PHYSICAL REVIEW D VOLUME 13, NUMBER 8 15 APRIL 1976

Renormalizability and lepton-hadron universality in pion β decay in a gauge field theory

Kasinath Niyogi, Partha Ghose,* and Asim K. Ray
Department of Physics, Visva-Bharati University, Santiniketan 731235, West Bengal, India
(Received 15 October 1975)

The divergent contributions in the second-order weak and electromagnetic corrections to pion β decay are calculated in the symmetrical theory of weak and electromagnetic interactions using the unitary gauge and dimensional regularization. It is shown that, after renormalization, these divergences cancel and the Z_2/Z_1 ratios for leptons and pions are equal as expected from lepton-hadron universality.

I. INTRODUCTION

Following the work of 't Hooft[1] and others,[2] several attempts[3] have been made to calculate higher-order corrections to the leptonic and semileptonic processes in the Salam-Weinberg[4] theory of the weak and electromagnetic interactions. Recently, Sirlin,[5] using Ward identities and current algebra in the context of $SU(2)\times U(1)$ gauge models, has shown in the 't Hooft-Feynman gauge that after renormalization of strong-coupling constants and masses the divergent part of the second-order corrections to the leptonic and semileptonic amplitudes mediated by the W meson is a universal multiple of the lowest-order amplitude independent of the strong interactions. It would be interesting, therefore, to check lepton-hadron universality for a semileptonic process explicitly without using current algebra, and we have done that in the present paper.

Using the Salam-Weinberg model of leptons and a model of pions[6] in which the strong interactions of the pions are not incorporated, we have calculated the second-order weak and electromagnetic corrections to pion β decay ($\pi^- \to \pi^0 + e^- + \bar{\nu}_e$) in the unitary gauge and shown the finiteness of second-order radiative corrections and the equality of Z_2/Z_1 ratios for leptons and pions after extracting the various renormalization constants.

In this presentation we have systematically neglected terms of the order of m_e/m_W^2, which are negligible for practical purposes. Although our approach is similar to those of the previous attempts,[3] it is somewhat more comprehensive and, perhaps, more explicit in demonstrating lepton-hadron universality together with renormalizability including the relevant counterterms.

The plan of the paper is as follows: Section II is devoted to the discussion of the relevant Lagrangian and the classification of diagrams. Sections III–VI contain the results of calculations of divergent parts of the matrix elements due to different classes of diagrams. In addition, various renormalization constants and the relevant counterterms are defined in these sections. The Higgs scalar interactions are discussed in Sec. VII. In Sec. VIII the principal results in regard to renormalizability, including the equality of the Z_2/Z_1 ratios for the leptons and pions, are presented. Section IX contains a brief résumé and some concluding remarks.

II. LAGRANGIANS AND DIAGRAMS

A. Relevant interaction Lagrangians

The relevant leptonic Lagrangian in the Salam-Weinberg model[4] is

$$\begin{aligned}
\mathcal{L}_{\text{lep}} = {} & (1+C_1)\bar{e}(\not{q}-m_e)e + C_2\bar{e}e - (1+C_3)\bar{\nu}\not{Q}\frac{1-\gamma_5}{2}\nu + g\sin\phi\,\bar{e}\gamma_\lambda e A^\lambda \\
& + \frac{g}{\cos\phi}\left(\tfrac{1}{2}\bar{\nu}\gamma_\lambda\frac{1-\gamma_5}{2}\nu - \tfrac{1}{2}\bar{e}\gamma_\lambda\frac{1-\gamma_5}{2}e\right)Z^\lambda + g\frac{\sin^2\phi}{\cos\phi}\bar{e}\gamma_\lambda e Z^\lambda + \frac{g}{\sqrt{2}}(1+C_4)\left(\bar{\nu}\gamma_\lambda\frac{1-\gamma_5}{2}eW^{+\lambda} + \bar{e}\gamma_\lambda\frac{1-\gamma_5}{2}\nu W^{-\lambda}\right) \\
& - (1+C_5)\left(\tfrac{1}{2}W^+_{\mu\nu}W^-_{\mu\nu} + m_W{}^2 W^+_\mu W^{-\mu}\right) \\
& + C_6 W^+_\mu W^{-\mu} + ig(\cos\phi Z^\nu - \sin\phi A^\nu)[W^{-\mu}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu) - W^{+\mu}(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) + \partial^\mu(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu)] \\
& - g^2 W^-_\mu W^+_\nu(\cos\phi Z_\rho - \sin\phi A_\rho)(\cos\phi Z_\sigma - \sin\phi A_\sigma)(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \\
& - \frac{g^2}{2}(W^-_\mu W^{-\mu}W^+_\nu W^{+\nu} - W^-_\mu W^{+\mu}W^-_\nu W^{+\nu}) - \frac{gm_e}{2m_W}\chi\bar{e}e - \frac{gm_W}{2}(1+\tan^2\phi)\chi Z_\mu Z^\mu - gm_W\chi W^-_\mu W^{+\mu} + \frac{gm_\chi}{m_W}\chi^3 \\
& - \frac{g^2}{8}(1+\tan^2\phi)\chi^2 Z_\mu Z^\mu - \frac{g^2}{4}\chi^2 W^-_\mu W^{+\mu} + \text{other counterterms}\,,
\end{aligned} \tag{1a}$$

where $\bar{e}$, e, W^+ and W^- denote creation of electrons, positrons, negative W bosons, and positive W bosons, respectively, and annihilation of the corresponding antiparticles. The parameters g, g', and ϕ are related to the electronic charge e, the Fermi coupling constant G, and the boson masses m_W and m_Z as follows:

$$g\sin\phi = e = g'\cos\phi\,,\quad \frac{g^2}{8m_W{}^2} = \frac{G}{\sqrt{2}}\,,\quad m_Z = \frac{m_W}{\cos\phi}\,. \tag{1b}$$

Terms involving C_i's ($i = 1, 2, \ldots, 6$) are the relevant counterterms necessary to show the finiteness of the second-order radiative corrections to pion β decay.

The difficulty of incorporating hadrons in the Salam-Weinberg theory is well known,[7] and several interesting schemes[8] have been proposed. However, in our calculation we have used a simple model of pions[6] in which a gauge-invariant pionic Lagrangian is constructed out of a doublet φ_1 and its conjugate $\varphi_2^\dagger$ formed with an appropriate combination of pions (π) and the η-meson fields, plus the gauge fields A_μ and B_μ coupled to weak isospin (T_L) and hypercharge (Y_L). The doublets φ_1 and φ_2 are given by

$$\varphi_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}}(\eta + i\pi^0) \\ \pi^- \end{pmatrix} \quad\text{and}\quad \varphi_2 = i\tau_2\varphi_1^C = \begin{pmatrix} \pi^+ \\ \frac{1}{\sqrt{2}}(\eta - i\pi^0) \end{pmatrix}\,, \tag{2a}$$

with the weak hypercharge $Y_L = +\frac12$ for φ_1 and $-\frac12$ for φ_2. The coupling strengths of φ_1 and φ_2 to the gauge fields A_μ and B_μ are precisely the same as the corresponding coupling constants of L and R which occur in Weinberg's leptonic Lagrangian. The interaction Lagrangian[6] is

$$\begin{aligned}
\mathcal{L}_{\text{pion}} = {} & \tfrac12(1+C_7)(\partial_\mu\pi^0\partial^\mu\pi^0 - m_{\pi^0}{}^2\pi^{0\,2}) + \tfrac12 C_8\pi^{0\,2} + (1+C_9)(\partial_\mu\pi^+\partial^\mu\pi^- - m_{\pi^+}{}^2\pi^+\pi^-) \\
& + C_{10}\pi^+\pi^- + \frac12\,\frac{g}{2\cos\phi}(\pi^0\overleftrightarrow{\partial}_\mu\eta)Z^\mu - ig\sin\phi(\pi^-\overleftrightarrow{\partial}_\mu\pi^+)A^\mu + \frac{ig}{2}\,\frac{1-2\sin^2\phi}{\cos\phi}(\pi^-\overleftrightarrow{\partial}_\mu\pi^+)Z^\mu \\
& + \tfrac12 g(1+C_{11})\{[i(\pi^-\overleftrightarrow{\partial}_\mu\eta) + (\pi^-\overleftrightarrow{\partial}_\mu\pi^0)]W^{+\mu} + [i(\eta\overleftrightarrow{\partial}_\mu\pi^+) + (\pi^+\overleftrightarrow{\partial}_\mu\pi^0)]W^{-\mu}\} \\
& + \tfrac14(\eta^2 + \pi^{0\,2})\left(g^2W_\mu^+W^{-\mu} + \frac{g^2}{2\cos^2\phi}Z_\mu Z^\mu\right) \\
& + \pi^+\pi^-\left[g^2\frac{(1-2\sin^2\phi)^2}{4\cos^2\phi}Z^\mu Z_\mu + g^2\sin^2\phi A_\mu A^\mu + \tfrac12 g^2W_\mu^+W^{-\mu} + g^2\tan\phi(1-2\sin^2\phi)A_\mu Z^\mu\right] \\
& + \frac12\left(g^2\frac{\sin^2\phi}{\cos\phi}Z^\mu + g^2\sin^2\phi A^\mu\right)[(i\pi^0\pi^- - \eta\pi^-)W_\mu^+ - (i\pi^0\pi^+ + \eta\pi^+)W_\mu^-] \\
& - \left(g^2\frac{m_{\pi^+}{}^2}{4m_W{}^2}\chi^2 + g\frac{m_{\pi^+}{}^2}{m_W}\chi\right)\pi^+\pi^- - \tfrac12(\eta^2 + \pi^{0\,2})\left(g^2\frac{m_{\pi^0}{}^2}{4m_W{}^2}\chi^2 + g\frac{m_{\pi^0}{}^2}{m_W}\chi\right) \\
& + \text{other counterterms}\,,
\end{aligned} \tag{2b}$$

where π^- and π^+ denote absorption of negative and positive pions, respectively, and creation of the corresponding antiparticles. Here again terms with C_i's are the relevant counterterms; the expression for C_i's in terms of different renormalization constants will be given in appropriate sections. The Higgs scalar (χ) interaction terms in Eq. (2b) arise from the SU(2)× U(1)-invariant interaction

$$\alpha(\varphi_1^\dagger\tau_\mu\varphi_1)(\chi^\dagger\tau^\mu\chi) + \beta(\varphi_2^\dagger\tau_\mu\varphi_2)(\chi^\dagger\tau^\mu\chi)\,, \tag{2c}$$

which gives rise to a zeroth-order mass difference between π^0 and π^+ through spontaneous symmetry breaking, i.e., when χ develops a vacuum expectation value [α and β in Eq. 2(c) are arbitrary constants]. It may be noted that the strong interactions of the pions are not incorporated in the pionic Lagrangian.

B. Classification of diagrams

The lowest-order diagram and all relevant Feynman diagrams of order g^4 for the process $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$ are shown in Fig. 1. The diagrams of order g^4 can be classified into four groups:

(1) two-boson exchange [diagrams (2)–(6) and (50)–(52)],

(2) boson-propagator modification [diagrams (7)–(16)],

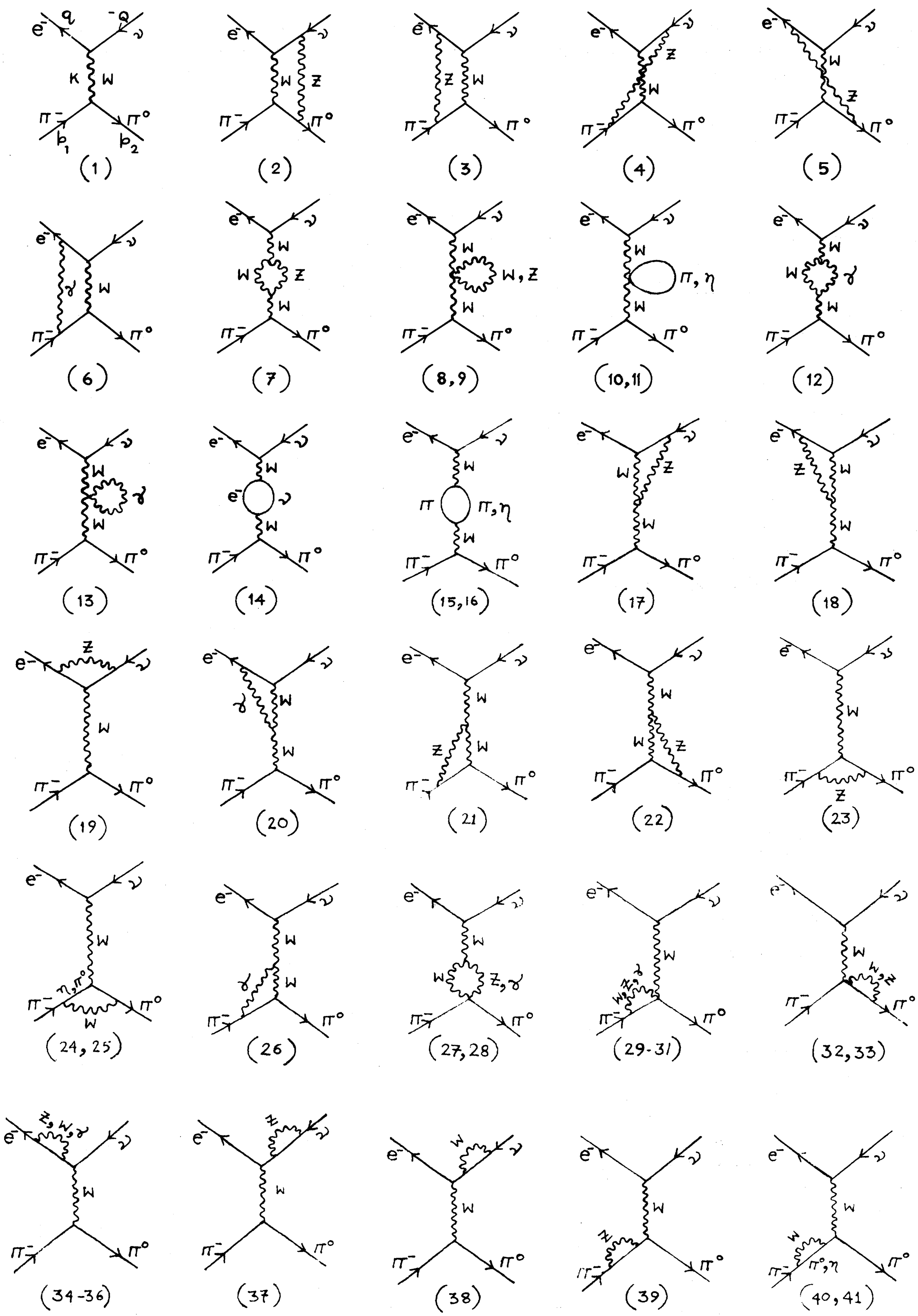

Fig. 1 *(Continued on following page)*

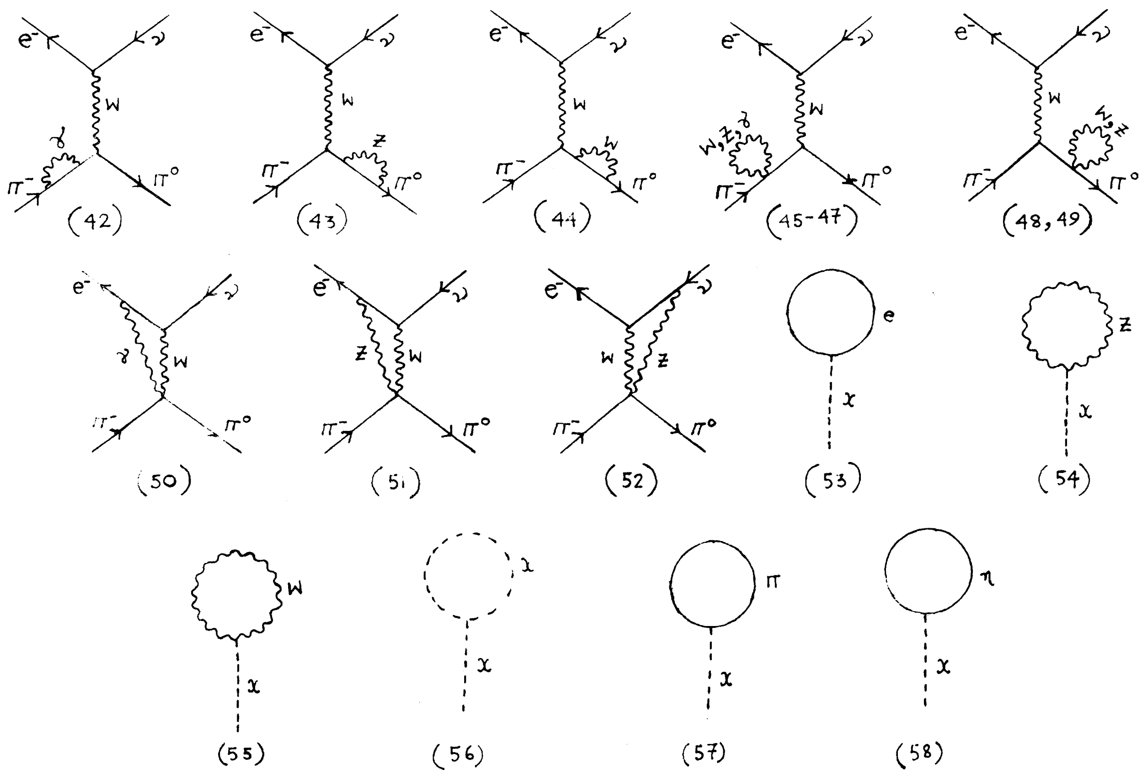

FIG. 1. Feynman diagrams for π^- β decay in orders g^2 and g^4 and tadpole diagrams (see Sec. VII).

(3) vertex modification [diagrams (17)–(33)],

(4) external-line modification [diagrams (34)–(49)].

The matrix element for the lowest-order process [diagram (1)] is

$$M_0 = -\frac{1}{4\sqrt{2}} g^2 (p_1 + p_2)^\nu \bar{u}(q)\gamma^\mu (1-\gamma_5) v(Q) \times \frac{(g_{\mu\nu} - K_\mu K_\nu / m_W{}^2)}{K^2 - m_W{}^2}, \tag{3}$$

where

$$p_1 = p_2 + q + Q\ ,\ K = p_1 - p_2\ .$$

It is to be noted that the divergent and finite contributions to the matrix element due to the diagrams (27), (28), (50), (51), and (52) depend on the momentum transfer $p_1 - p_2$ only and the former cancel with the divergent contributions from other diagrams leaving the finite part. In an earlier paper[9] we have shown that this finite contribution gives rise to possible violation of conservation of vector current, which was estimated to be 0.1% for pion β decay. However, in the present paper from now on we shall concentrate on the divergent contributions to the matrix element depending on $p_1 + p_2$ only. In evaluating the Feynman integrals we have used dimensional regularization.[10]

III. TWO-BOSON EXCHANGE [DIAGRAMS (2)–(6)]

Diagrams (2)–(5) contribute to the weak radiative corrections. All these diagrams are quadratically divergent, and the total divergent contribution from them is

$$M_1^{\text{weak}} = \frac{\pi^2 g^4}{(2\pi)^4} \frac{\cos^2\phi}{4\sqrt{2}} \left(\frac{2}{4-n}\right) (p_1 + p_2)^\nu \bar{u}(q) \times \gamma_\nu (1-\gamma_5) v(Q)(a+b+c)\ , \tag{4}$$

where

$$a = \frac{1}{m_W{}^2},\qquad b = \frac{1}{m_Z{}^2}, \tag{5}$$

and

$$c = \frac{1}{2m_W{}^2 m_Z{}^2}\left(\frac{K^2 - m_W{}^2}{2} - \tfrac{1}{2} m_Z{}^2 - \tfrac{1}{3}K^2\right).$$

Diagram (6) contributes to the electromagnetic radiative correction, and the corresponding divergent contribution to the matrix element is

$$M_1^{\text{em}} = \frac{\pi^2 g^4}{(2\pi)^4} \frac{\sin^2\phi}{4\sqrt{2}} \left(\frac{2}{4-n}\right) (p_1 + p_2)^\nu \bar{u}(q)\gamma_\nu (1-\gamma_5) v(Q)(a) + \gamma \text{ gauge terms}\ . \tag{6}$$

IV. VECTOR-BOSON-PROPAGATOR MODIFICATION [DIAGRAMS (7)–(17)] [11]

The W-boson self-energy function $\pi_{\text{weak}}^{\alpha\beta}(K)$ involved in the diagrams (7)–(11) and (14)–(16) has quartic, quadratic, and logarithmic divergences. $\pi_{\text{weak}}^{\alpha\beta}(K)$ can be expressed as

$$\pi_{\text{weak}}^{\alpha\beta}(K)=\frac{i\pi^2g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)[a'(K^2g^{\alpha\beta}-K^\alpha K^\beta)+b'(K^4g^{\alpha\beta}-K^2K^\alpha K^\beta)+c'(K^6g^{\alpha\beta}-K^4K^\alpha K^\beta)+d'K^2g^{\alpha\beta}+f'g^{\alpha\beta}], \quad (7)$$

where a', b', c', d', and f' are finite constants defined as

$$a'=\left(\frac{7}{6}\frac{m_W^4}{m_Z^4}+\frac{7}{2}\frac{m_W^2}{m_Z^2}-\frac{37}{3}\right),\quad b'=-\frac{7}{12}\left(\frac{m_W^2+m_Z^2}{m_Z^4}\right),$$
$$c'=-\frac{1}{12m_Z^4},\quad d'=\left(\frac{1}{4}\frac{m_W^4}{m_Z^4}+\frac{2}{3}\frac{m_W^2}{m_Z^2}+\frac{115}{12}\right), \quad (8)$$
$$f'=\left(-\frac{3}{4}\frac{m_W^6}{m_Z^4}+3\frac{m_W^4}{m_Z^2}-\tfrac{3}{4}m_Z^2+\tfrac{3}{4}m_W^2-m_\pi^2\right).$$

The modified W propagator is then

$$\Delta'_{\mu\nu}=(-i)\left(\frac{g_{\mu\nu}-K_\mu K_\nu/m_W^2}{K^2-m_W^2}\right)-\left(\frac{g_{\mu\alpha}-K_\mu K_\alpha/m_W^2}{K^2-m_W^2}\right)\pi_{\text{weak}}^{\alpha\beta}(K)\left(\frac{g_{\nu\beta}-K_\nu K_\beta/m_W^2}{K^2-m_W^2}\right). \quad (9)$$

This can be rewritten in the form [neglecting terms $O(g^4)$]

$$\Delta'_{\mu\nu}=(-i)\left(\frac{g_{\mu\nu}-K_\mu K_\nu/[m_W^2+(\delta m_W^2)_{\text{weak}}]}{K^2-m_W^2-(\delta m_W^2)_{\text{weak}}}\right)Z_3^{\text{weak}}+\frac{i\pi^2g^2\cos^2\phi}{(2\pi)^4}\left(\frac{2}{4-n}\right)(\tfrac{5}{6}a+b+c), \quad (10)$$

where the mass and wave-function renormalization constants $(\delta m_W^2)_{\text{weak}}$ and Z_3^{weak}, respectively, for the W boson are defined by

$$(\delta m_W^2)_{\text{weak}}=\frac{\pi^2g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{79}{12}\frac{m_W^4}{m_Z^2}-\tfrac{3}{4}m_Z^2-2m_W^2-m_\pi^2\right) \quad (11)$$

and

$$Z_3^{\text{weak}}=1+\frac{\pi^2g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(3\frac{m_W^2}{m_Z^2}-\frac{11}{4}\right). \quad (12)$$

Thus, even after the mass and wave-function renormalization, the propagator $\Delta'_{\mu\nu}$ is not finite; it has an additive infinite constant. The first term in (10) partially renormalizes m_W and g in the lowest-order matrix element [Eq. (3)], and the second term gives the following divergent contribution:

$$M_2^{\text{weak}}=\frac{\pi^2g^2}{(2\pi)^4}\frac{\cos^2\phi}{4\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu\bar{u}(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{5}{6}a+b+c). \quad (13)$$

The divergent parts of the W self-energy due to the electromagnetic interaction [diagrams (12) and (13)] contain quartic, quadratic, and logarithmically divergent terms and are dependent on the photon gauge. The self-energy function

$$\pi_{\text{em}}^{\alpha\beta}(K)=\frac{i\pi^2g^2\sin^2\phi}{(2\pi)^4}\left(\frac{2}{4-n}\right)[3(K^2g^{\alpha\beta}-K^\alpha K^\beta)-\tfrac{5}{6}a(K^4g^{\alpha\beta}-K^2K^\alpha K^\beta)+\tfrac{2}{3}K^2g^{\alpha\beta}+(15/4a)g^{\alpha\beta}]$$
$$+\gamma \text{ gauge terms} \quad (14)$$

also renormalizes the W-boson mass and wave function by $(\delta m_W^2)_{\text{em}}$ and $(Z_3^{\text{em}})^{1/2}$, respectively, and contributes an additive infinite constant to the matrix element. We obtain

$$(\delta m_W^2)_{\text{em}}=\frac{\pi^2g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{79}{12}\right)\left(m_W^2-\frac{m_W^4}{m_Z^2}\right)+\gamma \text{ gauge terms}, \quad (15)$$

$$Z_3^{\text{em}}=1+\frac{\pi^2g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)2\left(1-\frac{m_W^2}{m_Z^2}\right)+\gamma \text{ gauge terms}, \quad (16)$$

and

$$M_2^{\text{em}}=\frac{\pi^2g^4\sin^2\phi}{(2\pi)^44\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu\bar{u}(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{5}{6}a)+\gamma \text{ gauge terms}. \quad (17)$$

Thus the total contributions of weak and electromagnetic corrections to the mass and wave-function renormalization ($Z_3{}^W$) constants are the following:

$$\delta m_W{}^2 = (\delta m_W{}^2)_{\rm weak} + (\delta m_W{}^2)_{\rm em}$$

$$= \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)(\tfrac{35}{12}m_W{}^2 - \tfrac{3}{4}m_Z{}^2 - m_\pi{}^2) + \gamma \text{ gauge terms} \tag{18}$$

and

$$Z_3^W = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{m_W{}^2}{m_Z{}^2} - \frac{3}{4}\right) + \gamma \text{ gauge terms}. \tag{19}$$

Since we started with the renormalized Lagrangian, these W-boson mass and wave-function renormalization contributions can be canceled with the counterterms C_6 and C_5 respectively by choosing $C_6 = \delta m_W{}^2$ and $C_5 = Z_3^W - 1$. It should be emphasized that these counterterm prescriptions do not remove the divergent contributions (13) and (17) to the matrix element.

V. VERTEX MODIFICATION [DIAGRAMS (17)–(33)]

The vertex parts for both $e^-\bar{\nu}_e W$ and $\pi^-\pi^0 W$ vertices are quadratically divergent.

A. $e^-\bar{\nu}_e W$ vertex

The total divergent contribution to the $e^-\bar{\nu}_e W$ vertex due to W and Z exchanges [diagrams (17)–(19)] is

$$\Lambda^\alpha_{\rm weak}(-Q,q) = \frac{i\pi^2 g^3}{(2\pi)^4 2\sqrt{2}}\left(\frac{2}{4-n}\right)\{\cos^2\phi[3g^{\alpha\beta} + (a+b)(\tfrac{11}{12}K^2 - \tfrac{3}{4}m_W{}^2 - \tfrac{3}{4}m_Z{}^2)g^{\alpha\beta}$$

$$+ cK^2 g^{\alpha\beta} - \tfrac{2}{3}(a+b)K^\alpha K^\beta - cK^\alpha K^\beta]\}\gamma_\beta(1-\gamma_5). \tag{20}$$

The total vertex operator

$$\Gamma^\alpha_{\rm weak}(-Q,q) = \frac{ig}{2\sqrt{2}}\gamma^\alpha(1-\gamma_5) + \Lambda^\alpha(-Q,q) \tag{21}$$

can be rewritten as

$$\Gamma^\alpha_{\rm weak}(-Q,q) = \frac{ig}{2\sqrt{2}}\gamma^\alpha(1-\gamma_5)\frac{1}{Z^{\rm weak}_{1,l}}$$

$$+ \frac{i\pi^2 g^3 \cos^2\phi}{(2\pi)^4 2\sqrt{2}}\left(\frac{2}{4-n}\right)[(\tfrac{11}{12}a + b + c)(K^2 - m_W{}^2)g^{\alpha\beta} - \tfrac{2}{3}(a+b)K^\alpha K^\beta - cK^\alpha K^\beta]\gamma_\beta(1-\gamma_5), \tag{22}$$

where the renormalization constant for the $e^-\bar{\nu}_e W$ vertex is defined by

$$\frac{1}{Z^{\rm weak}_{1,l}} = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{13}{6}\frac{m_W{}^2}{m_Z{}^2} - \frac{3}{4}\right). \tag{23}$$

We see that the vertex renormalization does not make $\Gamma^\alpha_{\rm weak}(-Q,q)$ finite. The infinite contribution to the matrix element after the $e^-\bar{\nu}_e W$ vertex renormalization is

$$M_3^{\rm weak} = -\frac{\pi^2 g^4 \cos^2\phi}{(2\pi)^4 4\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu \bar{u}(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{11}{12}a + b + c). \tag{24}$$

The radiative correction to the $e^-\bar{\nu}_e W$ vertex due to photon exchange [diagram (20)] can be handled in the same way, and the vertex normalization constant for such a correction is given by

$$\frac{1}{Z^{\rm em}_{1,l}} = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(1 - \frac{m_W{}^2}{m_Z{}^2}\right)\left(\frac{19}{6}\right) + \gamma \text{ gauge terms}, \tag{25}$$

and the divergent contribution for such a process is

$$M_3^{\rm em} = -\frac{\pi^2 g^4 \sin^2\phi}{(2\pi)^4 4\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu \bar{u}(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{11}{12}a) + \gamma \text{ gauge terms}. \tag{26}$$

Thus the total renormalization constant due to the $e^-\bar{\nu}_e W$ vertex diagrams [(17)–(20)] is given by

$$\frac{1}{Z_{1,l}} = 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{m_W{}^2}{m_Z{}^2} - \frac{29}{12}\right) + \gamma \text{ gauge terms}. \tag{27}$$

This vertex renormalization can be canceled by choosing $C_4 = (Z_2/Z_1)_l\,(Z_3^{\,W})^{1/2} - 1$, leaving the divergent contributions (24) and (26) to the matrix element.

B. $\pi^-\pi^0 W$ vertex

The divergent contribution to the $\pi^-\pi^0 W$ vertex due to weak radiative corrections [diagrams (21)–(33)] is

$$\Lambda^{\alpha}_{\rm weak}(p_1,p_2) = -\frac{\pi^2 g^3}{2(2\pi)^4}\left(\frac{2}{4-n}\right)\Big\{\cos^2\phi\big[\tfrac{3}{2}g^{\alpha\beta} + (a+b)(\tfrac{11}{12}K^2 - \tfrac{3}{4}m_W{}^2 - \tfrac{3}{4}m_Z{}^2)g^{\alpha\beta} + cK^2 g^{\alpha\beta} - \tfrac{2}{3}(a+b)K^\alpha K^\beta - cK^\alpha K^\beta\big]$$
$$- \frac{3}{4}\,\frac{1-\cos^2\phi + \cos^4\phi}{\cos^2\phi}\,g^{\alpha\beta}\Big\}(p_1+p_2)_\beta\,. \tag{28}$$

The vertex operator is then

$$\Gamma^{\alpha}_{\rm weak}(p_1,p_2) = -\tfrac{1}{2}g(p_1+p_2)^\alpha + \Lambda^{\alpha}_{\rm weak}(p_1,p_2)$$
$$= -\tfrac{1}{2}g(p_1+p_2)^\alpha \frac{1}{Z^{\rm weak}_{1,\pi}}$$
$$- \frac{\pi^2 g^3\cos^2\phi}{2(2\pi)^4}\left(\frac{2}{4-n}\right)\big[(\tfrac{11}{12}a+b+c)(K^2 - m_W{}^2)g^{\alpha\beta} - \tfrac{2}{3}(a+b)K^\alpha K^\beta - cK^\alpha K^\beta\big](p_1+p_2)_\beta\,, \tag{29}$$

where $Z^{\rm weak}_{1,\pi}$, defined as the $\pi^-\pi^0 W$ vertex renormalization constant due to W and Z exchanges, is given by

$$\frac{1}{Z^{\rm weak}_{1,\pi}} = -\frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{3}{4}\frac{m_Z{}^2}{m_W{}^2} + \frac{1}{12}\frac{m_W{}^2}{m_Z{}^2}\right). \tag{30}$$

The infinite contribution to the matrix element that remains after renormalization of the pion vertex due to weak radiative corrections is then

$$M_4^{\rm weak} = -\frac{\pi^2 g^4\cos^2\phi}{(2\pi)^4 4\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu\,\bar u(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{11}{12}a+b+c)\,. \tag{31}$$

Similarly, the electromagnetic correction coming from diagram (26) to the $\pi^-\pi^0 W$ vertex is

$$\Lambda^{\alpha}_{\rm em} = -\frac{\pi^2 g^3\sin^2\phi}{2(2\pi)^4}\left(\frac{2}{4-n}\right)\big[\tfrac{3}{4}g^{\alpha\beta} + a(\tfrac{11}{12}K^2 - \tfrac{3}{4}m_W{}^2)g^{\alpha\beta} - \tfrac{1}{6}aK^\alpha K^\beta\big](p_1+p_2)_\beta + \gamma \text{ gauge terms}\,. \tag{32}$$

The corresponding vertex renormalization constant is given by

$$\frac{1}{Z^{\rm em}_{1,\pi}} = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(1 - \frac{m_W{}^2}{m_Z{}^2}\right)\left(\frac{11}{12}\right) + \gamma \text{ gauge terms}\,, \tag{33}$$

and so the infinite contribution that remains after renormalization is

$$M_4^{\rm em} = -\frac{\pi^2 g^4\sin^2\phi}{(2\pi)^4 4\sqrt{2}}\left(\frac{2}{4-n}\right)(p_1+p_2)^\nu\,\bar u(q)\gamma_\nu(1-\gamma_5)v(Q)(\tfrac{11}{12}a) + \gamma \text{ gauge terms}\,. \tag{34}$$

The total $\pi^-\pi^0 W$ vertex renormalization constant due to weak and electromagnetic corrections is therefore

$$\frac{1}{Z_{1,\pi}} = 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{m_W{}^2}{m_Z{}^2} + \frac{3}{4}\frac{m_Z{}^2}{m_W{}^2} - \frac{11}{12}\right) + \gamma \text{ gauge terms}\,. \tag{35}$$

This renormalization of the $\pi^-\pi^0 W$ vertex is canceled with the choice

$$C_{11} = \left(\frac{Z_2}{Z_1}\right)_\pi (Z_3^W)^{1/2} - 1\,.$$

VI. EXTERNAL-LINE MODIFICATION [DIAGRAMS (34)–(49)]

The modifications of the lepton lines [diagrams (34)–(38)] and the pion lines [diagrams (39)–49)] give rise to quadratic and logarithmic divergences which renormalize the masses and wave functions. The effect of the modification of the electron and neutrino lines due to weak and electromagnetic corrections is to renormalize the corresponding masses by δm_e and δm_ν and the wave functions by $(Z_{2,e})^{1/2}$ and $(Z_{2,\nu})^{1/2}$, which, within our approximation, are given by

$$\delta m_e = \frac{3\pi^2 g^2}{2(2\pi)^4}\left(\frac{2}{4-n}\right) m_e \left(\frac{m_Z^2}{m_W^2} - 1\right) + O\left(\frac{m_e^2}{m_{W,Z}^2}\right)$$
$$+\gamma \text{ gauge terms}, \quad (36)$$

$$\delta m_\nu = \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right) O\left(\frac{m_\nu^3}{m_{W,Z}}\right) \simeq 0, \quad (37)$$

$$Z_{2,e} = 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(1 - \frac{m_W^2}{m_Z^2}\right) + O\left(\frac{m_e^2}{m_{W,Z}^2}\right), \quad (38)$$

$$Z_{2,\nu} = 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right) O\left(\frac{m_\nu^2}{m_{W,Z}^2}\right) \simeq 1. \quad (39)$$

Similarly, choosing the photon gauge

$$\frac{\lambda}{2} = n - (n-1)\left(\frac{m_Z^2}{\mu^2}\right)^{n/2-1}, \quad (40)$$

where n is the dimension of space and μ is the photon mass, and using $p_1^2 = m_{\pi^-}^2$, $p_2^2 = m_{\pi^0}^2$, we obtain

$$\delta m_{\pi^-}^2 = \frac{3\pi^2 g^2}{4(2\pi)^4}\left(\frac{2}{4-n}\right) \times\left[\left(2 + \frac{m_Z^2}{m_W^2}\right) m_{\pi^-}^2 - 2\frac{m_Z^4}{m_W^2} - 4m_W^2\right], \quad (41a)$$

$$\delta m_{\pi^0}^2 = \frac{3\pi^2 g^2}{4(2\pi)^4}\left(\frac{2}{4-n}\right) \times\left[\left(2 + \frac{m_Z^2}{m_W^2}\right) m_{\pi^0}^2 - 2\frac{m_Z^4}{m_W^2} - 4m_W^2\right], \quad (41b)$$

and

$$Z_{2,\pi^-} = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{3}{4}\frac{m_Z^2}{m_W^2} + \frac{m_W^2}{m_Z^2} + \frac{1}{2}\right), \quad (42a)$$

$$Z_{2,\pi^0} = 1 + \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{3}{4}\frac{m_Z^2}{m_W^2} + \frac{3}{2}\right). \quad (42b)$$

It is to be noted that $Z_{2,\pi}$ is independent of the photon gauge. The counterterm prescription to remove these divergences is

$$C_7 = Z_{2,\pi^0} - 1, \quad C_8 = \delta m_{\pi^0}^2,$$
$$C_9 = Z_{2\pi^-} - 1, \quad C_{10} = \delta m_{\pi^-}^2.$$

VII. HIGGS SCALAR INTERACTIONS

When the Higgs scalar interactions with external particles and gauge bosons are considered we obtain the following additional graphs (not shown in Fig. 1) of order g^4:

(a) two two-boson exchange diagrams,
(b) two W-boson propagator modification diagrams,
(c) four vertex modification diagrams, and
(d) five external-line modification diagrams.

However, within our approximation, the divergent contribution comes only from the boson-propagator modification and pion-line modification diagrams. The contribution of the Higgs scalar interaction to the W-boson mass and wave-function renormalization constants are

$$(\delta m_W^2)_{\text{scalar int}} = -\frac{\pi^2 g^2}{4(2\pi)^4}\left(\frac{2}{4-n}\right)(3m_\chi^2 + \tfrac{10}{3}m_W^2) \quad (43)$$

and

$$(Z_3^W)_{\text{scalar int}} = 1 - \frac{\pi^2 g^2}{12(2\pi)^4}\left(\frac{2}{4-n}\right). \quad (44)$$

Similar contributions for the pions are

$$(\delta m_{\pi^-}^2)_{\text{scalar int}} = -\frac{\pi^2 g^2}{(2\pi)^4}\frac{m_{\pi^-}^2}{2m_W^2}\left(\frac{2}{4-n}\right)(2m_{\pi^-}^2 - m_\chi^2), \quad (45)$$

$$(\delta m_{\pi^0}^2)_{\text{scalar int}} = -\frac{\pi^2 g^2}{(2\pi)^4}\frac{m_{\pi^0}^2}{2m_W^2}\left(\frac{2}{4-n}\right)(\tfrac{1}{2}m_{\pi^0}^2 - m_\chi^2), \quad (46)$$

and

$$(Z_2^\pi)_{\text{scalar int}} = 1.$$

Higgs scalar interactions also give rise to tadpole diagrams (53)–(58) shown in Fig. 1 in lowest-order perturbation theory. These diagrams can be attached to all the lines (both internal and external except the neutrino line) of the diagram (1) of Fig. 1, and give extra divergent contributions to the mass counterterms C_2, C_6, C_8, and C_{10} and the W-boson wave-function counterterm C_5 only.

VIII. RENORMALIZABILITY AND Z_2/Z_1 FOR LEPTONS AND PIONS

It is easy to see from expressions (4), (6), (13), (17), (24), (26), (31), and (34) that

$$M_1^{\text{weak}} + M_2^{\text{weak}} + M_3^{\text{weak}} + M_4^{\text{weak}} = 0 \quad (47)$$

and

$$M_1^{\text{em}} + M_2^{\text{em}} + M_3^{\text{em}} + M_4^{\text{em}} = 0. \quad (48)$$

We have explicitly verified that the γ-gauge-dependent terms in (48) cancel among themselves. As pointed out earlier, the modification of the lepton and pion lines does not give rise to any divergent contribution to the matrix element after mass and wave-function renormalization. Therefore it follows from (47) and (48) that in the present theory of pion β decay the electromagnetic and the weak interactions are separately renormalizable.

We also find

$$\left(\frac{Z_2}{Z_1}\right)_l = \frac{(Z_{2,e}Z_{2,\nu})^{1/2}}{Z_{1,l}}$$

$$= 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{1}{2}\frac{m_W^{\,2}}{m_Z^{\,2}} - \frac{23}{12}\right) \qquad (49)$$

and

$$\left(\frac{Z_2}{Z_1}\right)_\pi = \frac{(Z_{2,\pi^-}Z_{2,\pi^0})^{1/2}}{Z_{1,\pi}}$$

$$= 1 - \frac{\pi^2 g^2}{(2\pi)^4}\left(\frac{2}{4-n}\right)\left(\frac{1}{2}\frac{m_W^{\,2}}{m_Z^{\,2}} - \frac{23}{12}\right). \qquad (50)$$

It may be noted that Eq. (49) can be obtained by subtracting Eq. (7) from Eq. (9) of Bollini *et al.*[12] From Eqs. (49) and (50) we conclude that the Z_2/Z_1 ratio is the same for leptons and pions even when the second-order radiative corrections are taken into account. This result is expected from lepton-hadron universality, and, consequently, $C_4 = C_{11}$, as prescribed by multiplicative renormalization.

IX. CONCLUSION

Using the Salam-Weinberg model[4] of leptons and a model of pions[6] in which the strong interactions are not taken into account, we have calculated the second-order radiative corrections to pion β decay in the unitary gauge for W and Z bosons and the gauge (40) for the photon. We have used dimensional regularization to evaluate the divergent integrals. We have shown that the second-order weak and electromagnetic radiative corrections are finite and the present theory of pion β decay is renormalizable. We have also demonstrated the equality of the Z_2/Z_1 ratios for leptons and pions as expected from lepton-hadron universality. We have verified that the above conclusions remain unchanged in the presence of the Higgs scalar interactions, which give additional divergent contributions to the mass counterterms (except that of the neutrino) and the W-boson wave-function counterterm only.

ACKNOWLEDGMENTS

We would like to thank Professor S. N. Biswas and Dr. G. Rajasekaran for many helpful discussions. K. N. is indebted to the Department of Atomic Energy, Government of India, for the award of a Junior Research Fellowship. A. R. wishes to acknowledge a book grant from the University Grants Commission.

---

*Present address: British Council Division, British High Commission, 5 Shakespeare Sarani, Calcutta, India, 700016.

[1]G. 't Hooft, Nucl. Phys. B35, 167 (1971).

[2]S. Weinberg, Phys. Rev. Lett. 27, 1688 (1971); A. Salam and J. Strathdee, Trieste Report No. IC/71/145 (unpublished); A. Slavnov, Kiev Report No. ITP/71/83E (unpublished); C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Orsay report, 1972 (unpublished); C. P. Korthals Altes and M. Perrottet, CNRS report, 1972 (unpublished); P. Budini and P. Furlan, Trieste Report No.IC/72/21 (unpublished); B. W. Lee, Phys. Rev. D 5, 823 (1972); B. W. Lee and J. Zinn-Justin, *ibid.* 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972); G. 't Hooft and M. T. Veltman, Nucl. Phys. B44, 189 (1972).

[3]G. Rajasekaran, Phys. Rev. D 6, 3032 (1972); S. Borchardt and K. T. Mahanthappa, Nucl. Phys. B63, 445 (1973); D. A. Ross, *ibid.* B51, 116 (1973); T. W. Appelquist, J. R. Primack, and H. R. Quinn, Phys. Rev. D 6, 2998 (1972); 7, 2998 (1973); T. W. Appelquist and H. R. Quinn, Phys. Lett. 39B, 229 (1972); D. Bailin, Nuovo Cimento 14A, 199 (1973); S. Y. Lee, UCSD report, 1972 (unpublished); D. A. Ross and J. C. Taylor, Nucl. Phys. B51, 125 (1973); T. Hagiwara, Phys. Rev. D 9, 1023 (1974).

[4]A. Salam, in *Elementary Particle Physics: Relativistic Groups and Analyticity* (Nobel Symposium No. 8), edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367; S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); 27, 1688 (1971).

[5]A. Sirlin, Phys. Rev. Lett. 32, 966 (1974).

[6]S. N. Biswas and P. Ghose, Visva-Bharati University report, 1974 (unpublished).

[7]S. Weinberg, Phys. Rev. D 5, 1412 (1972).

[8]J. Schechter and Y. Ueda, Phys. Rev. D 2, 736 (1970); S. L. Glashow, J. Iliopoulos, and L. Maiani, *ibid.* 2, 1285 (1970); K. Bardakci and M. B. Halpern, *ibid.* 6, 696 (1972); I. Bars, M. B. Halpern, and M. Yoshimura, Phys. Rev. Lett. 29, 969 (1971); D. A. Dicus and V. S. Mathur, Phys. Rev. D 7, 525 (1973); J. C. Pati and A. Salam, *ibid.* 8, 1240 (1973); B. de Wit, Nucl. Phys. B51, 237 (1973); K. Bardakci, *ibid.* B51, 174 (1973); S. Weinberg, Phys. Rev. D 5, 1412 (1972); J. D. Bjorken and S. L. Glashow, Phys. Lett. 11, 255 (1964); K. Tanaka, Ohio State Report No.COO-1545-103, 1972 (unpublished); B. W. Lee, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 4, p. 249; C. H. Llewellyn Smith, CERN Report No. CERN-

TH 1710, 1973 (unpublished); R. N. Mohapatra and S. Sakakibara, Phys. Rev. D 9, 429 (1974); E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[9]K. Niyogi and A. K. Ray, Phys. Rev. D 12, 892 (1975).

[10]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); CERN Report No. 73-9, 1973 (unpublished).

[11]Diagrams (14), (15), and (16) do not give any divergent contribution to the matrix element after W-boson mass and wave-function renormalization. The contribution of diagram (14) containing a fermion loop has been calculated as in S. Borchardt and K. T. Mahanthappa, Nucl. Phys. B63, 445 (1973).

[12]C. G. Bollini, J. J. Giambiagi, and A. Sirlin, Nuovo Cimento 16A, 423 (1973).