Three-pion scattering amplitudes and integral equations*

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The isospin structure of the three-pion-to-three-pion scattering amplitude is studied using a generalization of the tensor analysis which is well known in two-pion scattering. This leads to a natural definition of the various 3π-to-3π isoscalar amplitudes, a simple deduction of the consequences of Bose statistics for these amplitudes, and a straightforward technique for carrying out the isospin reduction of the full scattering amplitude and corresponding scattering integral equations without the use of recoupling techniques. Our general results are applicable to a very wide class of three-pion scattering integral equations; however, we only consider in explicit detail the so-called K-matrix formalism. A new derivation of the latter is given specifically for 3π-to-3π scattering in order to illustrate the generation of connected-kernel scattering equations for three identical particles without the usual introduction of (interim) unphysical operators and amplitudes which are defined in terms of the interactions of specific particles.

I. INTRODUCTION

Some years ago Basdevant and Kreps[1] made the first serious attempt to understand the resonant states of the three-pion system, such as the ω, in terms of a particular model for 3π-to-3π scattering. Given scattering integral equations for the 3π-to-3π amplitudes in each of the four three-pion total-isospin states one identifies the appearance of a 3π resonance with a relative maximum (for real energies) or a pole (for unphysical energies) in the inverse of the Fredholm determinants of these equations as a function of the total c.m. energy.

The basic physical picture, apart from all of the poorly understood approximations introduced to obtain tractable integral equations, underlying the model of Ref. 1 is the idea that elastic 3π scattering could be viewed solely in terms of the pair-wise (off-shell) scatterings. Given the full two-pion (off-shell) amplitude as input, one is then left with a standard three-particle description of the entire scattering process. While this picture apparently constitutes a fairly good description of reality for the low-energy three-nucleon problem, its extension to three-pion scattering is open to some question.

The calculations of Ref. 1 as well as those carried out along the same lines by Mennessier *et al.* [2] were unsuccessful in predicting the properties of the 3π system for c.m. energies less than 1.5 GeV, while for energies much above this the model is inapplicable owing to inelastic effects. One of the easily identifiable reasons for this was the neglect of all but the isovector two-pion scattering in the $I\neq 0$ states. In the preceding picture of the scattering process this leads to a degeneracy between the $I=1$ and $I=2$ resonances. Other unsatisfactory aspects of these calculations are not so readily explained.

A much more satisfactory formulation of 3π-to-3π scattering has been introduced recently by Brayshaw in terms of a relativistic three-particle version of the boundary-condition model.[3] Here physically reasonable assumptions are introduced in order to specify the nature of the off-shell propagation of pions under specific kinematical conditions and to handle (phenomenologically) those aspects of the dynamics which do not fall into the picture of the scattering as a succession of two-pion scatterings. The calculations carried out thus far using this model in the ω and A_1 channels are quite impressive and give strong support to the basic physical idea of the pair-collision picture of the 3π system at least for c.m. energies less than 1.5 GeV.[3]

Another calculation[4] using the so-called minimal K-matrix model[5] has also been carried out recently in conjunction with an analysis of $\pi^- p\to\pi^-\pi^+\pi^- p$ scattering. The A_1 channel was considered in detail but the results were quite uninformative. The other isospin states were not investigated nor were the properties of the 3π-to-3π scattering amplitude considered in isolation.

The minimal K-matrix model[4,5] yields a set of on-shell integral equations for the 3π-to-3π scattering amplitude in which the only input consists of the (on-shell) two-pion amplitudes. The primary virtues of the model are its simplicity and the fact that it yields unitary 3π-to-3π amplitudes below the inelastic threshold. Although its defects are obvious (most notably its misrepresentation of the double-scattering poles) more extensive calculations would be interesting for several reasons[5] and these calculations are in progress.[6]

It is not our intention to explore the relative virtues of either Brayshaw's model or the K-

matrix approach (either in its minimal or general forms) any further. In view of the importance of the 3π system it is of considerable interest to explicate the general features of the 3π-to-3π scattering amplitude and of a generic class of integral equations which are used to generate this amplitude.

We have specific reference, first, to a natural, or at least standardized, definition of the various total-isospin amplitudes and the implications of Bose statistics upon them, particularly for total isospins $I=1, 2$. In the latter case because the relevant representation spaces are reducible the choice of these amplitudes is hardly unique and several choices have appeared in the literature.[7,8]

In Sec. II we develop a Cartesian-tensor analysis of the 3π-to-3π scattering amplitude which is a (nontrivial) generalization of the formalism familiar in the two-pion problem.[9] Using this we are able to deduce canonical forms for the complete amplitude as well as its total-isospin projections and for which the consequences of Bose statistics are explicit. It is found that the complete amplitude is characterized by only two isoscalar functions and various permutations of their arguments. Also the tensor analysis is exploited to obtain in a very simple and explicit manner the total-isospin projections of any sort of 3π-to-3π scattering integral equation without explicit recourse to isospin recoupling schemes.

This formalism is applied in Sec. V, by way of example, to the K-matrix integral equations for 3π-to-3π scattering. Before doing this we give in Sec. IV a somewhat more satisfactory derivation of this formalism in which the pion identity is properly taken into account throughout. This avoids the usual benign artifice of assuming particle distinguishability initially and then imposing the symmetries required by identity afterwards. In multiparticle scattering problems this device is resorted to primarily to define the various Faddeev decompositions[10] of the full scattering amplitude. Essentially, we show how to carry out this decomposition in three-particle scattering without the introduction of particle-indexed scattering operators. The latter do not arise in any natural fashion in field-theoretic representations of the scattering amplitude, for example; however, the present technique can be directly related to such representations.

We do not consider the partial-wave analysis of the 3π-to-3π amplitudes or of the corresponding integral equations. This has been done most recently by Lock[8] who has introduced a new relativistic partial-wave decomposition which involves a consistent definition of the two-particle relative momentum in the arbitrary Lorentz frames required to describe the various two-particle scatterings in the over-all three-particle c.m. frame.

II. THREE-PION TENSOR ANALYSIS

Let us outline the general problem of extracting the consequences of charge independence upon the structure of the 3π-to-3π scattering amplitude, $\langle 3\pi|T|3\pi\rangle$. The complexity of the problem arises from the fact that the irreducible subspaces resulting from the reduction of the direct product of the single-pion spaces are not uniquely labeled by the total isospin I:

$$[1]\times[1]\times[1] = [0]+[1]+[1]'+[1]''+[2]+[2]'+[3].$$

Charge independence does not forbid transitions from, say, $[1]$ to $[1]'$. The subspaces degenerate in I are distinguished by their transformation properties with respect to S_3, the permutation group on three objects. It is elementary to deduce from charge independence that there are 15 independent isoscalars appearing in the reduction of $\langle 3\pi|T|3\pi\rangle$ onto subspaces of I, one each for the irreducible spaces $I=0, 3$, nine corresponding to $I=1$, and four to $I=2$.

Several different approaches for excuting the isospin reduction of $\langle 3\pi|T|3\pi\rangle$ exist, and they all depend upon one isospin recoupling scheme or another.[7,8] The present treatment, which differs markedly in style to these, is based on the generalization of the tensor technique which is well-known in standard treatments of two-pion scattering.[9] In the two-pion case this technique is introduced to explicate the implications of generalized Bose statistics (including crossing) while the aspects of the isospin reduction are entirely trivial. In the three-pion case the results of the application of this technique to the isospin reduction problem are far from obvious. We will find, as in the two-pion case, that this approach readily lends itself to the explication of the symmetries implied by Bose statistics. Overall, this method appears to possess several advantages in simplicity and generality over the standard recoupling techniques.

We chose a Cartesian basis for the single-pion isospin states, $|\alpha\rangle$, $\alpha=\bar{1},\bar{2},\bar{3}$, where

$$|\bar{1}\rangle = \frac{1}{\sqrt{2}}(|\pi^+\rangle + |\pi^-\rangle),$$

$$|\bar{2}\rangle = \frac{-i}{\sqrt{2}}(|\pi^+\rangle - |\pi^-\rangle),$$

$$|\bar{3}\rangle = |\pi^0\rangle,$$

and where we have introduced bars over the integers to avoid confusion with the notation used below for the enumeration of the elements of S_3.

We denote a three-pion ("in" or "out") state by $|\alpha, \beta, \mu\rangle$. In this section we completely suppress any explicit dependences upon the single-pion kinematic variables (such as momentum); in later sections α will be taken to be a complete set of single-pion variables.

The simplicity of our entire formalism depends crucially upon the introduction of a convenient notation for the permutations on the single-pion variables α, β, μ. The triplet $(\alpha_j, \beta_j, \mu_j)$ refers to the cyclic permutation j $(=1, 2, 3)$ of the ordered set (α, β, μ), where $j=1$ will be taken to correspond to the identity permutation. A caret over an index, e.g., $\hat{j}$, will denote the permutation (cyclic) inverse to the permutation denoted by that index. Remembering that α, e.g., is a member of an ordered set, we define

$$\alpha_{jk} \equiv (\alpha_j)_k = \alpha_{kj}$$

and then note that

$$\alpha_{j1} = \alpha_j, \quad \alpha_{jj} = \alpha_{\hat{j}}, \quad \alpha_{j\hat{j}} = \alpha,$$

which define the multiplication rules for the permutation *indices*, namely $jk = kj$, $j1 = j$, $jj = \hat{j}$, $j\hat{j} = 1$.

We will deal solely with what we term three-pion operators, $\mathcal{O}$. Suppose we have an arbitrary operator, $\mathcal{O}_H$, such as the S or T operators, defined on the entire hadronic space. Let $P_{3\pi}$ be a projection operator on the three-pion (in or out) space.[5] Then the three-pion operator $\mathcal{O}$ corresponding to $\mathcal{O}_H$ is defined by $\mathcal{O} \equiv P_{3\pi}\mathcal{O}_H P_{3\pi}$.

The full content of charge independence of an arbitrary three-pion operator $\mathcal{O}$ is contained in the stipulation that its matrix elements satisfy the transformation law for a tensor of rank 6:

$$\begin{aligned}\langle \gamma, \rho, \nu|\mathcal{O}|\alpha, \beta, \mu\rangle &= \sum (D^{-1})_{\gamma, a'}(D^{-1})_{\rho, b'}(D^{-1})_{\nu, c'} \\ &\quad \times \langle a', b', c'|\mathcal{O}|a, b, c\rangle D_{a,\alpha} D_{b,\beta} D_{c,\mu},\end{aligned} \tag{2.1}$$

where the sum is over repeated indices and $D_{\alpha,\mu}$, for instance, is an element of a real unitary rotation matrix. In view of our previous discussion it is evident that there are at most 15 linearly independent solutions of (2.1).

Let us introduce the operators, which are taken to act like the identity with respect to the kinematic variables,

$$\langle \gamma, \rho, \nu|\kappa(j)|\alpha, \beta, \mu\rangle \equiv \delta^{\gamma}_{\alpha_j}\delta^{\rho}_{\beta_j}\delta^{\nu}_{\mu_j} = \delta^{\gamma_{\hat{j}}}_{\alpha}\delta^{\rho_{\hat{j}}}_{\beta}\delta^{\nu_{\hat{j}}}_{\mu}, \tag{2.2a}$$

$$\langle \gamma, \rho, \nu|p(j)|\alpha, \beta, \mu\rangle \equiv \delta^{\gamma}_{\alpha_j}\delta^{\rho}_{\mu_j}\delta^{\nu}_{\beta_j} = \delta^{\gamma_j}_{\alpha}\delta^{\rho_j}_{\mu}\delta^{\nu_j}_{\beta}, \tag{2.2b}$$

$$\langle \gamma, \rho, \nu|\tau_{i,j}|\alpha, \beta, \mu\rangle \equiv \delta^{\gamma_i}_{\alpha_j}\delta^{\rho_i, \nu_i}\delta_{\beta_j, \mu_j}. \tag{2.2c}$$

Evidently $\kappa(j)$ $[p(j)]$ introduces a cyclic [anticyclic] permutation of the isospin variables while $\tau_{i,j}$ induces a pair-wise trace.[11] It is trivial to verify that $\{\kappa(j), p(j), \tau_{i,j}\}$ constitutes a set of 15 linearly independent operators each of which satisfies (2.1).

Thus, for any $\mathcal{O}$ which satisfies (2.1) we have the general decomposition into the fundamental tensor operators (2.2):

$$\mathcal{O} = \sum_{i,j} A_{i,j}\tau_{i,j} + \sum_i [B_i p(i) + C_i \kappa(i)]. \tag{2.3}$$

Let $\mathcal{O}_I$ be an operator which satisfies

$$\begin{aligned}\langle \gamma, \rho, \nu|\vec{I}^2\mathcal{O}_I|\alpha, \beta, \mu\rangle &= \langle \gamma, \rho, \nu|\mathcal{O}_I\vec{I}^2|\alpha, \beta, \mu\rangle \\ &= I(I+1)\langle \gamma, \rho, \nu|\mathcal{O}_I|\alpha, \beta, \mu\rangle,\end{aligned} \tag{2.4}$$

where $\mathcal{O}_I$ need not necessarily satisfy (2.1) although later we will confine ourselves only to those operators that do. The square of the total three-pion isospin operator $\vec{I}^2$ can be expressed as

$$\vec{I}^2 = \sum_{a=1}^{3} \vec{I}^{(a)2} + \Gamma, \tag{2.5}$$

where $\vec{I}^{(a)}$ is the isospin operator of the ath pion, $\langle \alpha|I^{(a)}_{\gamma}|\beta\rangle = i\epsilon_{\alpha\gamma\beta}$, and

$$\Gamma \equiv 2\sum_i [p(i) - \tau_{i,i}].$$

Using (2.5) it follows from Eq. (2.4) that

$$[I(I+1) - 6]\mathcal{O}_I = \Gamma\mathcal{O}_I = \mathcal{O}_I\Gamma, \tag{2.6}$$

which in turn implies that

$$[I(I+1) - 2]\mathcal{O}_I\tau_{i,j} = 0, \tag{2.7a}$$

$$[I(I+1) - 2]\tau_{i,j}\mathcal{O}_I = 0, \tag{2.7b}$$

and that $\tau_{i,j}$ is an $\mathcal{O}_I$-type operator corresponding to $I=1$. Clearly

$$\mathcal{O}_I\tau_{i,j} = \tau_{i,j}\mathcal{O}_I = 0, \quad \text{for } I \neq 1. \tag{2.7c}$$

Equations (2.6) and (2.7c) together imply the following relations:

$$\mathcal{O}_0\kappa(j) = \kappa(j)\mathcal{O}_0 = \mathcal{O}_0, \quad \mathcal{O}_3\kappa(j) = \kappa(j)\mathcal{O}_3 = \mathcal{O}_3, \tag{2.8a}$$

$$\mathcal{O}_0 p(j) = p(j)\mathcal{O}_0 = -\mathcal{O}_0, \quad \mathcal{O}_3 p(j) = p(j)\mathcal{O}_3 = +\mathcal{O}_3, \tag{2.8b}$$

$$\sum_i p(i)\mathcal{O}_2 = \mathcal{O}_2 \sum_i p(i) = 0, \tag{2.8c}$$

$$\sum_i \kappa(i)\mathcal{O}_2 = \mathcal{O}_2 \sum_i \kappa(i) = 0. \tag{2.8d}$$

If $\mathcal{O}_I$ satisfies (2.1), it of course has the decomposition

$$\mathcal{O}_I = \sum_{i,j} A_{i,j}^{(I)} \tau_{i,j} + \sum_i B_i^{(I)} p(i) + \sum_i C_i^{(I)} \kappa(i), \tag{2.9}$$

which can be regarded as the projection of (2.3) onto the subspace of total isospin I. Equations (2.6)–(2.8) are constraints on the coefficients in (2.9), and we find with a convenient regrouping of terms

$$\mathcal{O}_0 = C^{(0)} \sum_i [\kappa(i) - p(i)], \tag{2.10a}$$

$$\mathcal{O}_1 = \sum_{i,j} A_{i,j}^{(1)} \tau_{i,j}, \tag{2.10b}$$

$$\mathcal{O}_2 = \sum_i \left\{ B_i^{(2)} \left[p(i) - \tfrac{1}{2} \sum_j \tau_{i\hat{i},j} \right] + C_i^{(2)} \left[\kappa(i) - \tfrac{1}{2} \sum_j \tau_{\hat{i}j,j} \right] \right\}, \tag{2.10c}$$

where

$$\sum_i B_i^{(2)} = \sum_i C_i^{(2)} = 0, \tag{2.10c$'$}$$

and

$$\mathcal{O}_3 = A^{(3)} \left\{ \sum_{i,j} \tau_{i,j} - \tfrac{5}{2} \sum_i [p(i) + \kappa(i)] \right\}. \tag{2.10d}$$

We note that in the $I=2$ case because of the constraint (2.10c$'$) only four independent isoscalars enter into (2.10c). In connection with (2.10a) it is interesting to note that

$$\left\langle \gamma, \rho, \nu \left| \sum_i [\kappa(i) - p(i)] \right| \alpha, \beta, \mu \right\rangle = \epsilon_{\gamma\rho\nu} \epsilon_{\alpha\beta\mu}.$$

Now the projection operators P_I on the subspaces of total isospin I must have the form (2.10) but in addition they also satisfy

$$P_I^2 = P_I. \tag{2.11}$$

Conditions (2.10) and Eq. (2.11) yield the explicit projections

$$P_0 = \tfrac{1}{6} \sum_i [\kappa(i) - p(i)], \tag{2.12a}$$

$$P_1 = \sum_{i,j} (\hat{\delta}^{-1})_{i,j} \tau_{i,j}, \tag{2.12b}$$

$$P_2 = \tfrac{1}{6} \sum_{i,j} (1 - 3\delta_{i,j}) \tau_{i,j} + \sum_i (\delta_{1,i} - \tfrac{1}{3}) \kappa(i), \tag{2.12c}$$

$$P_3 = \tfrac{1}{6} \sum_i [p(i) + \kappa(i)] - \tfrac{1}{15} \sum_{i,j} \tau_{i,j}, \tag{2.12d}$$

where we have from the definition of $\hat{\delta}$ in the Appendix,

$$(\hat{\delta}^{-1})_{i,j} = \tfrac{1}{10}(5\delta_{i,j} - 1).$$

It is evident from Eqs. (2.12) that

$$\sum_{I=0}^{3} P_I = \kappa(1)$$

and that all of the P_I's are Hermitian.

With the aid of the explicit expressions (2.12) for the P_I we can relate the isoscalars appearing in the decomposition (2.3) for $\mathcal{O}$ to those in the decompositions (2.10) for $\mathcal{O}_I$, where

$$\mathcal{O} = \sum_{I=0}^{3} \mathcal{O}_I, \tag{2.13a}$$

$$\mathcal{O}_I \equiv P_I \mathcal{O} P_I. \tag{2.13b}$$

One finds that

$$C^{(0)} = \tfrac{1}{6} \sum_i (C_i - B_i), \tag{2.14a}$$

$$A_{i,j}^{(1)} = A_{i,j} + \sum_k [B_k (\hat{\delta}^{-1})_{i,k\hat{j}} + C_k (\hat{\delta}^{-1})_{i,j\hat{k}}], \tag{2.14b}$$

$$B_i^{(2)} = B_i - \tfrac{1}{3} \sum_k B_k, \quad C_i^{(2)} = C_i - \tfrac{1}{3} \sum_k C_k, \tag{2.14c}$$

$$A^{(3)} = -\tfrac{1}{15} \sum_i (B_i + C_i). \tag{2.14d}$$

Equations (2.3) and (2.10) represent the complete isospin structure of a charge-independent three-pion operator $\mathcal{O}$ and its projections $\mathcal{O}_I$ in terms of the independent isoscalar amplitudes and the 15 fundamental tensor operators. Equations (2.14) provide the connection between the two sets of isoscalar amplitudes. This is all we require for the isospin analysis of the three-pion scattering amplitude as well as the various three-pion scattering integral equations.

Let us consider a generic example of the latter which we write in the condensed operator form

$$T = H + RT, \tag{2.15a}$$

where all quantities in (2.15a) are three-pion operators; in particular, the sum over states implicit in the product RT is restricted to only three-pion states. The total isospin projection of (2.15) is [cf. Eqs. (2.13)]

$$T_I = H_I + R_I T_I. \quad (2.15b)$$

The structure of (2.15b) is trivial in the (irreducible) $I=0$ and $I=3$ cases. In the $I=1$ and $I=2$ cases let us introduce a notation for the isoscalar amplitudes such that $A^{(I)}_{i,j}(T)$, for example, denotes the isoscalar coefficients of the operators $\tau_{i,j}$ in the decomposition of T_I. Then we find that (2.15b) leads to the integral equations

$$A^{(1)}_{i,j}(T) = A^{(1)}_{i,j}(H) + \sum_{k,l} A^{(1)}_{i,k}(R)\hat{\delta}_{k,l} A^{(1)}_{l,j}(T) \quad (2.16)$$

in the $I=1$ case and to

$$B^{(2)}_i(T) = B^{(2)}_i(H) + \sum_j [B^{(2)}_{\hat{i}\hat{j}}(R) C^{(2)}_j(T) + C^{(2)}_{\hat{i}j}(R) B^{(2)}_j(T)], \quad (2.17a)$$

$$C^{(2)}_i(T) = C^{(2)}_i(H) + \sum_j [B^{(2)}_{\hat{i}j}(R) B^{(2)}_j(T) + C^{(2)}_{\hat{i}\hat{j}}(R) C^{(2)}_j(T)], \quad (2.17b)$$

for $I=2$. We will give explicit realizations of these integral equations in Sec. V.

III. BOSE STATISTICS

It remains to investigate the implications of Bose statistics on the isoscalar coefficients appearing in the decompositions (2.3), (2.9), and (2.10) of the 3π-to-3π scattering amplitude $\langle \gamma, \rho, \nu | T | \alpha, \beta, \mu \rangle$ or its total isospin projections $\langle \gamma, \rho, \nu | T_I | \alpha, \beta, \mu \rangle$. In this section the index α, e.g., refers to a complete set of single-pion observables including, of course, the (Cartesian) isospin index.

We recall that the unitary operators $\kappa(j)$ and $p(j)$ induce even and odd permutations, respectively, on the single-pion isospin indices. We denote by $e(j)$ and $o(j)$ the unitary operators which induce the corresponding even and odd permutations on the single-pion kinematic variables; the product rules for these operators are tabulated in the Appendix. We see then that S_3 is realized on the three-pion space by the set of six unitary operators, $U_\eta(i)$, which are defined by

$$U_e(j) \equiv e(j)\kappa(j) = U^\dagger_e(\hat{j}),$$

$$U_o(j) \equiv o(j)p(j) = U^\dagger_o(j),$$

where $\eta = e, o$ refers to even (e) or odd (o) permutations.

Bose statistics implies, then, that for all j and η

$$T = U_\eta(j) T = T U_\eta(j), \quad (3.1)$$

namely, the 3π-to-3π amplitude is invariant with respect to permutations of the initial- or of the final-particle indices. The general class of amplitudes which result from the Faddeev decompositions of T, which are introduced to obtain connected-kernel equations, will not, of course, possess the full symmetry (3.1).

We can define a generalized, albeit nonunique, Faddeev decomposition for the case of three bosons by relating T to another three-pion operator $\overline{T}$ by means of

$$T = \sum_j U^\dagger_e(j) \overline{T}. \quad (3.2)$$

We infer from (3.1), then, that for an operator related to T by (3.2) Bose statistics requires

$$\overline{T} = U_o(1)\overline{T} \quad (3.3a)$$

$$= \overline{T} U_\eta(j). \quad (3.3b)$$

Let us write T in the form (2.3):

$$T = \sum_{k,l} A_{k,l}\, \tau_{k,l} + \sum_k [B_k p(k) + C_k \kappa(k)]. \quad (3.4)$$

Then Eqs. (3.1) require the following transformation properties with respect to $e(j)$ and $o(j)$:

$$A_{k\hat{j},l} = e(j) A_{k,l} = A_{k\hat{j},l\hat{j}}\, e(j), \quad (3.5a)$$

$$B_{kj} = e(\hat{j}) B_k = B_k e(j), \quad (3.5b)$$

$$C_{kj} = e(j) C_k = C_k e(j), \quad (3.5c)$$

$$A_{\hat{k}j,l} = o(j) A_{k,l} = A_{\hat{k}j,\hat{l}j}\, o(j), \quad (3.5d)$$

$$C_{k\hat{j}} = o(j) B_k = B_{\hat{k}}\, o(\hat{j}), \quad (3.5e)$$

$$B_{kj} = o(j) C_k = C_{\hat{k}}\, o(j). \quad (3.5f)$$

Equations (3.5) have several interesting consequences. It is obvious from (3.5a) that all of the $A_{k,l}$ can be generated from $A_{1,1}$ by cyclic permutations on the initial and final variables. From (3.5d) we see that

$$A_{1,1} = o(1) A_{1,1} = A_{1,1} o(1), \quad (3.6)$$

from which we infer that $A_{1,1}$ has the general form

$$A_{1,1} = [1 + o(1)]\, \mathfrak{A}\, [1 + o(1)], \quad (3.7)$$

where $\mathfrak{A}$ has no constraints with respect to permutations. We obtain, then, from (3.7)

$$A_{k,l} = [e(\hat{k}) + o(k)]\, \mathfrak{A}\, [e(l) + o(l)], \quad (3.8)$$

which in matrix form provides a canonical expression for all of the $A_{k,l}$ in terms of a single isoscalar amplitude, $\mathfrak{A}$.

One can deduce similar canonical forms for B_k and C_k. We note that all of the B_k and C_k can be generated from C_1 using Eqs. (3.5). It follows from (3.5c) and (3.5f) that

$$C_1 = e(\hat{j})C_1 e(j) = o(j)C_1 o(j), \tag{3.9}$$

namely C_1 is invariant with respect to S_3. Equations (3.9) have the trivial consequence

$$C_1 = \frac{1}{6}\sum_j [e(\hat{j})C_1 e(j) + o(j)C_1 o(j)]. \tag{3.10}$$

What is not trivial is the observation that Eqs. (3.9) are satisfied identically using (3.10) for C_1 without the use of any of the properties of the C_1 which appears on the right-hand side of (3.10). This along with Eqs. (3.9) implies that C_1 has the general form

$$C_1 = \sum_j [e(\hat{j})\mathfrak{C}e(j) + o(j)\mathfrak{C}o(j)], \tag{3.11}$$

where $\mathfrak{C}$ is an arbitrary isoscalar. Then from (3.5) and (3.11) we obtain

$$B_k = \sum_j [o(k\hat{j})\mathfrak{C}e(j) + e(\hat{k}j)\mathfrak{C}o(j)], \tag{3.12a}$$

$$C_k = \sum_j [e(k\hat{j})\mathfrak{C}e(j) + o(\hat{k}j)\mathfrak{C}o(j)], \tag{3.12b}$$

the matrix forms of which yield canonical forms for all of the B_k and C_k in terms of a single isoscalar amplitude. This completes our deduction of the consequences of charge independence and Bose statistics upon the full 3π-to-3π scattering amplitude.

It is of considerable interest to apply the preceding results on the full amplitude to the projections, T_I, on the total-isospin subspaces. T_I, of course, has the decomposition

$$T_I = \sum_{k,l} A_{k,l}^{(I)}\tau_{k,l} + \sum_k [B_k^{(I)}p(k) + C_k^{(I)}\kappa(k)]. \tag{3.13}$$

This can be written in the forms given by Eqs. (2.10). Relative to the latter we introduce the following changes in notation pertinent to (3.13), $\tau_0 = 6C^{(0)}$, $T_{i,j} = A_{i,j}^{(1)}$, $F_i = B_i^{(2)}$, $G_i = C_i^{(2)}$, and $\tau_3 = -15A^{(3)}$. We find using Eqs. (3.5), (3.8), and (3.12) that in the $I = 0, 3$ cases

$$\begin{aligned}\tau_0 &= e(j)\tau_0\\ &= \tau_0 e(j)\\ &= -o(j)\tau_0\\ &= -\tau_0 o(j)\\ &= \sum_{k,l}[e(k) - o(k)]\mathfrak{C}[e(l) - o(l)],\end{aligned} \tag{3.14a}$$

$$\begin{aligned}\tau_3 &= e(j)\tau_3\\ &= \tau_3 e(j)\\ &= +o(j)\tau_3\\ &= +\tau_3 o(j)\\ &= \sum_{k,l}[e(k) + o(k)]\mathfrak{C}[e(l) + o(l)].\end{aligned} \tag{3.14b}$$

For $I = 1$ we see that the $T_{k,l}$ satisfy Eqs. (3.5a) and Eqs. (3.5d) so

$$T_{k,l} = [e(\hat{k}) + o(k)]\bar{\mathfrak{A}}[e(l) + o(l)], \tag{3.15}$$

where

$$\bar{\mathfrak{A}} = \mathfrak{A} + \frac{1}{2}\sum_j o(j)\mathfrak{C}o(j) - \frac{1}{40}\sum_{k,l}[e(k) + o(k)]\mathfrak{C}[e(l) + o(l)].$$

Finally, for $I = 2$ it is easily seen that

$$F_{ij} = e(\hat{j})F_i = F_i e(j), \tag{3.16a}$$

$$G_{\hat{i}j} = o(\hat{j})F_{\hat{i}} = F_i o(j), \tag{3.16b}$$

$$\sum_i F_i = \sum_i G_i = 0, \tag{3.16c}$$

where (3.16c) is entirely independent of the implications of Bose statistics. Evidently, both F_i and G_i are expressible in terms of $\mathfrak{C}$, e.g.,

$$F_i = \sum_i [o(i\hat{j})\mathfrak{C}e(j) + e(\hat{i}j)\mathfrak{C}o(j)] - \frac{1}{3}\sum_{i,j}[o(k)\mathfrak{C}e(j) + e(k)\mathfrak{C}o(j)], \tag{3.16d}$$

and the corresponding expression for G_i can be obtained by using Eq. (3.16b).

Next, we investigate the consequences of Eqs. (3.3) on $\bar{T}$. We employ the same notation for the various isoscalar components of $\bar{T}$ as we have for those of T except for the addition of an overbar. First of all, from Eq. (3.2) we deduce the connection between the various full amplitudes and the Faddeev-type amplitudes for the various isospin states:

$$\tau_0 = \sum_j e(j)\bar{\tau}_0, \quad \tau_3 = \sum_j e(j)\bar{\tau}_3, \tag{3.17a}$$

$$T_{k,l} = \sum_j e(j)\bar{T}_{kj,l}, \tag{3.17b}$$

$$F_i = \sum_j e(j)\bar{F}_{ij}, \tag{3.17c}$$

$$G_i = \sum_j e(j)\bar{G}_{i\hat{j}}. \tag{3.17d}$$

The consequences of Eqs. (3.3) which lead to relationships similar to Eqs. (3.5) are rather easily derived. It should be noted that these transformation equations are not as extensive as Eqs.

(3.5) because of the limited symmetry (3.3a) nor do they have as many interesting implications. Of the latter only the following will be of any use to us:

$$\overline{\tau}_0 = -o(1)\overline{\tau}_0, \quad \overline{\tau}_3 = +o(1)\overline{\tau}_3, \tag{3.18a}$$

$$\overline{T}_{k,l} = \overline{T}_{k,1} e(l), \tag{3.18b}$$

$$\overline{T}_{\hat{k},l} = o(1)\overline{T}_{k,l}, \tag{3.18c}$$

$$\overline{G}_i = o(1)\overline{F}_i . \tag{3.18d}$$

The demonstration of how one exploits Eqs. (3.18) will be deferred until Sec. V. Of course, we must have the $I=2$ constraints for these components of $\overline{T}$,

$$\sum_i \overline{F}_i = \sum_i \overline{G}_i = 0, \tag{3.19}$$

and as with (3.16c) this is merely the rephrasing of the consequences of charge independence, rather than Bose statistics, in terms of our new notation.

IV. *K*-MATRIX FORMALISM FOR THREE-PION SCATTERING

Our development follows along the lines of that in Ref. 5 except for the incorporation of Bose statistics. This is done, however, without the usual artifice of assuming, initially, that the particles are distinguishable and, thus, also without the introduction of unphysical operators (or amplitudes) which are defined in terms of the interactions of two specific particles.[1-4,8,12-14] The initial assumption of distinguishability was used in Ref. 4 for the special case of the minimal K-matrix formalism; we present an alternative derivation in that case as well as the treatment of the full K-matrix formalism. It should become evident in the course of our discussion that the present procedure of generating connected-kernel three-particle scattering integral equations can be easily adapted to any of the standard off-shell definitions of the three-particle scattering operator.

The disconnected portion T_D of the 3π-to-3π scattering amplitude can be written as

$$\langle \gamma, \rho, \nu | T_D | \alpha, \beta, \mu \rangle = \tfrac{1}{3} \sum_{f,i} \delta(\gamma_f, \alpha_i) t(\rho_f, \nu_f | \beta_i, \mu_i), \tag{4.1}$$

where α, e.g., refers to a complete set of pion variables as in Sec. III. $t(\rho_f, \nu_f|\beta_i, \mu_i)$ denotes the properly symmetrized two-pion elastic scattering amplitude,

$$t(\rho_f, \nu_f|\beta_i, \mu_i) = t(\nu_f, \rho_f|\beta_i, \mu_i) = t(\rho_f, \nu_f|\mu_i, \beta_i), \tag{4.2}$$

which contains, in a plane-wave representation, a δ function arising from two-particle energy-momentum conservation, and $\delta(\gamma_f, \alpha_i)$ is the Dirac-Kronecker δ function in the indicated single-pion variables corresponding to a freely moving particle. The normalization in (4.1) is chosen so that the no-scattering term in the complete 3π-to-3π S matrix is

$$\langle \gamma, \rho, \nu | \alpha, \beta, \mu \rangle = \frac{1}{3!} \sum_i \delta(\gamma_i, \alpha)[\delta(\rho_i, \beta)\delta(\nu_i, \mu) + \delta(\nu_i, \beta)\delta(\rho_i, \mu)].$$

It will prove convenient to write (4.1) in three-pion operator form (see Sec. II) as

$$T_D = \tfrac{1}{3} \sum_{f,i} U_e^\dagger(f) t U_e(i), \tag{4.3}$$

which will also serve to define our notation for the two-particle transition matrices on the three-pion space.

The two-pion t and k matrices are related by the on-shell equation

$$\delta(\gamma, \alpha) t(\rho, \nu|\beta, \mu) = \delta(\gamma, \alpha) k(\rho, \nu|\beta, \mu) - i\pi \sum \delta(\gamma, c) k(\rho, \nu|a, b)\delta(c, \alpha) t(a, b|\beta, \mu) \tag{4.4}$$

and an equation identical to (4.4) except for the interchange of k and t under the summation sign. The sums in (4.4) are over the repeated indices and represent discrete sums and integrations over the three-pion states. Equation (4.4) is defined for arbitrary two-pion relative energy, but below the four-pion threshold, in this energy, two-particle unitary requires that

$$k(\rho, \nu|\beta, \mu)^* = k(\beta, \mu|\rho, \nu).$$

Evidently, $k(\rho, \nu|\beta, \mu)$ also satisfies Eqs. (4.2). Corresponding to the relationship between (4.1) and (4.3) we write instead of (4.4) the three-pion operator equation

$$t = k - i\pi k t, \tag{4.5a}$$

and similarly

$$t = k - i\pi t k. \tag{4.5b}$$

The general (hadronic) T_H and K_H operators are related by

$$T_H = K_H - i\pi K_H T_H = K_H - i\pi T_H K_H, \tag{4.6}$$

and, unitarity implies and is implied by the Hermiticity of K_H.[5] Evidently, we are only interested in the three-pion projection $T = P_{3\pi} T_H P_{3\pi}$ which satisfies

$$T = K - i\pi KT - i\pi P_{3\pi}K_H(1-P_{3\pi})T_H P_{3\pi}, \tag{4.7}$$

where

$$K = P_{3\pi}K_H P_{3\pi} = K^\dagger .$$

For energies below the 5π-threshold, (4.7) reduces to

$$T = K - i\pi KT. \tag{4.8}$$

One can, as in Ref. 4, use (4.8) above the inelastic threshold but if K is Hermitian T will not satisfy the unitarity constraints unless one also accepts rather specific models for the other amplitudes which then become coupled to T by unitarity.[5] Henceforth, we will work only with (4.8) without any stipulation as to whether we are above or below the three-pion inelastic threshold. We comment upon the treatment of the general case represented by (4.7) at the end of the section.

Clearly, Bose statistics requires that K satisfy Eqs. (3.1). Since we can decompose K into disconnected and connected parts,[5]

$$K = \tfrac{1}{3}\sum_{f,i} U_e^\dagger(f) k U_e(i) + K_c, \tag{4.9}$$

we see that K_c satisfies Eqs. (3.1) as well. Next we introduce a Faddeev-like decomposition of K_c,

$$K_c = \sum_f U_e^\dagger(f)\overline{K}_c, \tag{4.10}$$

which can always be done since a possible choice for $\overline{K}_c$ is simply $\frac{1}{3}K_c$. We note that $\overline{K}_c$ satisfies Eqs. (3.3).

We now define

$$\overline{T} \equiv \overline{K}(1 - i\pi T), \tag{4.11}$$

where

$$\overline{K} \equiv \tfrac{1}{3} k \sum_i U_e(i) + \overline{K}_c. \tag{4.12}$$

It follows from Eqs. (4.9)–(4.12) that [cf. Eq. (3.2)]

$$T = \sum_f U_e^\dagger(f)\overline{T} \tag{4.13}$$

so that $\overline{T}$ satisfies Eqs. (3.3).

With the aid of the group multiplication laws for $U_e(f)$ (see Appendix), Eqs. (4.5b) and (4.12), and the fact that $\overline{K}_c$ satisfies (3.3b), it is easily demonstrated that the identity

$$(1 - i\pi t)\overline{T} = (1 - i\pi t)\overline{K}(1 - i\pi T)$$

reduces to the on-shell, connected-kernel equation ($\overline{\delta}_{i,j} = 1 - \delta_{i,j}$)

$$\overline{T} = \tfrac{1}{3}\sum_i tU_e(i) + (1 - i\pi t)\overline{K}_c - i\pi\left\{t\left[\sum_j \overline{\delta}_{j,1}U_e(j)\right] + 3(1 - i\pi t)\overline{K}_c\right\}\overline{T}, \tag{4.14}$$

or with the choice $\overline{K}_c = \frac{1}{3}K_c$,

$$\overline{T} = \frac{1}{3}\left[\sum_i tU_e(i) + \hat{K}\right] - i\pi\left\{t\left[\sum_j \overline{\delta}_{j,1}U_e(j)\right] + \hat{K}\right\}\overline{T}, \tag{4.14$'$}$$

where $\hat{K} = (1 - i\pi t)K_c$. Reverting back to our matrix notation, (4.14′) can be written as

$$\overline{T}(\gamma,\rho,\nu|\alpha,\beta,\mu) = H(\gamma,\rho,\nu|\alpha,\beta,\mu) + \sum R(\gamma,\rho,\nu|a,b,c)\overline{T}(a,b,c|\alpha,\beta,\mu), \tag{4.15}$$

where

$$H = \frac{1}{3}\left[\sum_j t(\gamma,\rho,\nu|\alpha_j,\beta_j,\mu_j) + K_c(\gamma,\rho,\nu|\alpha,\beta,\mu) - i\pi\sum t(\gamma,\rho,\nu|a,b,c)K_c(a,b,c|\alpha,\beta,\mu)\right], \tag{4.16a}$$

$$R = -i\pi\Big[t(\gamma,\rho,\nu|\beta,\mu,\alpha) + t(\gamma,\rho,\nu|\mu,\alpha,\beta) + K_c(\gamma,\rho,\nu|\alpha,\beta,\mu) - i\pi\sum t(\gamma,\rho,\nu|a,b,c)K_c(a,b,c|\alpha,\beta,\mu)\Big]. \tag{4.16b}$$

The minimal equations[4,5] are obtained by setting $K_c = 0$ in the preceding relations. For the kinematical properties of Eq. (4.15), in particular the fact that after a partial-wave analysis one obtains on-shell, finite-domain integral equations, we refer to the previous literature.[4,5,8]

The major complication involved in carrying out a similar analysis in the general case represented by (4.7) is the presence of additional disconnected structure arising from the $(1 - P_{3\pi})$ term on the right-hand side of (4.7). A consistent treatment, which will be given elsewhere, involves the generalization of the disconnected two-particle Heitler equation (4.4) to include sums over the $(1 - P_{3\pi})$ states.

V. ISOSPIN ANALYSIS OF THE K-MATRIX EQUATIONS

We now apply the formalism of Secs. II and III to the 3π-to-3π integral equation (4.15). This is done in full detail for the minimal case, $K_c=0$; it then suffices to outline the procedure for $K_c\neq 0$.

We require, first of all, the (three-pion) isospin projections of the two-pion amplitudes. The isospin structure of these amplitudes is well-known,[9]

$$t(j)\equiv\langle \gamma,\rho,\nu|t|\alpha_j,\beta_j,\mu_j\rangle$$
$$=\delta(\overline{\gamma},\overline{\alpha}_j)\delta^{\gamma}_{\alpha_j}[A(\overline{\rho},\overline{\nu}|\overline{\beta}_j,\overline{\mu}_j)\delta^{\rho}_{\beta_j}\delta^{\nu}_{\mu_j}+B(\overline{\rho},\overline{\nu}|\overline{\beta}_j,\overline{\mu}_j)\delta^{\rho,\nu}\delta_{\beta_j,\mu_j}+C(\overline{\rho},\overline{\nu}|\overline{\beta}_j,\overline{\mu}_j)\delta^{\rho}_{\mu_j}\delta^{\nu}_{\beta_j}]. \quad (5.1)$$

In (5.1) we have distinguished the single-pion Cartesian isospin indices α and the corresponding kinematic variables $\overline{\alpha}$. Then, for example, $\delta(\overline{\gamma},\overline{\alpha})\delta^{\gamma}_{\alpha}$ is the product of a δ function in the kinematic variables and a Kronecker δ in the isospin indices. The isoscalars A, B, and C are related to the two-pion total-isospin amplitudes M_I by

$$A=\tfrac{1}{2}(M_1+M_2),$$
$$B=\tfrac{1}{3}(M_0-M_2), \quad (5.2)$$
$$C=\tfrac{1}{2}(M_2-M_1).$$

Our definitions (5.1) and (5.2) for A, B, C, and M_I are unconventional[9] in the sense that they include the δ function corresponding to two-pion energy-momentum conservation. Actually, for notational brevity it is convenient to reabsorb the factor $\delta(\overline{\gamma},\overline{\alpha}_j)$ which appears in (5.1) back into the isoscalars so that (5.1) can be written in the compact form

$$t(j)=A(j)\kappa(j)+B(j)\tau_{1,j}+C(j)p(j), \quad (5.3)$$

where, e.g.,

$$A(j)\equiv\delta(\overline{\gamma},\overline{\alpha}_j)A(\overline{\rho},\overline{\nu}|\overline{\beta}_j,\overline{\mu}_j).$$

It is useful to note that by definition

$$A(j)=A(1)e(j),$$
$$B(j)=B(1)e(j), \quad (5.4)$$
$$C(j)=C(1)e(j),$$

and, in addition, from Bose statistics

$$B(j)=o(1)B(j)=B(\hat{j})o(1), \quad (5.5a)$$
$$C(j)=o(1)A(j)=A(\hat{j})o(1). \quad (5.5b)$$

Let us define

$$t(j)_I\equiv P_I\,t(j)P_I\,.$$

Then using Eqs. (2.9), (2.10), (2.12), and (2.14) we easily see that

$$t(j)_0=[A(j)-C(j)]P_0,$$
$$t(j)_3=[A(j)+C(j)]P_3, \quad (5.6a)$$

$$t(j)_1=\sum_{k,l}\mathfrak{M}_{k,l}(j)\tau_{k,l}, \quad (5.6b)$$

where

$$\mathfrak{M}_{k,l}(j)=A(j)(\hat{\delta}^{-1})_{kj,l}+C(j)(\hat{\delta}^{-1})_{kl,j}+B(j)\delta_{k,1}\delta_{l,j},$$

and

$$t(j)_2=A(j)\sum_k(\delta_{k,j}-\tfrac{1}{3})\Big[\kappa(k)-\tfrac{1}{2}\sum_l\tau_{\hat{k}l,l}\Big]+C(j)\sum_k(\delta_{k,j}-\tfrac{1}{3})\Big[p(k)-\tfrac{1}{2}\sum_l\tau_{k\hat{l},l}\Big]. \quad (5.6c)$$

It is clear from Eqs. (5.2) and Eqs. (5.6) that $t(j)_0$ and $t(j)_3$ involve only the $I=1$ and $I=2$ two-pion amplitudes, respectively, while $t(j)_2$ does not contain the isoscalar two-pion amplitudes. $t(j)_1$, of course, incorporates all of the possible two-pion isospin states.

Using the notation (5.1) we can rewrite (4.15)–(4.16) as

$$\overline{T}=\frac{1}{3}\Big[\sum_j t(j)+K_c-i\pi t(1)K_c\Big]-i\pi\Big[\sum_j\overline{\delta}_{j,1}t(j)+K_c-i\pi t(1)K_c\Big]\overline{T}, \quad (5.7a)$$

which in the miminal case becomes, simply,

$$\overline{T}=\tfrac{1}{3}\sum_j t(j)-i\pi\Big[\sum_j\overline{\delta}_{j,1}t(j)\Big]\overline{T}. \quad (5.7b)$$

Now using Eqs. (2.15)–(2.17), Eqs. (5.6), and the notation introduced in connection with Eqs. (3.17), we see immediately that the minimal equation (5.7b) reduces to

$$\overline{\tau}_0=\tfrac{1}{3}\sum_j[A(j)-C(j)]-i\pi\sum_j\overline{\delta}_{j,1}[A(j)-C(j)]\overline{\tau}_0, \quad (5.8a)$$

$$\overline{\tau}_3=\tfrac{1}{3}\sum_j[A(j)+C(j)]-i\pi\sum_j\overline{\delta}_{j,1}[A(j)+C(j)]\tau_3 \quad (5.8b)$$

for $I=0,3$. For $I=1$ we obtain the nine coupled equations

$$\overline{T}_{i,l}=\tfrac{1}{3}\sum_j\mathfrak{M}_{i,l}(j)-i\pi\sum_j R^{(1)}_{i,j}\overline{T}_{j,l}, \quad (5.9a)$$

where

$$R^{(1)}_{i,j} \equiv \sum_k \bar{\delta}_{k,1}[A(k)\delta_{ik,j} + C(k)\delta_{k,ij} + \delta_{i,1}B(k)\hat{\delta}_{k,j}]. \quad (5.9b)$$

Finally, for $I=2$ (5.7b) reduces to the six coupled equations

$$\bar{F}_i = \frac{1}{3}\Big[C(i) - \tfrac{1}{3}\sum_j C(j)\Big] - i\pi\sum_j\Big\{\Big[\bar{\delta}_{i\hat{j},1}C(i\hat{j}) - \tfrac{1}{3}\sum_k \bar{\delta}_{k,1}C(k)\Big]\bar{G}_j + \Big[\bar{\delta}_{\hat{i}j,1}A(\hat{i}j) - \tfrac{1}{3}\sum_k \bar{\delta}_{k,1}A(k)\Big]\bar{F}_j\Big\}, \quad (5.10a)$$

$$\bar{G}_i = \frac{1}{3}\Big[A(k) - \tfrac{1}{3}\sum_j A(j)\Big] - i\pi\sum_j\Big\{\Big[\bar{\delta}_{\hat{i}j,1}C(\hat{i}j) - \tfrac{1}{3}\sum_k \bar{\delta}_{k,1}C(k)\Big]\bar{F}_j + \Big[\bar{\delta}_{i\hat{j},1}A(i\hat{j}) - \tfrac{1}{3}\sum_k \bar{\delta}_{k,1}A(k)\Big]\bar{G}_j\Big\}. \quad (5.10b)$$

The uncoupled Eqs. (5.8) for the amplitudes $\bar{\tau}_0(\bar{\gamma},\bar{\rho},\bar{\nu}|\bar{\alpha},\bar{\beta},\bar{\mu})$ and $\tau_3(\bar{\gamma},\bar{\rho},\bar{\nu}|\bar{\alpha},\bar{\beta},\bar{\mu})$ are ready for the application of a partial-wave analysis as in Ref. 8. The constraints (3.18a) are clearly satisfied.

The nine coupled Eqs. (5.9) can easily be reduced to the solution of two coupled integral equations. By virtue of Eq. (3.18b) we need only consider $l=1$. Also from Eq. (3.18c) we can eliminate, say, $\bar{T}_{3,l}$ $[=o(1)\bar{T}_{2,l}]$. Then, using the fact that

$$R^{(1)}_{i,j}o(1) = R^{(1)}_{i,\hat{j}},$$

we obtain as our irreducible system of $I=1$ equations in the minimal case

$$\bar{T}_{1,1} = \tfrac{1}{3}\sum_j \mathfrak{M}_{1,1}(j) - i\pi R^{(1)}_{1,1}\bar{T}_{1,1} - 2i\pi R^{(1)}_{1,2}\bar{T}_{2,1}, \quad (5.11a)$$

$$\bar{T}_{2,1} = \tfrac{1}{3}\sum_j \mathfrak{M}_{2,1}(j) - i\pi R^{(1)}_{2,1}\bar{T}_{1,1} - 2i\pi R^{(1)}_{2,2}\bar{T}_{2,1}. \quad (5.11b)$$

The $I=2$ equations (5.10) simplify to a system of two coupled equations as well. We can eliminate $\bar{G}_i$, say, with the aid of (3.18d). Equations (5.10b) are then superfluous and Eqs. (5.10a) reduce to

$$\bar{F}_i = \frac{1}{3}\Big[C(i) - \tfrac{1}{3}\sum_j C(j)\Big] - i\pi\sum_j R^{(2)}_{i,j}\bar{F}_j, \quad (5.12a)$$

where

$$R^{(2)}_{i,j} = 2\Big[\bar{\delta}_{\hat{i}j,1}A(\hat{i}j) - \tfrac{1}{3}\sum_k \bar{\delta}_{k,1}A(k)\Big], \quad (5.12b)$$

and we have made use of Eq. (5.5b). It is evident from Eqs. (5.12) that the constraint (3.19) is satisfied as it must be. We can use (3.19) to eliminate $\bar{F}_3$, say, from (5.12a) and we obtain, finally, our irreducible $I=2$ equations for $\bar{F}_1$ and $\bar{F}_2$,

$$\bar{F}_1 = \frac{1}{3}\Big[C(1) - \tfrac{1}{3}\sum_j C(j)\Big] + 2i\pi A(2)\bar{F}_1 - 2i\pi[A(2) - A(3)]\bar{F}_2, \quad (5.13a)$$

$$\bar{F}_2 = \frac{1}{3}\Big[C(2) - \tfrac{1}{3}\sum_j C(j)\Big] + 2i\pi[A(2) - A(3)]\bar{F}_1 + 2i\pi A(3)\bar{F}_2. \quad (5.13b)$$

Our irreducible sets of minimal equations (5.8), (5.11), and (5.13) simplify further if the scattering in some of the two-pion isospin states can be neglected. If $M_2=0$, then from (5.2) and (5.5b)

$$A(j) = -C(j) = -A(\hat{j})o(1). \quad (5.14)$$

Then $\bar{\tau}_3=0$ and Eqs. (5.13) reduce to the two uncoupled equations

$$\bar{F}_1 = \frac{1}{3}\Big[C(1) - \tfrac{1}{3}\sum_j C(j)\Big] - 2i\pi C(3)\bar{F}_1, \quad (5.15a)$$

$$\bar{F}_2 = \frac{1}{3}\Big[C(2) - \tfrac{1}{3}\sum_j C(j)\Big] - 2i\pi C(3)\bar{F}_2, \quad (5.15b)$$

and we note that $\bar{F}_i = -\bar{G}_i$ in this case. No essential simplification of the $I=1$ equations, (5.11), occurs as a consequence of (5.14).

If in addition to (5.14) we also have $M_0=0$, then

$$B(j) = 0. \quad (5.16)$$

No further simplifications over that implied by (5.14) obtain for $I\neq 1$ but in the isovector case we see from (5.11) that (5.14) and (5.16) imply

$$\bar{T}_{1,1} = 0, \quad (5.17a)$$

$$\bar{T}_{2,1} = \tfrac{1}{6}[C(2) - C(3)] - 2i\pi C(3)\bar{T}_{2,1}, \quad (5.17b)$$

namely a single integral equation for $\bar{T}_{2,1}$. In this case we observe that the kernels of the $I=1$ and $I=2$ integral equations are identical.

In the general case (5.7a) it should be clear that as a consequence of the Bose symmetry and isospin constraints (3.18) and (3.19), respectively, we must obtain equations of a complexity no greater than (5.8), (5.11), and (5.13). Namely, for $I=0$ and $I=3$ we obtain a single integral equation in each case. For $I=1$ we get two irreducible coupled integral equations and similarly for $I=2$.

In order to write down some of these equations in the general case we need a notation for the various isoscalar components of K_c. Let $\bar{K}_0$ and

$\bar{K}_3$ bear the same relation to K_c that $\bar{\tau}_0$ and $\bar{\tau}_3$ do to $\bar{T}$. Then, it follows by inspection of (5.7a) and (5.8) that

$$\bar{\tau}_0 = \frac{1}{3}\left\{\sum_j [A(j) - C(j)] + \bar{K}_0 - i\pi[A(1) - C(1)]\bar{K}_0\right\} - i\pi\left\{\sum_j \bar{\delta}_{j,1}[A(j) - C(j)] + \bar{K}_0 - i\pi[A(1) - C(1)]\bar{K}_0\right\}\bar{\tau}_0, \tag{5.18a}$$

$$\bar{\tau}_3 = \frac{1}{3}\left\{\sum_j [A(j) + C(j)] + \bar{K}_3 - i\pi[A(1) + C(1)]\bar{K}_3\right\} - i\pi\left\{\sum_j \bar{\delta}_{j,1}[A(j) + C(j)] + \bar{K}_3 - i\pi[A(1) + C(1)]\bar{K}_3\right\}\bar{\tau}_3. \tag{5.18b}$$

The $I=1$ case is interesting in that it provides another application of the product rule implicit in Eq. (2.16). One finds, with $K_{i,j}$ bearing the same relation to K_c that $\bar{T}_{i,j}$ does to $\bar{T}$,

$$\bar{T}_{i,l} = H^{(1)}_{i,l} - i\pi\sum_j \bar{R}^{(1)}_{i,j}\bar{T}_{j,l}, \tag{5.19a}$$

where

$$H^{(1)}_{i,l} = \tfrac{1}{3}\sum_j \mathfrak{M}_{i,k}(j) + K_{i,l} - i\pi\sum_{j,k}\mathfrak{M}_{i,j}(1)\hat{\delta}_{j,k}K_{k,l}, \tag{5.19b}$$

$$\bar{R}^{(1)}_{i,j} = R^{(1)}_{i,j} + \sum_k K_{i,k}\hat{\delta}_{k,j} - i\pi\sum_{k,l,m}\mathfrak{M}_{i,k}(1)\hat{\delta}_{k,l}K_{l,m}\hat{\delta}_{m,j}. \tag{5.19c}$$

Again the nine coupled equations (5.19a) can be reduced to two coupled equations with the same general form as Eqs. (5.11). So long as $K_{i,j} \neq 0$ we obtain no simple case such as Eqs. (5.17) when $M_0 = M_2 = 0$; similar remarks apply to Eq. (5.18).

The same general form of equations as (5.13) obtain in the $I=2$ case after the use of the Bose and isospin constraints on $\bar{F}_i$ and $\bar{G}_i$.

It is worth pointing out in connection with the use of the general K-matrix equations that since K_c is fully symmetric, i.e., it satisfies Eqs. (3.1), its isoscalar amplitude in each of the total isospin states can be placed in the canonical forms (3.14), (3.15), and (3.16d).

VI. GENERAL LINEAR THREE-PION INTEGRAL EQUATIONS

The preceding section consisted of the application of the general formalism of Secs. II and III to the K-matrix formulation of 3π-to-3π scattering. The general forms of the equations in each isospin state and the number of independent integral equations in each total-isospin state have an applicability which obviously transcends the K-matrix model.

The essential ingredients of our discussion in Sec. V were the general properties of the quite arbitrary Faddeev decomposition defined by (3.2) and (3.3) and the fact that $\bar{T}$, which is the seminal Faddeev subamplitude, satisfies a linear integral equation of the form (4.15) where H and R are three-pion-type amplitudes which possess symmetry properties which suffice for the satisfaction of Eqs. (3.3). No special kinematical conditions, such as the on-shell constraints which appear in the K-matrix formalism, entered into the discussion of Sec. V. Correspondingly, whether or not (4.15) is an off-shell equation plays any role in our general results concerning the isospin analysis.

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APPENDIX

The products of the operators $\tau_{i,j}$, $\kappa(i)$, and $p(i)$ are easily derived from their definitions (2.2). Our primary purpose in displaying them in detail is to make explicit our index notation for permutations in relation to manipulations with these operators. We see, not unexpectedly, that the $\{\kappa(i), p(i)\}$ constitute a representation of S_3. We find from (2.2) that

$$\tau_{i,j}\tau_{k,l} = \hat{\delta}_{j,k}\tau_{i,l},$$

where

$$\hat{\delta}_{j,k} \equiv 2\delta_{i,j} + 1,$$

$$\tau_{i,k}\kappa(j) = \tau_{i,jk}, \quad \tau_{i,k}p(j) = \tau_{i,j\hat{k}},$$

$$\kappa(j)\tau_{i,k} = \tau_{i\hat{j},k}, \quad p(j)\tau_{i,k} = \tau_{\hat{i}j,k},$$

$$\kappa(i)\kappa(j) = \kappa(ij), \quad p(i)p(j) = \kappa(\hat{i}j),$$

$$p(i)\kappa(j) = p(ij), \quad \kappa(i)p(j) = p(\hat{i}j).$$

The operators $e(j)$ and $o(j)$ also constitute a representation of S_3:

$$e(i)e(j) = e(ij) = e(ji), \quad o(i)o(j) = e(\hat{i}j),$$

$$o(i)e(j) = o(ij), \quad e(i)o(j) = o(\hat{i}j).$$

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