\lambda \equiv -\frac{3}{2}$. In addition, they found the following:

(a) For the Dirac representation: $F=\frac{3}{4}$, $D=1$, $\alpha=-2$, $G^2 = c$ number, $G=-\frac{3}{4}i\gamma_5$, with $V_\mu = \frac{1}{2}\gamma_\mu$. Here γ_μ are the Dirac matrices satisfying the Clifford algebra condition $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$.

(b) For the Majorana representation: $F=-\frac{3}{8}$, $D=-\frac{1}{2}$, $\alpha=1$, $G=0$. V_μ and $S_{\mu\nu}$ are realized not as finite matrices but as differential operators[1] acting on two internal coordinates q_1 and q_2.

(c) For the trivial representation: $F=D=\alpha=G=0$, $V_\mu=0$, $S_{\mu\nu}=0$.

Now consider the equations

$$T_\mu \psi = 0 \,. \tag{7}$$

Here T_μ is given by

$$T_\mu \equiv -\Pi_\mu + \alpha(iS_{\mu\nu}\Pi^\nu + mV_\mu) \,. \tag{8}$$

Inspection of Eq. (7) shows that it reduces to Staunton's equation[1] when the Majorana representation is used, while it reduces to Dirac's equation[7] in the case of the Dirac representation.

For the trivial representation we find that Eq. (7) reduces to $\Pi_\mu\psi = 0$. But *in this case* $[\Pi_\mu, \Pi_\nu] = ieF_{\mu\nu}$, where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Note that even in curved space-time, for the trivial representation, no terms in the Riemann tensor appear in this commutator since $S_{\mu\nu}=0$. Therefore, Eq. (7) implies $[\Pi_\mu, \Pi_\nu]\psi = 0$, or $ieF_{\mu\nu}\psi = 0$. Thus the ψ field, in the trivial case, either is chargeless or it vanishes whenever an electromagnetic field is present. Choosing $e=0$, we find that Eq. (7) reduces to

$$\psi_{,\mu} = 0 \quad \text{(trivial representation)} \,. \tag{9}$$

The solution of Eq. (9) is ψ is a constant.

We have shown that no physical influence may be propagated via the ψ field in the trivial representation. Furthermore, the condition that $\int \psi^\dagger \psi \, d^3x$ be finite in Minkowski space-time requires that ψ be zero. We shall ignore the trivial representation in the remainder of the paper.

III. ACTION PRINCIPLE

We desire a Lagrangian action principle whose variation will yield the vector equations (7) for the space-time scalar field ψ. This is achieved by allowing both ψ *and* $V_\mu\psi$ to be independently varied. This is possible only if the representation, of which ψ and V_μ constitute a realization, is not specified before the variation. The group structure must be determined concurrently with the dynamical equations. If this viewpoint is taken, the action of V_μ on ψ cannot be initially specified in terms of ψ since the internal coordinate dependence of neither the operator nor the function is initially specified. After the variation two nontrivial representations are found to be allowed. One is in terms of finite (4×4) matrices. The other is an infinite-dimensional representation that may be labeled by two internal coordinates, q_1 and q_2.

Before presenting the action principle, we will consider in greater detail the points outlined in the preceding paragraph. For this purpose it will be useful to clearly distinguish between the situation in special relativity (SR) and that in general relativity (GR).[8] In the present case (GR) there is no relationship between the parameters of the internal transformation group and the descriptors of the space-time manifold-mapping group. Only in the flat-space limit (SR) can a connection between them be made. This is done in terms of the symmetry group of the theory using the condition that $\bar{\delta} V_\mu$ be zero.[8]

In SR the specification that V_μ be a space-time vector restricts $C(3)$ and $C(4)$ in Eq. (1) to be the values listed in Eq. (2). In GR this is not the case since no statement concerning the internal properties of the operator V_μ is contained in the specification "space-time vector." This means that an SR variational principle cannot contain $C(3)$ and $C(4)$ as independent variants while a GR variational principle may contain them. In either case the structure constants $C(5)$ and $C(6)$ may not be incorporated in the Lagrangian as independent variants. The commutation relation they specify must be known so that covariant derivatives[2] may be constructed.

The answer to the question of whether or not ψ and $(V_\mu\psi)$ may be considered as independent variants hinges on the independence of $C(3)$ and $C(4)$. In SR the Lorentz transformation group applies and we may specify the matrix elements of a vector operator in terms of its irreducible representation content.[9] Thus in SR only the components of V_μ in the direct integral over the principal series of the Lorentz group are left unspecified. One may write

$$V_\mu = \sum_{\oplus j_0} \int d\nu A(j_0, \nu)\Gamma_\mu(j_0, \nu),$$

where the Γ_μ are *vectors*[9] within a single irreducible representation and are completely determined. The weight factors $A(j_0, \nu)$ for each irreducible representation are arbitrary. Whether or not this constitutes an additional variational degree of freedom over and above that contained in ψ and its conjugate is not obvious. In GR the situation is different. Since $C(3)$ and $C(4)$ are not initially known, V_μ is not necessarily even a vector with respect to the internal transformation group. Accordingly, we shall consider both ψ and $(V_\mu\psi)$ as independent variants.

In the Lagrangian that we shall write, ψ is a space-time scalar field. Its particular representation structure, with respect to the internal group, will only be determined after the variation has been performed. Accordingly we may designate a conjugate to $\psi, \psi^\dagger$, by requiring that it be chosen so that $\psi^\dagger\psi$ transforms like a space-time scalar only. (It is understood that whenever $\psi^\dagger$ and ψ appear in the same expression, an inner product over the full range of the internal degrees of freedom is to be taken.) Questions concerning the dimensionality (finite or infinite) or the unitarity of the representation for the internal group before variation are all unanswerable. In this case it is appropriate that the internal group's integration manifold be left as an unspecified inner product since a full knowledge of the group will not be available until after the variation.

Consider an action A given by

$$A = \int (-g)^{1/2}\mathcal{L}\, d^4x, \tag{10}$$

where

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_C + \mathcal{L}_E + \mathcal{L}_G, \tag{11}$$

and

$$\begin{aligned}\mathcal{L}_\psi \equiv\, & (\Pi_\mu\psi)^\dagger\beta V^\mu\psi + (V_\mu\psi)^\dagger\beta\Pi^\mu\psi \\ & - (iS_{\mu\nu}\Pi^\nu\psi)^\dagger\alpha\beta V^\mu\psi \\ & - (V^\mu\psi)^\dagger\alpha\beta iS_{\mu\nu}\Pi^\nu\psi + \tfrac{1}{2}m\psi^\dagger\alpha^2\beta\psi \\ & - m(V^\mu\psi)^\dagger\alpha\beta V_\mu\psi,\end{aligned} \tag{12}$$

where

$$\beta \equiv \tfrac{1}{2}\alpha(1+\alpha\lambda)^{-1}.$$

Note that $S_{\mu\nu}$ and Π_μ commute with the operators α and β. Also note that

$$(iS_{\mu\nu}\psi)^\dagger\psi = -\psi^\dagger iS_{\mu\nu}\psi,$$

$$\begin{aligned}\mathcal{L}_C \equiv\, & [C(2,\mu,\nu,\rho\sigma) - \tfrac{1}{2}i(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})]^2 \\ & + [C(3,\mu,\rho\sigma,\nu) - i(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\rho}g_{\nu\sigma})]^2 + [C(1)]^2 \\ & + [C(4)]^2 + [C(7)]^2 + [C(8)]^2,\end{aligned} \tag{13}$$

$$\mathcal{L}_E \equiv -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{14}$$

$$\mathcal{L}_G \equiv -(16\pi G)^{-1}R. \tag{15}$$

The squared quantities in Eq. (13) are shorthand for the scalars formed by contracting like indices (i.e., $[C(1)]^2 \equiv C(1,\mu,\rho,\tau)C(1,\nu,\sigma,\kappa)g^{\mu\nu}g^{\rho\sigma}g^{\tau\kappa}$).

Variations of A must be taken independently with respect to ψ, $\psi^\dagger$, $(V^\mu\psi)$, $(V^\mu\psi)^\dagger$, A_μ, the *vierbein* field $h_\mu{}^a$, and each of the C's present in Eq. (13).

Variation of A with respect to the C's yields Eqs. (2) and (6). Of course Eqs. (2) and (6), and any relations derived from them, may not be used in A until after the other variations have been performed. After the variation, all the results of the Staunton-Browne theorem[3] may be utilized.

Variation of A with respect to $(V_\mu\psi)^\dagger$ (or its conjugate) yields Eq. (7) (or its conjugate). Variation of A with respect to $\psi^\dagger$ (or its conjugate) yields the

Majorana equation

$$V^\mu \Pi_\mu \psi - \alpha D(1+\alpha\lambda)^{-1} m\psi = 0 \tag{16}$$

(or its conjugate). Equation (16) is obtainable from Eq. (7) by multiplication by V^μ. It represents no new information.

Variation of A with respect to A_μ yields Maxwell's equations with a source term j^μ. We obtain

$$F^{\mu\nu}{}_{;\nu} = \alpha e(\psi^\dagger V^\mu \psi) \equiv j^\mu \,. \tag{17}$$

In the case of the Dirac representation j^μ is $-e\psi^\dagger\gamma^\mu\psi$, while in the case of the Majorana representation $j^\mu = e(\psi^\dagger V^\mu \psi)$.

The final variation of A is taken with respect to the *vierbein* field $h_\mu{}^a$. It yields Einstein's equations with both matter and electromagnetic stress-energy source terms. This variation is algebraically straightforward, but tedious. An outline of the procedure is presented in the Appendix. The new result, expressed in terms of a matter stress-energy tensor $T_{\alpha\lambda}$, is

$$\begin{aligned} T_{\alpha\lambda} &\equiv (-g)^{-1/2} h_{\alpha a} \frac{\delta(-g)^{1/2}\mathcal{L}_\psi}{\delta h^\lambda{}_a} \\ &= -D((1+\alpha^{-1})\{\psi^\dagger(V_\alpha\Pi_\lambda + V_\lambda\Pi_\alpha)\psi \\ &\qquad + [(V_\alpha\Pi_\lambda + V_\lambda\Pi_\alpha)\psi]^\dagger\psi\} \\ &\qquad - 2m\psi^\dagger[V_\alpha V_\lambda + V_\lambda V_\alpha - \tfrac{1}{2}\alpha^2(1+\alpha\lambda)^{-1} g_{\alpha\lambda}]\psi) \,. \end{aligned} \tag{18}$$

The terms in Eq. (18) proportional to m are already known[2] to satisfy the requirement of having zero divergence. However, for the Dirac representation we note that these terms are only a restatement of the Clifford algebra condition and, accordingly, are zero. For the Majorana representation these terms are not zero. Their presence is necessary to correctly fix the magnitude of the stress-energy tensor. By using the Majorana equation and the Klein-Gordon equation

$$\{\Pi^2 - m^2 - \tfrac{1}{2}i\alpha S_{\mu\nu}[\Pi^\mu, \Pi^\nu]\}\psi = 0 \,, \tag{19}$$

which is obtained[1,2] from forming $[T_\mu, T_\nu]\psi = 0$, we may easily verify that $T_{\alpha\lambda}$ has zero divergence.

In the representations of specific interest, Eq. (18) reduces to

$$T_{\alpha\lambda}(\text{Majorana representation}) = \{\psi^\dagger(V_\alpha\Pi_\lambda + V_\lambda\Pi_\alpha)\psi + [(V_\alpha\Pi_\lambda + V_\lambda\Pi_\alpha)\psi]^\dagger\psi - m\psi^\dagger(V_\alpha V_\lambda + V_\lambda V_\alpha + g_{\alpha\lambda})\psi\} \,, \tag{20}$$

and

$$T_{\alpha\lambda}(\text{Dirac representation}) = -\tfrac{1}{4}\{\psi^\dagger(\gamma_\alpha\Pi_\lambda + \gamma_\lambda\Pi_\alpha)\psi + [(\gamma_\alpha\Pi_\lambda + \gamma_\lambda\Pi_\alpha)\psi]^\dagger\psi\} \,. \tag{21}$$

The contraction of $T_{\alpha\lambda}$, $T \equiv T_\alpha{}^\alpha$, is in both cases $T = m\psi^\dagger\psi$.

IV. THE ENERGY OPERATOR

In this section we shall restrict our attention to free particles in Minkowski space-time. The energy operator, H, is given by

$$\begin{aligned} H \equiv P^0 &= \int d^3x\, T^{00} \\ &= \int d^3x\, \{(\alpha+1)[\psi^\dagger V_0\Pi_0\psi + (V_0\Pi_0\psi)^\dagger\psi] \\ &\qquad - m\psi^\dagger(2\alpha V_0 V_0 + 1)\psi\} \,. \end{aligned} \tag{22}$$

In Eq. (22) use was made of the fact that $2(1+\alpha\lambda) = -\alpha^3$ in both the Dirac and Majorana representations. We may rewrite Eq. (22), using the fact that the Clifford algebra condition holds only in the Dirac representation, and obtain

$$\begin{aligned} H = \int d^3x\, &\{(\alpha+1)[\psi^\dagger V_0\Pi_0\psi + (V_0\Pi_0\psi)^\dagger\psi] \\ &- m\delta^1_\alpha\psi^\dagger(2V_0V_0 + 1)\psi\} \,. \end{aligned}$$

Now by using the Majorana equation (16) to eliminate $V_0\Pi_0$, and then by integrating by parts, we find

$$\begin{aligned} H = \int d^3x\, &[-2(\alpha+1)\psi^\dagger V^i\Pi_i\psi \\ &+ 2m\alpha^{-1}(1+\alpha)\psi^\dagger\psi \\ &- m\delta^1_\alpha\psi^\dagger(2V_0V_0 + 1)\psi] \,. \end{aligned} \tag{23}$$

In the case of the Dirac representation, Eq. (23) is the standard energy operator and evaluation may proceed in the usual manner.[10] Accordingly, our interest will center on the Majorana representation. In this case Eq. (23) reduces to

$$\begin{aligned} H\,(\text{Majorana}) = \int d^3x\, &[-4\psi^\dagger V^i\Pi_i\psi + 4m\psi^\dagger\psi \\ &- m\psi^\dagger(2V_0V_0 + 1)\psi] \,. \end{aligned} \tag{24}$$

Now consider a momentum eigensolution of Staunton's equations. Without loss of generality we may transform to the particle's rest frame. In this case the solution[1] is given by

$$\psi = (Aq_1 + Bq_2)\exp[-\tfrac{1}{2}(q_1{}^2 + q_2{}^2) - imx^0] \,, \tag{25}$$

where A and B are arbitrary. In this realization the z component of the internal angular momentum operator, J_3,[11] may be applied to Eq. (25). The eigenfunctions of J_3, $\psi_{+1/2}$, and $\psi_{-1/2}$, with eigenvalues $\pm\frac{1}{2}$, as expected, are given by

$$\psi_{+1/2} = Na(q_1 - q_2)\exp[-\tfrac{1}{2}(q_1^2 + q_2^2) - imx^0], \quad (26)$$

$$\psi_{-1/2} = Nb(q_1 + q_2)\exp[-\tfrac{1}{2}(q_1^2 + q_2^2) - imx^0], \quad (27)$$

where N is a normalization constant. The quantities a and b are annihilation operators for particles of spin up and down, respectively. With $N = (\frac{1}{2}\pi)^{-1/2}$ we find

$$\psi_n^\dagger \psi_m = \delta_n^{+1/2}\delta_m^{+1/2} a^\dagger a + \delta_n^{-1/2}\delta_m^{-1/2} b^\dagger b \quad (n, m = \pm\tfrac{1}{2})\,. \quad (28)$$

In the case of a general momentum eigensolution of Staunton's equation,[1] the same procedure as described here yields

$$\psi = N[a(q_1 - iq_2) + b(q_1 + iq_2)]\exp\{-\tfrac{1}{2}(p_0 + p_3)^{-1}[m(q_1^2 + q_2^2) + ip_1(q_1^2 - q_2^2) - 2ip_2q_1q_2] - ip_\mu x^\mu\}\,. \quad (29)$$

Straightforward calculation using Eq. (29) yields

$$[\psi^\dagger(V_a V_b + V_b V_a + \eta_{ab})\psi] = 3m^{-2}p_a p_b(\psi^\dagger\psi)\,, \quad (30)$$

$$(\psi^\dagger V_a \psi) = e^{-1}j_a = m^{-1}p_a(\psi^\dagger\psi)\,, \quad (31)$$

and

$$(\psi^\dagger\psi) = a^\dagger a + b^\dagger b\,. \quad (32)$$

Note that $a^\dagger a$ and $b^\dagger b$ are the number operators for particles with spin up and spin down, respectively.[12]

Evaluation of H (Majorana) is completed by substituting Eqs. (30), (31), and (32) into Eq. (24). We find

$$H\ (\text{Majorana}) = \int d^3x[m^{-1}p_0^2(a^\dagger a + b^\dagger b)]\,. \quad (33)$$

V. DISCUSSION

A variational principle that will yield both known spin-$\frac{1}{2}$ equations has been found. To achieve this objective, a portion of the group structure of the theory had to be left undetermined until after the variation. The unification made possible by this approach was illustrated by our finding the matter stress-energy tensors and the energy operators for the second-quantized free fields.

The work developed in this paper opens up four areas for further research. These are as follows:

1. *General relativity*. With the matter stress-energy tensor for Staunton's spin-$\frac{1}{2}$ equation determined, the coupled Einstein-Staunton equations may be solved. Since the Minkowski space-time interpretation of Staunton's equations[1,4] may be visualized in terms of two particles interacting on the light cone, the behavior of the solutions near a singularity is of special interest.

2. *Group theory*. Groups larger than SO(3, 2) may be considered in the same manner as used here. While the SO(3, 2) group is interesting because the Lorentz generators S_{ab} may be expressed as commutators in terms of the V_a, other groups may enable one to incorporate other interesting internal symmetries into the formalism [i.e., SU(3), SU(6)][13] (see Ref. 13) in a natural manner.

3. *Quantum field theory*. Since Staunton's equation describes a positive-energy particle, the representation of operators such as the time-reversal operator may be different from the usual manner used in quantum field theory. The applicability of the *PCT* theorem and the connection between spin and statistics are two of the many important areas that the author believes should be investigated in detail.

4. *Spin*. At present Staunton's spin-$\frac{1}{2}$ equation is the only interacting equation of its type known. However, it is believed[1,4] that a set of equations, linear in the momentum operator, should exist for each integer and half-integer spin value. By writing the now known Lagrangians for particles of Dirac type (i.e., using Dirac matrices, one example of which is the spin-$\frac{3}{2}$ Rarita-Schwinger equation), it may be possible to allow the group to be determined by the variation and thereby obtain the equations of the Majorana type for various spins.[14] If this proves possible, it will be, to the author's knowledge, the first time that equations of motion have been discovered from an action principle.

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APPENDIX

Terms of the form $(-g)^{1/2}A^{d\dagger}h^\mu{}_d\Pi_\mu\psi$ appear in the Lagrangian. The quantity $A^{d\dagger}$ represents any desired functions lumped together. We shall calculate here the contribution to the variation from the $(-g)^{1/2}h^{\mu}{}_{d}\Pi_{\mu}\psi$ term when the variation is taken with respect to $h^{\lambda}{}_{a}$.[15] Since $g_{\mu\nu}=h_{\mu}{}^{a}h_{\nu a}$, we may quickly obtain the useful relations

$$\begin{aligned}\delta g^{\mu\nu}&=(\delta^{\mu}{}_{\lambda}h^{\nu a}+h^{\mu a}\delta^{\nu}{}_{\lambda})\delta h^{\lambda}{}_{a},\\ \delta g_{\mu\nu}&=-(g_{\mu\lambda}h_{\nu}{}^{a}+g_{\nu\lambda}h_{\mu}{}^{a})\delta h^{\lambda}{}_{a},\\ \delta h_{\mu}{}^{b}&=-h_{\mu}{}^{a}h_{\lambda}{}^{b}\delta h^{\lambda}{}_{a}.\end{aligned} \tag{A1}$$

On expansion of Eq. (4) and substitution with Eq. (3) into Eq. (5), we find

$$\begin{aligned}\Pi_{\mu}&=i\partial_{\mu}+\tfrac{1}{2}h^{\nu c}h_{\nu}{}^{b}{}_{,\mu}S_{bc}\\ &\quad-\tfrac{1}{2}h^{\nu c}h^{\tau b}g_{\tau\mu,\nu}S_{bc}+eA_{\mu}.\end{aligned} \tag{A2}$$

Now taking variations with respect to $h^{\lambda}{}_{a}$ we find

$$\begin{aligned}h_{\alpha a}\frac{\delta}{\delta h^{\lambda}{}_{a}}(-g)^{1/2}A^{d\dagger}h^{\mu}{}_{d}\Pi_{\mu}\psi&=(-g)^{1/2}A^{\dagger}_{\alpha}\Pi_{\lambda}\psi-g_{\alpha\lambda}(-g)^{1/2}A^{d\dagger}h^{\mu}{}_{d}\Pi_{\mu}\psi+(-g)^{1/2}\frac{\delta A^{d\dagger}}{\delta h^{\lambda}{}_{a}}h^{\mu}{}_{d}h_{\alpha a}\Pi_{\mu}\psi\\ &\quad+\tfrac{1}{2}(-g)^{1/2}[(A^{\mu\dagger}S_{\lambda\alpha}\psi)_{;\mu}-(A_{\alpha}{}^{\dagger}S_{\lambda}{}^{\nu}\psi)_{;\nu}-(A_{\lambda}{}^{\dagger}S_{\alpha}{}^{\nu}\psi)_{;\nu}].\end{aligned} \tag{A3}$$

When A_{λ} is any function of V_{μ}, $S_{\mu\nu}$, and ψ

$$i[A^{\mu\dagger}S_{\lambda}{}^{\nu}\psi]_{;\rho}=-(\Pi_{\rho}A^{\mu})^{\dagger}S_{\lambda}{}^{\nu}\psi+A^{\mu\dagger}S_{\lambda}{}^{\nu}\Pi_{\rho}\psi. \tag{A4}$$

To verify Eq. (A4) use the fact that $S_{\mu\nu}$ and V_{μ} commute with Π_{ρ} and that

$$\Pi_{\rho}S_{\mu\nu}=iS_{\mu\nu;\rho}+\lambda_{\rho}{}^{ab}[S_{ab},S_{\mu\nu}]+eA_{\rho}S_{\mu\nu}, \tag{A5}$$

$$\Pi_{\rho}V_{\mu}=iV_{\mu;\rho}+\lambda_{\rho}{}^{ab}[S_{ab},V_{\mu}]+eA_{\rho}V_{\mu} \tag{A6}$$

to convert semicolons into Π's. Substitute Eq. (A4) into Eq. (A3) and find

$$\begin{aligned}h_{\alpha a}\frac{\delta}{\delta h^{\lambda}{}_{a}}(-g)^{1/2}A^{d\dagger}h^{\mu}{}_{d}\Pi_{\mu}\psi&=(-g)^{1/2}A^{\dagger}_{\alpha}\Pi_{\lambda}\psi-g_{\alpha\lambda}(-g)^{1/2}A^{d\dagger}h^{\mu}{}_{d}\Pi_{\mu}\psi+(-g)^{1/2}\frac{\delta A^{d\dagger}}{\delta h^{\lambda}{}_{a}}h^{\mu}{}_{d}h_{\alpha a}\Pi_{\mu}\psi\\ &\quad+\tfrac{1}{2}(-g)^{1/2}[(\Pi_{\mu}A^{\mu})^{\dagger}iS_{\lambda\alpha}\psi-A^{\mu\dagger}iS_{\lambda\alpha}\Pi_{\mu}\psi-(\Pi_{\nu}A_{\alpha})^{\dagger}iS_{\lambda}{}^{\nu}\psi+A_{\alpha}{}^{\dagger}iS_{\lambda}{}^{\nu}\Pi_{\nu}\psi\\ &\quad-(\Pi_{\nu}A_{\lambda})^{\dagger}iS_{\alpha}{}^{\nu}\psi+A_{\lambda}{}^{\dagger}iS_{\alpha}{}^{\nu}\Pi_{\nu}\psi].\end{aligned} \tag{A7}$$

Specific application of Eq. (A7) to Eq. (12) yields Eq. (18). To illustrate, we consider the special case where $A^{\mu}=V^{\mu}\psi$. Then Eq. (A7) may be simplified using Eqs. (7) and (16) to read

$$\begin{aligned}h_{\alpha a}\frac{\delta}{\delta h^{\lambda}{}_{a}}(-g)^{1/2}(V^{\mu}\psi)^{\dagger}\Pi_{\mu}\psi&=-g_{\alpha\lambda}(-g)^{1/2}(V^{\mu}\psi)^{\dagger}\Pi_{\mu}\psi\\ &\quad+(-g)^{1/2}\{\tfrac{1}{2}(1+\alpha^{-1})\psi^{\dagger}(V_{\alpha}\Pi_{\lambda}+V_{\lambda}\Pi_{\alpha})\psi+\tfrac{1}{2}(1+\alpha^{-1})[(V_{\alpha}\Pi_{\lambda}+V_{\lambda}\Pi_{\alpha})\psi]^{\dagger}\psi\\ &\quad-m\psi^{\dagger}(V_{\alpha}V_{\lambda}+V_{\lambda}V_{\alpha})\psi-\alpha Dm(1+\alpha\lambda)^{-1}g_{\alpha\lambda}\psi^{\dagger}\psi\}.\end{aligned} \tag{A8}$$

The first term on the right-hand side of (A8) is from the variation of $(-g)^{1/2}$. Since $\mathfrak{L}_{\psi}\equiv 0$, when the equations of motion hold, the contribution from all the terms involving the variation of $(-g)^{1/2}$ ultimately cancels out.

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[1]L. P. Staunton, Phys. Rev. D 10, 1760 (1974).

[2]H. F. Ahner, Phys. Rev. D 11, 3384 (1975).

[3]L. P. Staunton and S. Browne, Phys. Rev. D 12, 1026 (1975).

[4]L. C. Biedenharn, M. Y. Han, and H. van Dam, Phys. Rev. D 8, 1735 (1973).

[5]J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), p. 358–360. Also see Ref. 2.

[6]Latin lower case letters are used for local Lorentz indices while the Greek alphabet is reserved for space-time indices. Comma and semicolon denote ordinary derivatives and space-time covariant derivatives, respectively. $h_{\mu}{}^{a}h_{\nu a}=g_{\mu\nu}$ and $h_{\mu}{}^{a}h^{\mu b}=\eta^{ab}$ $=\mathrm{diag}(1,-1,-1,-1)$. $S_{ab}=h^{\mu}{}_{a}h^{\nu}{}_{b}S_{\mu\nu}$, and $V_{a}=h^{\mu}{}_{a}V_{\mu}$. The relations (1) hold with either Greek or Latin indices.

[7]See Appendix of Ref. 1. Actually the positron equation is obtained instead of the electron equation but this does not affect the generality of the discussion.

[8]The concepts outlined here are detailed for the case of Dirac's equation in Ref. 5.

[9]R. C. Hwa, Nuovo Cimento 56A, 107 (1968); 56A, 127 (1968).

[10]See, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), but note that his sign conventions differ slightly from those used here.

[11]The particular realization of the Majorana representation that we use for calculations has $\eta_i=-i\,\partial_i$ and in Minkowski space-time is $J_1=-S_{23}=\frac{1}{2}(q_1q_2+\eta_1\eta_2)$, $J_2=-S_{31}=\frac{1}{4}(q_1{}^2+\eta_1{}^2-q_2{}^2-\eta_2{}^2)$, $J_3=-S_{12}=\frac{1}{2}(q_2\eta_1$

$-q_1\eta_2)$, $K_1=S_{10}=\frac{1}{4}(q_1{}^2-\eta_1{}^2-q_2{}^2+\eta_2{}^2)$, $K_2=S_{20}=\frac{1}{2}(\eta_1\eta_2-q_1q_2)$, $K_3=S_{30}=\frac{1}{2}(q_1\eta_1+\eta_2q_2)$, $V_0=\frac{1}{4}(q_1{}^2+q_2{}^2+\eta_1{}^2+\eta_2{}^2)$, $V_1=\frac{1}{2}(-q_1\eta_1+q_2\eta_2)$, $V_2=\frac{1}{2}(q_1\eta_2+q_2\eta_1)$, $V_3=\frac{1}{4}(q_1{}^2+q_2{}^2-\eta_1{}^2-\eta_2{}^2)$. See P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[12]No commutation relations between a, b, $a^\dagger$, $b^\dagger$ have been used in obtaining these results. While the author assumes that anticommutation relations hold, he has found nothing in the formalism that ensures a link between spin and statistics as is true in the Dirac spin-$\frac{1}{2}$ case.

[13]L. P. Staunton and H. van Dam, Lett. Nuovo Cimento 7, 371 (1973).

[14]Equations for other spins that do not allow electromagnetic interaction have been found by S. Browne, Nucl. Phys. B79, 70 (1974). We desire a set of equations of the Staunton-Majorana type that provide for an electromagnetic interaction.

[15]The methods used here are described in detail in S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 365–373.