Geometrical approach to local gauge and supergauge invariance: Local gauge theories and supersymmetric strings*

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A geometrical analysis of local gauge invariance is presented. By extending the four directions in space-time which can define a light cone to an enlarged tangent space, we construct a superspace in which the intimate relationship between purely kinematic invariances and those of local internal symmetry groups are explored. Three applications are considered: (1) Local gauge theories in flat or curved space-time are derived from a scalar action in superspace. As a result, the theory has an extra dimensional parameter. (2) Born-Infeld electrodynamics and its non-Abelian counterpart are extended to curved space-time. This suggests a new set of gravitational field equations. (3) The formalism is extended to the case in which the connection forms anticommute and supersymmetric string models are constructed, which give action-principle formulations of the known dual models involving fermions and suggest possibilities for constructing new models. Applications to gravitation and local supersymmetry are pointed out.

I. INTRODUCTION

Theories unifying weak and electromagnetic interactions offer an attractive framework for the analysis of particle interactions. The basic principle underlying all such schemes is that of local gauge invariance. The differences in the strengths of different interactions arise from some sort of spontaneous symmetry breaking which leaves one or more of these invariances exact at the final stage. This is especially true if one attempts to extend this description to strong interactions.

Exact local gauge invariance is defined over Minkowski space-time. As a consequence, gauge theories endow space-time with a richer structure than that implied by the relativity theory and suggest the existence of intimate relationships between purely kinematic invariances, arising from the geometry of space-time, and internal-symmetry groups. The study of such relationships would entail a geometrical analysis of local gauge invariance. In this paper we present the results of one such analysis.

Special relativity defines a particular kind of geometry for space-time by associating with each point four directions from which a light cone can be constructed. The light-cone structures at different points are all equivalent, however, since Poincaré transformations act transitively in Minkowski space. One then constrains the dynamics to preserve such a light-cone structure. For local gauge theories, which must necessarily satisfy this requirement, we argue that the natural geometry is defined in a larger manifold and has a more complicated structure than that described above. We shall associate with each point a tangent space which contains more than four directions and which may be imagined to be broken into "horizontal" and "vertical" sectors. The horizontal sector contains the previous four directions from which a light cone can be constructed. Once again we require the equivalence of these light cones with respect to Poincaré transformations. There is no *a priori* reason to require such equivalence for the directions in the vertical sector, and we shall impose no such restrictions. We then study the general transformations which correlate such a structure at different points. An important feature of these transformations which we wish to emphasize is that they contain local gauge transformations of Yang-Mills fields, thus justifying such a geometrical description of local gauge invariance. Our description of gauge transformations as rotations of the base vectors in the enlarged tangent space at each point is similar to the one in general relativity. Indeed, our quantitative discussion below will utilize many of the techniques of this field where it is imperative that the light cone be treated as a local concept.

It will be useful at this point to compare the present approach to gauge invariance with the more conventional one. Within the latter scheme, one argues that[1-3] the phases of various functions may be defined locally and looks for ways of correlating such phases at different points, thus generating gauge fields. In the present framework, the differential elements defining the extended tangent space are precisely identifiable with the differentials of local phase in the conventional approach.[4] However, the geometrical picture is advantageous for at least three reasons. Firstly, it enables one to follow more closely the dynamics of gauge fields, since the possible singularities are now amenable to classification by powerful differential geometric methods. Secondly, the generalizations to local gauge theories in which the relevant algebras are not Lie algebras but graded Lie or nonassociative algebras, appear to be more tractable in our approach. In this paper we shall consider one such generalization to supergauge symmetries.[5] Finally, the framework is well suited for the inclusion of gravitation, both from the point of view in which it is itself viewed as a gauge theory and from one in which it is a prescribed curved manifold.[6-15] A detailed discussion of gravitation and local supersymmetries from this point of view will be given in a separate paper.

This paper is organized as follows: In Sec. II we introduce, from an intuitive point of view, various concepts necessary for a quantitative development of the scheme. A more complete description of the technical apparatus which goes along with these concepts will be postponed until Sec. V, in which we identify our superspace as a prototype of what is known mathematically as a fiber bundle and discuss its differential geometry. In particular, we obtain expressions for its connection coefficients and scalar curvature, identify the gauge potentials and field tensors, and show explicitly that gauge transformations are just rotations of the basis vectors of the superspace.

In Sec. III, we apply these concepts to non-Abelian gauge theories with or without gravity. One of the interesting results which emerges is that the actions dictated by our geometry are endowed with an additional dimensional parameter which is the curvature scalar of the group manifold. One can, if one wishes, relate this parameter to the bag constant.[16]

In Sec. IV we extend the Born-Infeld electrodynamics[17] and its non-Abelian counterpart to curved space-time without altering the geometrical structure discussed in previous sections. The action of this theory suggests a new set of gravitational field equations. The presence of the extra dimensional parameter is crucial in giving both the correct Born-Infeld theory with no gravitation and Einstein's theory in the first approximation.

In Sec. VI, we apply the same techniques to manifolds with metric tensors in which the analogs of gauge potentials anticommute. In particular, we construct a supersymmetric string (superstring for short) model[5] which bears the same relation to the dual models involving fermions as the conventional string model has to Bose-type dual models. In this model the action is invariant not only under the general coordinate transformations of the world sheet of the superstring but also under supergauge transformations in the superspace. Moreover, the supergauge constraints arise naturally from the geometry of the superspace.

Finally, Sec. VII contains our conclusions.

II. THE CONSTRUCTION OF THE SUPERSPACE

We want to describe from a physical point of view how a superspace associated with a geometrically unified theory is to be constructed. For the moment we shall ignore the fact that such a superspace is in fact a fiber bundle.

Let $g_{\mu\nu}(X)$ = components of the metric tensor of a 4-dimensional space-time manifold. Also let $\bar{g}_{AB}(\theta)$ = components of a metric tensor of an n-dimensional group manifold associated with some exact local symmetry (or supersymmetry) group. Given these, we want to construct an $(n+4)$-dimensional manifold characterized by a metric tensor $G_{ij}(Y)$, where

$$Y^i = \{X^\mu, \theta^A\}, \quad \mu = 0, \ldots, 3, \quad A = 1, \ldots, n.$$

Clearly, the knowledge of $g_{\mu\nu}(X)$ and $g_{AB}(\theta)$ is not sufficient to determine $G_{ij}(Y)$ completely. So we must supply further information for the determination of the components G_{ij}. This we do in the following way. Define (for a more rigorous definition see Sec. V)

$$h_i^\mu = \frac{\partial X^\mu}{\partial Y^i}, \quad i = 0, \ldots, n+3, \quad \mu = 0, \ldots, 3 \tag{2.1}$$

with their inverses h_ν^i defined such that

$$h_i^\mu h_\nu^i = \delta_\nu^\mu . \tag{2.2}$$

Then construct the normal vectors N_A^i such that

$$h_i^\mu N_A^i = 0 \tag{2.3}$$

and their inverses such that

$$h_\mu^i N_i^A = 0, \quad N_A^i N_i^B = \delta_A^B . \tag{2.4}$$

We take $g_{\mu\nu}(X)$ and $\bar{g}_{AB}(\theta)$ to be symmetric. If we also require that $G_{ij}(Y)$ be symmetric then it is easy to check that $g_{\mu\nu}, \bar{g}_{AB}, N_\mu^A$, together, have the same number of components as G_{ij}, so that we can use them to characterize the components of G_{ij}. The superspace so constructed is modeled after the way in which a hollow cylinder is constructed from a horizontal circle and a vertical line. In our case the analog of the circle is the space-time manifold (referred to as "base manifold") and the analog of the vertical line is the group manifold (referred to as "fiber"). Thus, by construction the superspace is the local direct product of the space-time and group manifolds, and the projection of the superspace into the base manifold gives the connection between G_{ij} and $g_{\mu\nu}$:

$$g_{\mu\nu}=h_\mu^i h_\nu^j G_{ij}\,,\qquad g^{\mu\nu}=h_i^\mu h_j^\nu G^{ij}\,. \tag{2.5}$$

Similarly, the projection into the fiber gives

$$\bar{g}_{AB}=N_A^i N_B^j G_{ij}\,,\qquad \bar{g}^{AB}=N_i^A N_j^B G^{ij}\,. \tag{2.6}$$

These relations can then be solved for the components of G_{ij}. To be more explicit, let us take the Y^i's such that

$$\frac{\partial X^\mu}{\partial Y^\nu}=\delta_\nu^\mu,\quad \nu=0,\ldots,3\qquad \frac{\partial X^\mu}{\partial Y^A}=0,\quad A=4,\ldots,n+3\,. \tag{2.7}$$

Then we have

$$h_i^\mu=\begin{cases}\delta_\nu^\mu, & i=\nu\\ 0, & i=A\,.\end{cases} \tag{2.8}$$

By (2.3)

$$N_A^\mu=0\,,$$

so that

$$N_A^i=(0,N_A^B)=(0,\delta_A^B)\,. \tag{2.9}$$

From (2.4) we find the h_μ^i to be

$$h_\mu^i=\begin{cases}\delta_\mu^\nu, & i=\nu\\ -N_\mu^A, & i=A\,.\end{cases} \tag{2.10}$$

Using these results in, for example, (2.6), one finds

$$G_{AB}=\bar{g}_{AB}\,.$$

The other components of G_{ij} can be obtained in a similar fashion. Then one gets

$$G_{ij}=\begin{pmatrix} g_{\mu\nu}+\bar{g}_{AB}N_\mu^A N_\nu^B & \bar{g}_{AB}N_\mu^B\\ N_\nu^A\bar{g}_{AB} & \bar{g}_{AB}\end{pmatrix},\qquad G^{ij}=\begin{pmatrix} g^{\mu\nu} & -g^{\mu\nu}N_\nu^B\\ -N_\mu^A g^{\mu\nu} & \bar{g}_{AB}+N_\mu^A N_\nu^B g^{\mu\nu}\end{pmatrix}, \tag{2.11}$$

$$\det(G_{ij})=\det(g_{\mu\nu})\det\bar{g}_{AB}$$

or

$$G=g\bar{g}\,. \tag{2.12}$$

For an Abelian theory $\bar{g}_{AB}=1$ and (2.11) reduces to the metric given by Kaluza[7] for electrodynamics and gravitation. With G_{ij} given by (2.11) it follows that in superspace

$$dS^2=G_{ij}\,dY^i\,dY^i=dX^\mu g_{\mu\nu}dX^\nu+(d\theta^A+N_\mu^A\,dX^\mu)\bar{g}_{AB}(d\theta^B+N_\nu^B dX^\nu)\,. \tag{2.13}$$

From this the manner in which the extended tangent space is partitioned is quite clear.

It will be shown in Sec. V that the quantities N_A^μ have all the properties required of the potentials in gauge theories. Since they appear explicitly in the metric tensor, the metric tensor and the connection coefficients are in general gauge dependent and their complete specification involves a choice of gauge. Since the components of the metric tensor are specified in a given basis, one may inquire if there exists a basis in which the components of the metric tensor as well as the connection coefficients depend only on the gauge-covariant or -invariant quantities. It is shown in Sec. V that such a basis in fact exists.

Given the properties we have outlined above, it is now straightforward to compute the connection coefficients, the Ricci tensor, and the scalar curvature of the superspace. These have been carried out in Sec. V. Here we want to draw attention to the form of the scalar curvature of the superspace

$$R=R_{st}+R_G-\tfrac{1}{4}F_{\mu\nu}{}^A F_A^{\mu\nu}\,, \tag{2.14}$$

where

R_{st} = scalar curvature of the space-time manifold,

R_G = scalar curvature of the group manifold,

and

$$F_{\mu\nu}^A=N_{\mu,\nu}{}^A-N_{\nu,\mu}{}^A-N_{\mu,B}{}^A N_\nu^B \tag{2.15}$$

$$=N_{\mu,\nu}{}^A-N_{\nu,\mu}{}^A+f^A_{BC}N_\mu^B N_\nu^C\,. \tag{2.16}$$

From these expressions the relevance of this formalism to local gauge theories is already clear. Note in particular the appearance of R_G in the expression for R.

III. LOCAL GAUGE THEORIES IN FLAT OR CURVED SPACE-TIME

In the spirit of our geometrical unification, the action of a theory based on our superspace must depend only on the geometrical quantities characteristic of this manifold. The prime candidates are $G=\det(G_{ij})$, R, and R_{ij}. Since G is independent of $F_{\mu\nu}^{A}$, the action must also depend on R or R_{ij} or both. Beyond this the geometry alone cannot distinguish among the Lagrangians

$$\mathcal{L}_1=\sqrt{-G}R\,, \tag{3.1}$$

$$\mathcal{L}_2=\sqrt{-G}\,R^{\alpha}\,,\quad \alpha=\text{real number} \tag{3.2}$$

$$\mathcal{L}_3=\sqrt{-G}R_{ij}R^{ij}\,, \tag{3.3}$$

etc.

If we demand that in the limit of flat space-time the equations of motion for $F_{\mu\nu}^{A}$ be the same as those for non-Abelian gauge theories and that with $F_{\mu\nu}^{A}=0$ Einstein's equation be obtained, then $\mathcal{L}_1$ is the only choice. Thus let us consider the action

$$I_1=\int d^4X\,dV_G\sqrt{-G}R$$

$$=\int d^4X\,dV_G\sqrt{-g\bar{g}}\,(R_{st}+R_G-\tfrac{1}{4}F_{\mu\nu}^{A}F_{A}^{\mu\nu})\,. \tag{3.4}$$

One may find it tempting to integrate formally over the group parameters, leaving only the integration with respect to the space-time coordinates. Although this can be done for the action (3.4), we find it contrary to the idea of constructing a superspace. Therefore, we obtain the equations of motion directly from (3.4) without averaging over the bundle. Let us consider some of the important features of the action (3.4):

(a) The direct coupling of the gauge field to gravity is completely fixed. This is a general feature of the geometrically unified theories and is in particular true for theories based on Lagrangians (3.1)–(3.3).

(b) The curvature of the group manifold R_G provides the theory with another dimensional parameter. But since in the action (3.4) R appears linearly, this dependence on R_G is not essential in the sense that it can be eliminated by the replacement

$$\mathcal{L}_1\rightarrow\mathcal{L}_1'=\sqrt{-G}(R-R_G)\,. \tag{3.5}$$

Otherwise, it plays the role of a cosmological constant in curved space time. Since there is no direct connection between the scale of length in space-time and that in group manifold, the constant R_G is not fixed by the theory and is arbitrary.

(c) Consider the flat–space-time limit of the action (3.4):

$$I_1^F=\int d^4X\,dV_g(R_G-\tfrac{1}{4}F_{\mu\nu}^{A}F_{A}^{\mu\nu})\,. \tag{3.6}$$

Here again the term proportional to R_G can be discarded by working with the flat-space limit of the Lagrangian (3.5):

$$\tilde{I}_1^F=-\tfrac{1}{4}\int d^4X\,dV_g\,F_{\mu\nu}^{A}F_{A}^{\mu\nu}\,. \tag{3.7}$$

This is indeed the correct choice for theories in which the fields extend over the entire space-time. However, in theories in which the fields extend over a limited region of space-time, such as in the bag model,[16] it is no longer necessary to eliminate the group curvature term, and R_G will contribute to or may be interpreted as the bag constant. Thus, fiber-bundle manifolds provide a natural theoretical framework for theories with bag constants.

(d) Let us consider the field equations which follow from the action (3.7) where $F_{\mu\nu}^{A}$ is given by (2.15). Variation of the action in both the group and the space-time directions gives

$$\partial_\mu F_A^{\mu\nu}-N_\mu^B\partial_B F_A^{\mu\nu}=0\,. \tag{3.8}$$

Using the structural equation

$$\partial_B F_{\mu\nu}^{A}=-f_{BC}^{A}F_{\mu\nu}^{C} \tag{3.9}$$

obtained in Sec. V from the Jacobi identity, we find

$$\partial_\mu F_A^{\mu\nu}+f_{BA}^{C}N_\mu^B F_C^{\mu\nu}=0\,. \tag{3.10}$$

Thus Eq. (3.8) breaks up into two, one for the variation of $F_{\mu\nu}^{A}$ in the group parameters and the other for the variation in space-time. The latter is of course the familiar equation for $F_A^{\mu\nu}$ in non-Abelian gauge theories. Because (3.8) breaks up into (3.9) and (3.10) the classical theory based on the superspace becomes equivalent to the usual non-Abelian gauge theories.

Consider the structural equation (3.9) in more detail. Applying the operator ∂_D on both sides and contracting B and D indices we get

$$\eta^{BD}\partial_D\partial_B F_{\mu\nu}^{A}(\theta)=\bar{\eta}^{BD}f_{BC}^{A}f_{DE}^{C}F_{\mu\nu}^{E}(\theta)\,. \tag{3.11}$$

The operator $\bar{\eta}^{BD}\partial_D\partial_B$ is the Laplace-Beltrami operator of the relevant group. Therefore, the θ dependence of $F_{\mu\nu}^{A}$ can be expressed in terms of a generalized spherical harmonic associated with the local symmetry group. Because of the decoupling of (3.9) from (3.10), one can substitute these solutions in (3.7) and integrate over the group degrees of freedom to obtain an action which depends only on space-time parameters.

IV. A NEW EXTENSION OF BORN-INFELD THEORY TO CURVED SPACE-TIME

We want to show in this section that the fiber bundle superspace suggests a natural way of extending the Born-Infeld theory[17] and its non-Abelian counterpart to curved space-time. This in turn yields a new set of gravitational field equations.

The physical requirements we must keep in mind are (a) that for a flat-space-time manifold the formalism automatically reduces to the usual Born-Infeld theory and (b) that the resulting gravitational theory must give Einstein's theory to a first approximation, so that it be compatible with weak field consequences of Einstein's theory. To these we add the further requirement that the action of this theory depend only on the geometrical quantities associated with our fiber bundle. That is, to achieve this objective we do not alter the geometry from that of the previous section. We merely make a different choice of action. This means, in particular, that the structural equations which follow from Jacobi identity, etc. remain the same as those used in the previous section.

The appropriate actions satisfying our requirements are

$$I_2 = \int \sqrt{-G}\, R^{1/2} = \int \sqrt{-G}\, (R_{st} + R_G - \tfrac{1}{4} F_{\mu\nu}^{A} F_{A}^{\mu\nu})^{1/2} d^4X\, dV_g\,, \tag{4.1}$$

$$I_2' = \int d^4X\, dV_g \sqrt{-G}\, R_G^{1/2}(R^{1/2} - R_G^{1/2}) = \int d^4X\, dV_g \sqrt{-G}\, \{[R_G(R_{st} + R_G - \tfrac{1}{4} F^2)]^{1/2} - R_G\}\,. \tag{4.2}$$

In flat space-time, these reduce effectively to the Born-Infeld theory. For example,

$$I_2' \to \int d^4X\, dV_g \left[R_G \left(1 - \frac{1}{4R_G} F^2\right)^{1/2} - R_G\right], \tag{4.3}$$

so that R_G is related to the cutoff length or the maximum field strength of this theory.

On the other hand, when $F_{\mu\nu}^{A} \to 0$

$$I_2' \to \int d^4X\, dV_g \sqrt{-G}\, [R_G (1 + R_{st}/R_G)^{1/2} - R_G]\,. \tag{4.4}$$

It is clear that now, in contrast to the last section, the quantity R_G is an essential parameter of the theory and cannot be eliminated from the Lagrangian. In fact, its presence is crucial for the agreement of the present theory with the weak-field consequences of Einstein's theory. Taking R_G to be large, we have from (4.4)

$$I_2' \to V_g \int d^4X \sqrt{-g}\, [R_G(1 + R_{st}/R_G)^{1/2} - R_G] \approx V_g \int d^4X \sqrt{-g}\, R_{st}. \tag{4.5}$$

R_G must also be large if we want to obtain the conventional gauge theories in flat space-time as a first approximation to the action (4.3). This criterion can be used to estimate R_G in terms of the maximum field strengths allowed in (4.3) by relating it to the characteristic charges and masses of the exact extended solutions of the equations of motion. For the Abelian theory it is easy to check that R_G does turn out to be large.

Since the justification for the appearance of R_G lies in the fiber-bundle geometry, which in turn came about as a result of our geometrical unification of space-time and color symmetries, the theory presented in this section is one of a set of results which could not have been arrived at otherwise. For one thing, without a fiber-bundle manifold it would be tempting to take for the generalized theory the Lagrangian

$$\mathfrak{L} = \sqrt{-g}\, [(1 - F^2)^{1/2} + R_{st}]\,.$$

We hope to return to a more detailed discussion of our version of the theory elsewhere.

V. THE GEOMETRY OF A FIBER BUNDLE AND ITS RELATION TO GAUGE THEORIES

As was mentioned in Sec. II, the construction of a fiber bundle is modeled after the way in which a cylinder can be constructed from a horizontal ring and a vertical line. For a more formal definition we refer the reader to books on modern differential geometry. In the bundle manifolds of interest to us the horizontal cross section is identified with the space-time manifold and the vertical fiber with the group manifold, so that locally the bundle manifold is the direct product of the space-time and group manifolds.

To describe the properties of the bundle manifold, we will introduce a basis which is related to the bases in space-time and group manifolds. So we begin with a description of bases.

In modern differential geometry[18] the basis vectors are taken to be a set of vector fields identical with directional derivatives. Consider, for example, in some neighborhood the n quantities X^μ ($\mu = 1, \ldots, n$) whose values $X^\mu(P)$ are the coordinates of the point P. The operator $\vec{\partial}_\mu$ defined by

$$\vec{\partial}_\mu f = \frac{\partial f(X^1, \ldots, X^n)}{\partial X^\mu}$$

is the vector tangent to the lines $X^k = \text{const}$ ($k \neq \mu$). The n operators

$$\{\vec{e}_\mu\}=\{\vec{\partial}_\mu\}$$

are called basis vectors. Here the index μ specifies not which component but which vector field. For obvious reasons this set is called a coordinate-induced basis. Generally, a basis is coordinate-induced if

$$[\vec{e}_i,\vec{e}_j]=0 \quad \text{(coordinate basis)}.$$

It is often convenient to choose a basis for which this requirement is not satisfied. That is, in general,

$$[\vec{e}_i,\vec{e}_j]=C_{ij}^{\;k}\vec{e}_k,$$

where $C_{ij}^{\;k}$ are the commutation coefficients of the basis $\vec{e}_i$. For example, consider the components of the velocity vector in spherical coordinates. In the coordinate basis

$$\{\vec{\partial}_r,\vec{\partial}_\theta,\vec{\partial}_\phi\}\equiv\left\{\frac{\partial}{\partial r},\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}\right\},$$

$$\vec{V}=V^r\vec{\partial}_r+V^\theta\vec{\partial}_\theta+V^\phi\vec{\partial}_\phi,$$

where

$$V^r=\frac{dr}{dt},\quad V^\theta=\frac{d\theta}{dt},\quad V^\phi=\frac{d\phi}{dt}.$$

However, we usually find it more convenient to write $\vec{V}$ as

$$\vec{V}=\hat{V}^r\vec{e}_r+\hat{V}^\theta\vec{e}_\theta+\hat{V}^\phi\vec{e}_\phi,$$

where

$$\{\hat{V}^r,\hat{V}^\theta,\hat{V}^\phi\}=\left\{\frac{dr}{dt},r\frac{d\theta}{dt},r\sin\theta\frac{d\phi}{dt}\right\}.$$

Since $\vec{V}$ is the same vector as the one above, then we must have

$$\{\vec{e}_r,\vec{e}_\theta,\vec{e}_\phi\}=\left\{\partial_r,\frac{1}{r}\partial_\theta,\frac{1}{r\sin\theta}\partial_\phi\right\};$$

clearly,

$$[\vec{e}_\theta,\vec{e}_\phi]\neq 0.$$

Consider next the notion of a metric. Physically, the metric is a structure which determines distances between nearby points. More abstractly, it is defined as a bilinear nonsingular function which acts on pairs of vectors to produce a scalar. Since we are going to make use of the latter definition, we briefly sketch the equivalence of the two definitions. Consider a vector $\vec{V}$ connecting two nearby points P and Q with coordinate differences ΔX^μ. According to the first definition

$$\Delta S^2=g_{\mu\nu}\Delta X^\mu\Delta X^\nu,$$

where $g_{\mu\nu}$ are the components of the metric tensor. Alternatively, expanding V in the basis $\{\vec{e}_\mu\}=\{\vec{\partial}_\mu\}$, we get

$$\vec{V}=\Delta X^\mu\vec{e}_\mu,$$

$$\Delta S^2=\vec{V}\cdot\vec{V}=\Delta X^\mu\Delta X^\nu\vec{e}_\mu\cdot\vec{e}_\nu.$$

Comparing the two expressions for ΔS^2, we find

$$g_{\mu\nu}=\vec{e}_\mu\cdot\vec{e}_\nu\equiv g(\vec{e}_\mu,\vec{e}_\nu)$$

and by linearity of the operator g

$$\Delta S^2=\Delta X^\mu\Delta X^\nu g(\vec{e}_\mu,\vec{e}_\nu)\equiv g(\vec{V},\vec{V}).$$

In general one writes

$$d\vec{\omega}^i\otimes d\vec{\omega}^j g_{ij}\equiv g,$$

where the set $\{d\vec{\omega}^j\}$ (of one-forms) is a basis dual to $\{\vec{e}_i\}$:

$$\langle d\vec{\omega}^j,\vec{e}_i\rangle=\delta_i^j.$$

In the study of group manifolds it is convenient to choose the basis vectors and the metric such that they commute with the generators of the isometries (Killing vectors) associated with the manifold. The most straightforward way of doing this is to take the metric to be Euclidean or pseudo-Euclidean. One can take it to be, e.g., the Killing metric

$$\bar{\eta}_{AB}=f_{AC}^{\;D}f_{DB}^{\;C}$$

except when Abelian groups or subgroups are involved. By writing the differential quadratic form for the group manifold in the two equivalent forms

$$\Delta S^2=\bar{g}_{AB}(\theta)d\theta^A d\theta^B$$
$$=\bar{\eta}_{AB}d\vec{\Omega}^A\otimes d\vec{\Omega}^B$$

it can be seen that the simplification in the components of the metric tensor is achieved at the expense of making the dual basis $\{d\vec{\Omega}^A\}$ more complicated. That is, they are no longer simple differentials of the coordinates. Therefore, the basis vectors $\{\vec{E}_A\}$ to which $\{d\vec{\Omega}^A\}$ are dual are no longer simple partial derivatives of the coordinates, and

$$[\vec{E}_A,\vec{E}_B]\neq 0.$$

Even in cases such as that of the space-time manifold, where the choice of a coordinate basis has traditionally been employed, it often turns out to be more transparent and technically much simpler to work in a noncoordinate basis. In fact, the gauge-covariant basis we mentioned in Sec. II is one such noncoordinate basis.

The specification of a basis $\{\vec{E}_i\}$ involves the specification of the metric tensor

$$G_{ij}=\vec{E}_i\cdot\vec{E}_j$$

which reflects the symmetric properties of the basis and the specification of the commutators

$$[\vec{E}_i,\vec{E}_j]=C_{ij}^{\;k}\vec{E}_k$$

which reflect the antisymmetric properties of the basis.

Bases in space-time and group manifolds. As the simplest basis in the base manifold we take one that is induced by the coordinates. That is, we take[19]

$$\vec{e}_\mu = \frac{\partial}{\partial X^\mu}, \quad \mu = 0, \ldots, 3$$
$$[\vec{e}_\mu, \vec{e}_\nu] = 0 . \tag{5.1}$$

For the group manifold it is not convenient to work in a coordinate basis because the description of isometries in such manifolds can best be carried out in terms of an invariant basis. For one thing, this guarantees that they commute with the generators of isometries (Killing vectors). There are also technical advantages, as can be seen by the ease with which the components of the metric tensor and connection coefficients are evaluated below.

An invariant basis $\{\vec{e}_A^{(0)}\}$ can be set up at some point P_0 of the group manifold. Its elements may be taken to be isomorphic to the algebra of the group (the Killing vectors). Then the basis at other points of the manifold can be set up by translating the set $\{\vec{e}_A^{(0)}\}$ from P_0. Suppose, e.g., that we identify P_0 with the identity I of the group manifold. Then the transformation which takes P_0 to some other point Q can also be used to relate the bases at Q to those at P_0. This relation may be established in many ways, such as via Lie derivatives with Killing vectors, right translations, left translations, etc. Thus, if, e.g., the basis is invariant under right translations, then the commutation coefficients of the basis will not change from point to point, and we have at every point in the group manifold a basis $\{\vec{e}_A\}$ such that

$$[\vec{e}_A, \vec{e}_B] = f_{AB}^{\ C}\vec{e}_C . \tag{5.2}$$

To specify the properties of the basis completely, its symmetric properties are given by specifying a metric

$$g_{AB} = \vec{e}_A \cdot \vec{e}_B . \tag{5.3}$$

Bases for the bundle manifold. Since locally the bundle is the direct product of the base and the group manifolds, the tangent space at each point of the bundle is the direct sum of a horizontal subspace and a vertical subspace. So one possibility is to take the bases in these subspaces to be, respectively, isomorphic to those in the space-time and group manifolds. That is, for the basis in the tangent spaces to the points of the bundle we take the set $\{\vec{h}_i\}$ $(i = \{\mu, A\})$, with

$$[\vec{h}_\mu, \vec{h}_\nu] = 0 ,$$
$$[\vec{h}_A, \vec{h}_B] = f_{AB}^{\ C}\vec{h}_C , \tag{5.4}$$
$$[\vec{h}_\mu, \vec{h}_A] = 0 .$$

This is not a convenient basis for physical interpretation as well as for calculations, however, and we shall look for a basis in which the line element in the bundle manifold breaks up into a line element in the base manifold and one in the group manifold, with no cross terms. In other words, we want a basis $\{\vec{E}_i\}$ in which the components G_{ij} of the metric tensor G are in block diagonal form:

$$G_{\mu\nu} = \vec{E}_\mu \cdot \vec{E}_\nu = g_{\mu\nu} ,$$
$$G_{AB} = \vec{E}_A \cdot \vec{E}_B = \bar{g}_{AB} , \tag{5.5}$$
$$G_{A\mu} = \vec{E}_A \cdot \vec{E}_\mu = 0 .$$

One clear advantage of such a basis is that the raising and lowering indices with respect to G_{ij} are the same as those with respect to space-time and group indices. It will also be seen below that in this basis the metric tensor, the connection coefficients, the components of the Ricci tensor, etc. are all given in terms of either gauge-covariant or gauge-invariant quantities. For this reason we call this the "gauge-covariant basis."

The vertical part of the basis $\{\vec{E}_i\}$, i.e. the $\{\vec{E}_A\}$, can still be taken to be isomorphic to $\{\vec{h}_A\}$. But with requirements (5.5) it is no longer possible to demand that $\{\vec{E}_\mu\}$ be a coordinate basis. So in general we have

$$[\vec{E}_\mu, \vec{E}_\nu] = -F_{\mu\nu}^{\ k}\vec{E}_k = -F_{\mu\nu}^{\ \lambda}\vec{E}_\lambda - F_{\mu\nu}^{\ A}\vec{E}_A . \tag{5.6}$$

The right-hand side of this expression is not completely arbitrary because the $\vec{E}_\mu$'s are linear combinations of the vectors $\vec{h}_\mu$ and $\vec{h}_A$, so that the horizontal projection of $\vec{E}_\mu$ must be equal to $\vec{h}_\mu$. That is,

$$\Pi(\vec{E}_\mu) = \vec{h}_\mu ,$$
$$\Pi[\vec{E}_\mu, \vec{E}_\nu] = [\vec{h}_\mu, \vec{h}_\nu] = 0 , \tag{5.7}$$

where Π is the projection map of the fiber into a point of the base manifold. To satisfy this requirement we must set $F_{\mu\nu}^{\ \lambda} = 0$, which amounts to requiring that the corresponding gauge potentials satisfy the Maurer-Carten equations. Thus, in the gauge-covariant basis we have

$$[\vec{E}_\mu, \vec{E}_\nu] = -F_{\mu\nu}^{\ A}\vec{E}_A ,$$
$$[\vec{E}_A, \vec{E}_B] = f_{AB}^{\ C}\vec{E}_C , \tag{5.8}$$
$$[\vec{E}_\mu, \vec{E}_A] = 0 .$$

Next, consider the basis $\{\vec{N}^i\}$ dual to $\{\vec{h}_i\}$ and the basis $\{\vec{\Omega}^i\}$ dual to $\{\vec{E}_i\}$, i.e.,

$$\langle \vec{N}^i, \vec{h}_j \rangle = \delta^i_j , \tag{5.9}$$

$$\langle \vec{\Omega}^i, \vec{E}_j \rangle = \delta^i_j . \tag{5.10}$$

We can use these normalization conditions to find the relation between the direct product and the gauge-covariant basis vectors. Expanding one in terms of the other we write

$$\vec{h}_A = h^i_A \vec{E}_i, \quad \vec{h}_\mu = h^i_\mu \vec{E}_i, \tag{5.11}$$

$$\vec{N}^A = N^A_i \vec{\Omega}^i, \quad \vec{N}^\mu = N^\mu_i \vec{\Omega}^i . \tag{5.12}$$

In order that $\vec{h}_A$ satisfy the algebra of the gauge group, we must have

$$h^i_A = \delta^i_A, \quad \text{i.e.,} \quad h^B_A = \delta^B_A, \; h^\mu_A = 0 . \tag{5.13}$$

Using now (5.9) and (5.10) for different values of i and j, it is easy to show that

$$\begin{aligned} &N^B_A = \delta^B_A, \quad N^\mu_A = 0 , \\ &h^\mu_\lambda = N^\mu_\lambda = \delta^\mu_\lambda, \quad h^A_\mu = -N^A_\mu . \end{aligned} \tag{5.14}$$

The results (5.13) and (5.14) are the same as those in (2.8)–(2.10). Here we have obtained them by more formal considerations. By means of these, we simplify (5.11) and (5.12) to

$$\vec{h}_A = \vec{E}_A, \quad \vec{h}_\mu = \vec{E}_\mu + h^A_\mu \vec{E}_A = \vec{E}_\mu - N^A_\mu \vec{E}_A , \tag{5.15}$$

$$\vec{N}^A = \vec{\Omega}^A + N^A_\mu \vec{\Omega}^\mu, \quad \vec{N}^\mu = \vec{\Omega}^\mu . \tag{5.16}$$

Solving (5.15) for $\vec{E}_\mu$ and noting that $\vec{h}_\mu = \vec{\partial}_\mu$ it is clear that $\vec{E}_\mu$ has the form of a covariant derivative operator. This is further confirmed by looking at the commutator $[\vec{E}_\mu, \vec{E}_\nu]$ in (5.8).

The components of the metric tensor G. By definition, in the gauge-covariant basis the metric tensor is block diagonal, so we can write down trivially

$$G_{ij} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \bar{g}_{AB} \end{pmatrix} = \begin{pmatrix} \vec{E}_\mu \cdot \vec{E}_\nu & 0 \\ 0 & \vec{E}_A \cdot \vec{E}_B \end{pmatrix} , \tag{5.17}$$

$$G^{ij} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \bar{g}^{AB} \end{pmatrix} . \tag{5.18}$$

To obtain the components of G in the direct product basis, we make use of the relations (5.15) between the two bases. Thus

$$\begin{aligned} \vec{h}_\mu \cdot \vec{h}_\nu &= \vec{E}_\mu \cdot \vec{E}_\nu + N^A_\mu \vec{E}_A \cdot N^B_\nu \vec{E}_B \\ &= g_{\mu\nu} + \bar{g}_{AB} N^A_\mu N^B_\nu , \\ \vec{h}_A \cdot \vec{h}_B &= \vec{E}_A \cdot \vec{E}_B = \bar{g}_{AB} , \\ \vec{h}_\mu \cdot \vec{h}_A &= \bar{g}_{AB} h^B_\mu . \end{aligned} \tag{5.19}$$

These are identical to the metric tensor components (2.11) we obtained by our intuitive approach. In the present approach we know that although in the set $\{\vec{h}_i\}$ the bases $\vec{h}_\mu$ are coordinate-induced the group part $\vec{h}_A$ are noncoordinate bases. But in our derivation of (2.11) we did not appear to make any assumption about the basis in group space. In view of the identity of the two results, it may be asked where this assumption was made in Sec. II. The answer is that in addition to the symmetry of $g_{\mu\nu}$ we assumed that $\bar{g}_{AB}$ and G_{ij} were also symmetric. In a coordinate basis G_{ij} and $\bar{g}_{AB}$ need not be symmetric, as can be seen, e.g., from the nonsymmetric connection coefficients given below.

The structural relations. Next, using the Jacobi identities, etc. we derive a number of structural relations which follow from the properties of the basis vectors. Consider first the Jacobi identity

$$\begin{aligned} 0 &= [\vec{E}_A, [\vec{E}_\mu, \vec{E}_\nu]] + [\vec{E}_\mu, [\vec{E}_\nu, \vec{E}_A]] + [\vec{E}_\nu, [\vec{E}_A, \vec{E}_\mu]] \\ &= [\vec{E}_A, -F_{\mu\nu}{}^B \vec{E}_B] \end{aligned}$$

or

$$\vec{E}_A F_{\mu\nu}{}^B \equiv \vec{\partial}_A F_{\mu\nu}{}^B = -f_A{}^B{}_C F_{\mu\nu}{}^C . \tag{5.20}$$

This equation determines how $F_{\mu\nu}^A$ varies as a function of the group parameters. Note that $\vec{\partial}_A$ is not a partial derivative but a directional derivative or a generalized angular momentum operator. Also note that only for an Abelian gauge group, $F_{\mu\nu}^A$ is independent of the group parameters.

Replacing $\vec{E}_A$ by $\vec{E}_\mu$ in the above Jacobi identity, we obtain the familiar result

$$F_{\mu\nu;\lambda}^{A} + F_{\lambda\mu;\nu}^{A} + F_{\nu\lambda;\mu}^{A} = 0 . \tag{5.21}$$

An equation similar to (5.20) for the variation of h^A_μ (or N^A_μ) along the fiber can be obtained by using the relations (5.15) in the commutator $[\vec{h}_A, \vec{h}_\mu] = 0$. The result is

$$\vec{\partial}_A h^B_\mu = -f_{AC}^{\;B} h^C_\mu . \tag{5.22}$$

Finally, we want to relate the two functions $F_{\mu\nu}^A$ and N^A_μ which we have introduced into the structure of the fiber bundle. Using (5.15) to rewrite the commutator $[\vec{h}_\mu, \vec{h}_\nu] = 0$, we find

$$F_{\mu\nu}^A = N^A_{\mu,\nu} - N^A_{\mu,\nu} - \vec{\partial}_B N^A_\mu N^B_\nu \tag{5.23}$$

or by (5.22)

$$F_{\mu\nu}^A = N^A_{\mu,\nu} - N^A_{\nu,\mu} + f^A_{BC} N^B_\mu N^C_\nu . \tag{5.24}$$

Local gauge transformations. Let $\epsilon^A(x)$ ($A = i, \ldots, n$) be a set of arbitrary functions of space-time. We want to see the consequences of the transformation

$$\vec{E}_\mu \to e^{i\epsilon^A \vec{h}_A} \vec{E}_\mu e^{-i\epsilon^A \vec{h}_A} .$$

It is sufficient to consider infinitesimal transformations. Noting that

$$(1 + i\epsilon^A \vec{h}_A) \vec{h}_\mu (1 - i\epsilon^B \vec{h}_B) = \vec{h}_\mu - i\epsilon^A{}_{,\mu} \vec{h}_A$$

and that

$$(1+i\epsilon^A \vec{\mathrm{h}}_A)N^B_\mu \vec{\mathrm{h}}_B(1-i\epsilon^C \vec{\mathrm{h}}_C)=N^B_\mu \vec{\mathrm{h}}_B + iN^B_{\mu,A}\epsilon^A \vec{\mathrm{h}}_B$$

we find

$$\begin{aligned}(1+i\epsilon^A \vec{\mathrm{h}}_A)\vec{\mathrm{E}}_\mu(1-i\epsilon^B \vec{\mathrm{h}}_B) &= \vec{\mathrm{h}}_\mu + (N^B_\mu - i\epsilon^B{}_{,\mu} + iN^B_{\mu,A}\epsilon^A)\vec{\mathrm{h}}_B \\ &= \vec{\mathrm{h}}_\mu + N'^B_\mu \vec{\mathrm{h}}_B ,\end{aligned} \tag{5.25}$$

where

$$N'^B_\mu = N^B_\mu - i\epsilon^B{}_{\mu} - if^B_{AC}N^A_\mu \epsilon^C . \tag{5.26}$$

This is the familiar form of the infinitesimal local gauge transformations on vector potentials in gauge theories, which leave the field tensor $F^A_{\mu\nu}$ invariant. So gauge transformations are interpreted as local rotations of the base vectors $\vec{\mathrm{E}}_\mu$ of the bundle manifold, which change the relative admixture of vectors $\vec{\mathrm{h}}_\mu$ and $\vec{\mathrm{h}}_A$ in $\vec{\mathrm{E}}_\mu$ but leave the commutator $[\vec{\mathrm{E}}_\mu, \vec{\mathrm{E}}_\mu]$ gauge covariant.

This interpretation also helps clarify the underlying reason for the actions of the superstring models, which were taken to be invariant under general coordinate transformations of the bundle manifold, to be supergauge invariant.

The connection coefficients. The connection coefficients are defined as follows:

$$\Gamma_{ij}^{\,k} \equiv \langle \vec{\omega}^k, \nabla_{\vec{e}_i} \vec{e}_j \rangle ,$$

where $\langle \vec{\omega}^k, \vec{e}_i\rangle = \delta^k_i$ and $\nabla_{\vec{e}_i}$ is the directional covariant derivative in the $\vec{e}_i$ direction. If (a) the manifold over which $\Gamma_{ij}^{\,k}$ are defined admits a metric g such that $\nabla_{\vec{e}} g = 0$ and (b) the manifold is torsion-free, i.e.,

$$\nabla_{\vec{U}}\vec{V} - \nabla_{\vec{V}}\vec{U} = [\vec{U}, \vec{V}] ,$$

then

$$\begin{aligned}\Gamma_{ij}^{\,k} = \tfrac{1}{2}G^{kl}(&G_{li,j} + G_{lj,i} - G_{ij,l} \\ &+ C_{lij} + C_{lji} - C_{ijl}) ,\end{aligned}$$

where

$$G_{ij} = \vec{e}_i \cdot \vec{e}_j ,$$

$$[\vec{e}_i, \vec{e}_j] = C_{ij}^{\,k}\vec{e}_k .$$

Using this expression for $\Gamma_{ij}^{\,k}$, we evaluate the connection coefficients for the bundle manifold in the gauge-covariant basis:

$$\begin{aligned}&\Gamma^A_{BC} = \tfrac{1}{2}f^A_{BC}, \quad \Gamma^\mu_{AB} = 0 , \\ &\Gamma^A_{\mu B} = \Gamma^A_{B\mu} = 0, \quad \Gamma^A_{\mu\nu} = -\tfrac{1}{2}F^A_{\mu\nu} , \\ &\Gamma^\lambda_{\mu A} = \Gamma^\lambda_{A\mu} = \tfrac{1}{2}g^{\lambda\mu}\bar{g}_{AB}F_{\mu\nu}^{\;B} , \\ &\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \equiv \left\{ \begin{matrix}\lambda \\ \mu\nu\end{matrix} \right\} \\ &\quad = \tfrac{1}{2}g^{\lambda\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}) .\end{aligned} \tag{5.27}$$

Ricci tensor and the scalar curvature. The components of the Ricci tensor are given by

$$\begin{aligned}&R_{AB} = R^{(G)}_{AB} + \tfrac{1}{4}\bar{g}_{AC}\bar{g}_{BD}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}^{\;C}F_{\lambda\rho}^{\;D} , \\ &R_{\mu A} = R_{A\mu} = \tfrac{1}{2}\bar{g}_{AB}g^{\nu\lambda}\nabla_\nu F_{\mu\lambda}^{\;B} , \\ &R_{\mu\nu} = R^{\mathrm{st}}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{AB}g^{\lambda\rho}F_{\mu\lambda}^{\;A}F_{\nu\rho}^{\;B} ,\end{aligned} \tag{5.28}$$

where

$R^{(G)}_{AB}$ = Ricci tensor of the group manifold,

$R^{\mathrm{st}}_{\mu\nu}$ = Ricci tensor of space-time manifold

and we have defined the totally covariant derivative

$$\begin{aligned}\nabla_\lambda F_{\mu\nu}^{\;A} = F_{\mu\nu,\lambda}^{\;A} - \Gamma_{\lambda\mu}^{\;\rho}F_{\rho\nu}^{\;A} - \Gamma_{\lambda\nu}^{\;\rho}F_{\rho\mu}^{\;A} \\ + f^A_{BC}N^B_\lambda F_{\mu\nu}^{\;C} .\end{aligned} \tag{5.29}$$

Finally, the scalar curvature of the bundle is given by

$$\begin{aligned}R &= G^{ij}R_{ij} \\ &= R_{\mathrm{st}} + R_{\mathrm{G}} - \tfrac{1}{4}F_{\mu\nu}^{\;A}F^{\mu\nu}_A\end{aligned} \tag{5.30}$$

where

R_{st} = curvature scalar of the space-time,

R_{G} = curvature scalar of group manifold.

Abstraction and generalization. We have carried out the above analysis for Lie algebraic structures. However, insofar as the algebra provides a basis for the vertical sector of the tangent space, the formalism can be immediately generalized to other algebras, such as graded Lie or nonassociative algebras. We therefore list below a set of rules for obtaining locally gauge-invariant, supergauge-invariant, etc. theories based on various algebraic structures:

(a) Always begin the analysis in the gauge-covariant basis.

(b) Specify the block-diagonal metric components in the basis.

(c) Specify the commutation (or anticommutation) coefficients of the basis. This defines the field tensors of the theory.

(d) Relate the covariant differentials (one forms) of this basis to one in which the bases in the horizontal sector are coordinate-induced. This defines the gauge potentials and relates them to field tensors.

(e) If the vertical algebraic structure involves space-time, horizontal and vertical bases must be "interlocked."

VI. THE SUPERSYMMETRIC STRING MODELS

The string picture of the dual resonance models[20–23] has provided a very useful way of looking at hadron dynamics. However, over and beyond the problems which arise in a quantized string theory,[24] such as the tachyon and the 26-dimensional space-time, there is a major drawback within this framework: The conventional string only carries energy and momentum and is incapable of describing a system with internal quantum numbers, including fermion number.

To construct a more physical string model with these properties, one may argue that since the conventional string model arose from the Bose-type dual models, at least the fermion-number problem can be solved by considering models which involve fermionic excitations. Several such models now exist.[25-27] But attempts to construct string models with Fermi-type excitations from a geometric action principle have not yet been successful. The main reason is that the actions of the theories proposed so far[28-29] have been based entirely on the properties of the space-time manifold. As a result, they do not possess sufficient symmetry to provide a natural setting for the supergauge constraints. The farthest progress in this direction was made by Iwasaki and Kikkawa,[28] who succeeded in formulating a set of equations of motion and constraints for vectorlike fermions. Although this system of equations was used by Mandelstam[30] to construct an interacting string picture of the dual fermion model, it is not strictly derivable from an action principle.

To allow for the construction of a geometric action principle for these models, we associate additional degrees of freedom with each point on the world sheet of the string. Specifically, we increase the dimensionality of the tangent space for each point, so that as the string evolves in time it sweeps out a surface not just in space-time but in a higher-dimensional manifold. The correct action would then be a geometrical entity in this superspace. However, it is crucial that the time development take place in the space-time manifold, so there must be a partition of the superspace into a horizontal manifold along which the motion takes place and a vertical space which parametrizes the internal degrees of freedom.

From this description the close connection between the superspace in which the string motion takes place and the fiber-bundle geometry utilized in previous sections is quite clear. The main difference is that the analogs of the gauge potentials of the previous sections may now anticommute. We shall return to the full discussion of local supersymmetries in space-time elsewhere. Here we simply postulate a superspace characterized by the metric tensor

$$G_{ij}=\begin{pmatrix} g_{\mu\nu}+\bar{g}_{AB}N_\mu{}^A N_\nu{}^B & g_{AB}N_\mu{}^B \\ N_\nu{}^A g_{AB} & \bar{g}_{AB}\end{pmatrix}. \qquad (6.1)$$

If, in line with the previous sections, we had taken a full supergauge algebra for the vertical structure, we would have in G_{ij} not only $N_\mu{}^A$ which are gauge potentials of supersymmetry generators but also gauge potentials associated with translations and Lorentz transformations. We consider two cases:

(a) The fundamental fermionic dynamical variable of the theory is related to $N_\mu{}^A$ by

$$S_\mu{}^A=(-g)^{1/8}N_\mu{}^A, \qquad (6.2)$$

where g is as defined below. In this case $S_\mu{}^A$ and $N_\mu{}^A$ are anticommuting quantities even classically. We shall refer to $S_\mu{}^A$ as vectorlike fermions.

(b) $N_\mu{}^A$ is taken to be a composite field related to a fundamental fermionic field according to

$$N_\mu{}^A=(-g)^{-1/4}\,\bar\psi\Gamma^A\gamma_\mu\psi, \qquad (6.3)$$

where ψ is a 4-component Dirac spinor, γ_μ the Dirac matrices, and Γ^A the internal symmetry matrices. In this case we take the ψ's to anticommute, even classically, and refer to them as quarklike fermions.

Most of our geometrical arguments are insensitive to the choices (a) or (b) for $N_\mu{}^A$. We shall therefore continue the exposition in terms of $N_\mu{}^A$ until the distinction between the two cases becomes essential.

Consider the motion of a one-dimensional object, a "superstring," in the superspace characterized by (6.1). The evolution of the superstring traces a 2-dimensional surface in superspace, which is characterized by the metric tensor

$$G_{ab}=\frac{\partial Y^i}{\partial\eta^a}\frac{\partial Y^j}{\partial\eta^b}G_{ij}, \quad a,b=0,1 \qquad (6.4)$$

where η^a are intrinsic surface coordinates, G_{ij} is given by (6.1), and $\{Y^i\}=\{x^\mu,\theta^A\}$ are similar to those in Sec. II. We can now take over most of the arguments presented in Ref. 21. In particular, the coordinate conditions derived there are characteristics of a 2-dimensional manifold and do not depend on the special features of space-time or superspace. Therefore, it is always possible to choose the surface coordinates $\eta^0=\tau$ and $\eta^1=\sigma$ such that

$$G_{00}+G_{11}=0,\qquad G_{01}=0. \qquad (6.5)$$

Moreover, as pointed out above, the surfaces of physical interest are not completely arbitrary and are those which respect the partition of superspace into horizontal and vertical directions. Therefore the physically relevant surfaces automatically satisfy the additional condition

$$\frac{\partial Y^i}{\partial\eta^a}\cdot N_i{}^A=0, \qquad (6.6)$$

where $\partial Y^i/\partial\eta^a$ are tangent vectors to the world sheet, and $N_i{}^A$ are the analogs of normal vectors introduced in Secs. II and V. With our choice of Y^i this condition reduces to

$$\frac{\partial Y^\mu}{\partial\eta^a}\cdot N_\mu{}^A=0\,. \tag{6.7}$$

It will be seen from the following that these are the supergauge constraints of our theory. They control the orientation of the world sheet of the string in superspace, so that the horizontal and vertical properties remain distinct. Physically, this means that the internal properties of a *free* superstring remain constant in time.

From (6.1) and (6.4), we have

$$G_{ab}=g_{ab}+\left(\frac{\partial Y^\mu}{\partial\eta^a}N_\mu{}^A\right)\left(\frac{\partial Y^\nu}{\partial\eta^b}N_\nu{}^B\right)\bar{g}_{AB}\,, \tag{6.8}$$

where

$$g_{ab}=g_{\mu\nu}\frac{\partial Y^\mu}{\partial\eta^a}\frac{\partial Y^\nu}{\partial\eta^b} \tag{6.9}$$

with[31]

$$\det(g_{ab})=g\,\cdot \tag{6.10}$$

Thus instead of the constraints (6.5) and (6.7) we can take the combination

$$g_{00}+g_{11}=0\,,\quad g_{01}=0\,,\quad \frac{\partial Y^\mu}{\partial\eta^a}\cdot N_\mu{}^A=0\,. \tag{6.11}$$

Any linear combination of these will also do. This system of constraints is not yet complete because, as can be seen by analogy with conventional gauge potentials, the quantities $N_\mu{}^A$ which appear in the metric tensor (6.1) are gauge dependent. So eventually we have to add a further constraint to fix the gauge for $N_\mu{}^A$.

With the geometry completely specified, we now turn to the details of the dynamics. We take as dynamical variables the coordinates Y^μ and either $S_\mu{}^A$ or ψ. It goes without saying that since we are presently interested in a superstring theory, we need only specify $S_\mu{}^A$ or ψ on the world sheet of the superstring. So we write the action for the two cases, respectively, in the form

$$I_1=I_Y+I_S\,, \tag{6.12a}$$

$$I_2=I_Y+I_\psi\,. \tag{6.12b}$$

For I_Y we take

$$I_Y=\int d^2\eta\,\mathcal{L}_Y=\int d^2\eta\sqrt{-G}\equiv\int d^2\eta[-\det(G_{ab})]^{1/2}\,, \tag{6.13}$$

where for case (a) $N_\mu{}^A$ in I_Y is replaced with $S_\mu{}^A$ according to (6.2) and for case (b) it is replaced with ψ according to (6.3). The variation of I_Y with respect to Y leads in either case to the equation of motion

$$\frac{1}{\sqrt{-G}}\frac{\partial}{\partial\eta^a}\left(\sqrt{-G}\,G^{ab}\frac{\partial}{\partial\eta^b}Y^\mu\right)=0\,. \tag{6.14}$$

Despite its apparent complexity, in the (τ,σ) coordinate system in which the constraints (6.11) are satisfied (6.14) reduces to

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\sigma^2}\right)Y^\mu(\sigma,\tau)=0\,. \tag{6.15}$$

With our choices for $N_\mu{}^A$, I_Y does not depend on the derivatives of $S_\mu{}^A$ or ψ, so that in order for us to have equations of motion for them I_S and I_ψ must have these dependences, respectively. By construction $\mathcal{L}_Y$ is a scalar density under arbitrary general coordinate and supergauge transformations of the superstring parameters $\{\eta,\theta\}$. So for consistency we require that I_S and I_ψ be also invariant, at least under general coordinate transformations of η^a and of supergauge transformations. Moreover, we require that the contributions of I_S and I_ψ to the equation of motion for Y^μ in (τ,σ) coordinates be at most proportional to constraints and thus leave Eq. (6.15) unaltered. Beyond these the choice of I_S and I_ψ depends on the particular model one wishes to consider. Here we limit ourselves to two examples which lead to the known dual models. So we take the space-time to be the Minkowski space.

(a) Vectorlike fermions. From the definition of $S_\mu{}^A$ and the manner in which $S_\mu{}^A$ appears in various expressions, it is easy to see that it transforms not as a density but as a relative density. So although the system of equations and constraints that we obtain for $S_\mu{}^A$ and Y^μ are formally identical with those of Iwasaki and Kikkawa,[28] the transformation properties of our $S_\mu{}^A$ are different from theirs. This may be cited as one reason for the lack of success of the previous attempts to construct a geometrical theory involving vectorlike fermions.

To proceed further we replace the index A by the set (α, A'), where α takes on values 1 and 2 and A' is an internal symmetry index which we suppress from now on. Defining the two-component object

$$S_\mu=\begin{bmatrix}S_\mu^{(1)}\\ S_\mu^{(2)}\end{bmatrix} \tag{6.16}$$

we write for I_S in (6.12a)

$$I_S=\int d^2\eta\,\mathcal{L}_S\,, \tag{6.17}$$

with

$$\mathcal{L}_S = (-G)^{-1/4}\left(\sqrt{-G_{11}}\,\bar{S}\sigma^0 \frac{\partial S}{\partial \eta^0} + \sqrt{G_{00}}\,\bar{S}\,\sigma^1 \frac{\partial S}{\partial \eta^1}\right) - (-G)^{1/4}\left(\frac{1}{\sqrt{G_{00}}}\frac{\partial \bar{S}}{\partial \eta^0}\sigma^0 S + \frac{1}{\sqrt{-G_{11}}}\frac{\partial \bar{S}}{\partial \eta^1}\sigma^1 S\right) \tag{6.18}$$

where

$$\sigma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.19}$$

and

$$\bar{S} = S^T \sigma^0 \tag{6.20}$$

(S^T is the transpose of S).

With I_S given by (6.17) we may now vary I_1 given by (6.12a) to obtain the equations of motion for Y^μ and S_μ. Before we do this, we shall complete the list of constraints that S_μ and Y^μ must satisfy. Since S_μ is related to N_μ by (6.2) the constraints (6.11) remain in force, and since S_μ plays a role similar to vector potentials we must fix it by a gauge condition just as is done in conventional gauge theories. However, since there are already three gauge conditions, we must ensure that our choice of gauge is compatible with the ones given in (6.11). The particular gauge that we choose is dictated by this requirement and by the equations of motion for S_μ. Consider the gauge condition

$$g^{\mu\nu} S_\mu^{(\alpha)} S_\nu^{(\alpha)} = \text{const}, \quad \alpha = 1 \text{ or } 2 \text{ (no sum)}. \tag{6.21}$$

This is similar to the bilinear gauge considered by Dirac[32] and by Nambu[33] in classical and quantum electrodynamics, respectively. Projecting this on the world sheet of the string we get

$$g_{\mu\nu} S_\mu^{(\alpha)} \frac{\partial}{\partial \eta} S_\nu^{(\alpha)} = 0, \quad \eta = \tau \text{ or } \sigma. \tag{6.22}$$

It will be seen below that for each α only one combination of τ and σ will be independent of the corresponding equation of motion for $S_\mu^{(\alpha)}$. We shall take that combination as the gauge condition on $S_\mu^{(\alpha)}$.

Now we return to the action (6.12a). Variation with respect to $\bar{S}$ gives

$$\frac{\partial}{\partial \eta^0}\frac{\partial \mathcal{L}_S}{\partial \dot{\bar{S}}} + \frac{\partial}{\partial \eta^1}\frac{\partial \mathcal{L}_S}{\partial \bar{S}'} - \frac{\partial \mathcal{L}_S}{\partial \bar{S}} - \frac{\partial \mathcal{L}_Y}{\partial \bar{S}} = 0$$

or

$$-\frac{\partial}{\partial \eta^0}[(-G)^{1/4}(G_{00})^{-1/2}\sigma^0 S] - \frac{\partial}{\partial \eta^1}[(-G)^{1/4}(-G_{11})\sigma^1 S] - (-G)^{-1/4}\left[(-G_{11})^{1/2}\sigma^0\frac{\partial}{\partial \eta^0} + (G_{00})^{1/2}\sigma^1\frac{\partial}{\partial \eta^1}\right]S - \frac{\partial \mathcal{L}_Y}{\partial \bar{S}} = 0. \tag{6.23}$$

In the gauge specified by (6.11) this reduces to

$$\left(\sigma^0 \frac{\partial}{\partial \tau} + \sigma^1 \frac{\partial}{\partial \sigma}\right) S_\mu = 0. \tag{6.24}$$

Using the lightlike surface coordinates $u^\pm = \frac{1}{2}(\tau \pm \sigma)$, we get the equations of motion

$$\frac{\partial}{\partial u^+} S_\mu^{(1)} = \frac{\partial}{\partial u^-} S_\mu^{(2)} = 0. \tag{6.25}$$

This means that the independent combinations of the constraint equations (6.22) are

$$S_\mu^{(1)} \partial_- S_\mu^{(1)} = 0, \qquad S_\mu^{(2)} \partial_+ S_\mu^{(2)} = 0. \tag{6.26}$$

We must also check that in our special gauge the equation of motion (6.15 for Y^μ is not affected by the addition of I_S. It is straightforward to compute that this is indeed the case. For example, in the special gauge

$$\frac{\partial \mathcal{L}_S}{\partial \dot{Y}} = \frac{\dot{Y}}{g_{00}}\left(\bar{S}\sigma^1 \frac{\partial}{\partial \tau} S - \bar{S}\sigma^0 \frac{\partial}{\partial \tau} S\right)$$

and this vanishes by the constraints (6.26).

Along with the notion of a superspace, the equations of motion (6.15) and (6.25) together with the constraints (6.11) and (6.26) are the main results of the vectorlike superstring model. Since our constraints are of the Dirac type,[34] any linear combination of them may also be considered as constraints; the choice of a particular combination is a matter of convenience. Here we write down the combination which can be easily identified with those of the dual vectorlike fermion model:

$$iS^{(1)}\partial_- S^{(1)} + \tfrac{1}{2}(\partial_- Y)^2 = 0, \quad iS^{(2)}\partial_+ S^{(2)} + \tfrac{1}{2}(\partial_+ Y)^2 = 0, \quad S^{(1)}\partial_- Y = S^{(2)}\partial_+ Y = 0. \tag{6.27}$$

Except for suppressed internal symmetry indices, these relations have the same form as those given by Iwasaki and Kikkawa. Since a canonical quantization based on (6.15), (6.25), and (6.27) does not depend on how these relations were obtained, the usual results, such as the tachyon and the critical space-time dimension of 10, would also have to be attributed to the superstring model if it is quantized along the lines indicated by Iwasaki and Kikkawa. The critical dimension can be lowered to four if one interprets this model as a color string model[35] by replacing the usual commutation relations with paracommutation relations, but the tachyon persists. It remains an open question whether a consistent quantization scheme without these defects exists for the superstring model.

Although we have obtained a set of equations and constraints based on introducing a superspace and making geometrical arguments, we have not shown explicitly that the action we have written down is invariant under supergauge transformations. So our final task is the explicit demonstration of this property. The simplest way of doing this is to appeal to the parametrization invariance of the theory, i.e., invariance under the general coordinate transformations of the world-sheed parameters η^0 and η^1. Since the supergauge invariance of an action should not depend on the choice of (η^0, η^1), it is only necessary to demonstrate it for a specific choice of coordinates. For obvious reasons we make the choice $(\eta^0, \eta^1) = (\tau, \sigma)$ in which supergauge transformations are simple and the constraints (6.27) are in force. We emphasize that the action is first varied *without* any reference to the above coordinates or constraints. *After* varying the general action (6.12a), the first variation is evaluated in the above choice of coordinates. What we find is that the variation is at most proportional to the constraints (6.27) or to the equations of motion for Y^μ and $S_\mu^{(\alpha)}$. Thus, for the class of functions $\{Y^\mu, S_\mu^{(\alpha)}\}$ which satisfy (6.15), (6.25), and (6.27) the action is supergauge invariant.

More explicitly, consider the supergauge transformations[36]

$$\begin{aligned} S_\mu^{(1)} &\to S_\mu^{(1)} + f(u^-)\partial_- Y_\mu \,, \\ S_\mu^{(2)} &\to S_\mu^{(2)} + g(u^+)\partial_+ Y_\mu \,, \\ Y_\mu &\to Y_\mu + f S_\mu^{(1)} + g S_\mu^{(2)} \,, \end{aligned} \tag{6.28}$$

where f and g are small and anticommute with $S_\mu^{(\alpha)}$ and with each other. To apply these to the action (6.12a), we note that if in the expression (6.8) $\bar{g}_{AB} = \vec{e}_A \cdot \vec{e}_B$ is antisymmetric and the $N_\mu{}^A$'s anticommute, then

$$G_{00} = g_{00} + \frac{\partial Y^\mu}{\partial \eta^0} \frac{\partial Y^\nu}{\partial \eta^0} N_\mu{}^A N_\nu{}^B \bar{g}_{AB} \,,$$

$$G_{11} = g_{11} + \frac{\partial Y^\mu}{\partial \eta^1} \frac{\partial Y^\nu}{\partial \eta^1} N_\mu{}^A N_\nu{}^B \bar{g}_{AB} \,,$$

$$G_{10} = G_{01} = g_{01} + \frac{\partial Y^\mu}{\partial \eta^0} \frac{\partial Y^\nu}{\partial \eta^1} N_\mu{}^A N_\nu{}^B \bar{g}_{AB} \,.$$

To find the variation $\delta\sqrt{-G}$, we must find the variation of these quantities under the transformations (6.28). By straightforward computation the following statements can be verified:

$$g_{00} g_{11} \to g_{00} g_{11} + 2 g_{00} (f S^{(1)} + g S^{(2)}) \cdot (Y'' - \ddot{Y}) \,,$$

$$g_{01}{}^2 \to g_{01}{}^2 + 2 g_{01} \left[\dot{Y} \frac{\partial}{\partial \sigma} (f S^{(1)} + g S^{(2)}) + Y' \cdot \frac{\partial}{\partial \tau} (f S^{(1)} + g S^{(2)}) \right] ,$$

$$\begin{aligned} \dot{Y} \cdot S^{(2)} Y' \cdot S^{(2)} \to \dot{Y} \cdot S^{(2)} Y' \cdot S^{(2)} &+ \dot{Y} \cdot S^{(2)} \left[g Y' \partial_+ Y + \frac{\partial}{\partial \sigma} (f S^{(1)} + g S^{(2)}) \cdot S^{(2)} \right] \\ &+ \left[g \dot{Y} \partial_+ Y + \frac{\partial}{\partial \tau} (f S^{(1)} + g S^{(2)}) \cdot S^{(2)} \right] \cdot Y' \cdot S^{(2)} \,, \end{aligned}$$

and similarly for $\dot{Y} \cdot S^{(1)} Y' \cdot S^{(1)}$, etc. Since the variations in all of these expressions are proportional to constraints or equations of motion in our special gauge, $(\det G_{ab})$ is invariant under the infinitesimal supergauge transformations (6.28), as, of course, we expect it to be.

Similarly, by computing the variations of $\sqrt{G_{00}}$, $\sqrt{G_{11}}$, $S^\alpha \dot{S}^\alpha$, and $S^\alpha S'^\alpha$, one can verify that the action I_S is also invariant under the transformation (6.28), thus verifying the supergauge invariance of the action (6.12a). It is interesting to note that the above invariance also holds if $\bar{g}_{AB}$ is taken to be symmetric,

(b) Quarklike fermions. In this case we replace the $N_\mu{}^A$ appearing in the metric tensor (6.1) by the expression given in (6.3). Again we replace the index A by the set (α, A'), where we take $\alpha = 1$ or 2 (or just $\alpha = 1$, in which case we omit it) and suppress A'. Then we write for I_ψ in (6.12b)

$$I_\psi = \int d^2\eta \mathcal{L}_\psi \,, \tag{6.29}$$

where

$$\begin{aligned} \mathcal{L}_\psi = (-G)^{-1/4} &\left[\bar{\psi} \left(\sqrt{-G_{11}}\, \sigma^0 \frac{\partial}{\partial \eta^0} + \sqrt{G_{00}}\, \sigma^1 \frac{\partial}{\partial \eta^1} \right) \psi \right] \\ - (-G)^{1/4} &\left[\bar{\psi} \left(\frac{\overleftarrow{\partial}}{\partial \eta^0} \frac{\sigma^0}{\sqrt{G_{00}}} + \frac{\overleftarrow{\partial}}{\partial \eta^1} \frac{\sigma^1}{\sqrt{-G_{11}}} \right) \psi \right] . \end{aligned} \tag{6.30}$$

The analog of the constraint equations (6.11) for this case are

$$g_{00}+g_{11}=0\,,$$
$$g_{01}=0\,, \qquad (6.31)$$
$$\frac{\partial Y^\mu}{\partial \eta^a}\bar\psi\gamma_\mu\psi=0\,,\quad a=0,1$$

while the analog of the bilinear constraint (6.22) is

$$\bar\psi\frac{\overleftrightarrow{\partial}}{\partial\eta}\psi=0\,,\quad \eta=\sigma \text{ or } \tau\,. \qquad (6.32)$$

The equation of motion for ψ in the gauge (6.31) becomes

$$\left(\sigma^0\frac{\partial}{\partial\tau}+\sigma^1\frac{\partial}{\partial\sigma}\right)\psi=0\,. \qquad (6.33)$$

This in turn gives

$$\partial_+\psi^{(1)}=\partial_-\psi^{(2)}=0\,. \qquad (6.34)$$

If α in $\psi^{(\alpha)}$ takes on only one value then one of these equations should be omitted. From (6.32) and (6.34) we see that the independent bilinear constraints are

$$\bar\psi^{(1)}\overleftrightarrow{\partial}_-\psi^{(1)}=0\,,$$
$$\bar\psi^{(2)}\overleftrightarrow{\partial}_+\psi^{(2)}=0\,. \qquad (6.35)$$

These constraints as well as those given by (6.31) have the same structure as those found by Bardakci.[25] We note again that our ψ's transform not as densities but as relative densities.

We shall not go into further details of the quark-like superstrings in this paper. We only mention an interesting possibility that this model suggests for fiber bundle manifolds. Since (6.1) defines a fiber bundle, the elimination of $N_\mu{}^A$ in favor of $\bar\psi\gamma_\mu\psi$ may be regarded as interpreting the gauge fields themselves not as fundamental fields but as composites of more elementary Fermi fields.

VII. CONCLUSIONS

Our primary aim in this work was to give a geometrically unified description of space-time and exact local gauge symmetries which emphasizes the parallels with the geometry of gravitation. One result of such a description is that one can make use of powerful topological arguments to systematically obtain and analyze the classical solutions of gauge theories of a given symmetry.

In developing the geometrical picture we were also led to a number of concrete results which are interesting in their own right. These are (a) the appearance of an extra dimensional parameter in non-Abelian gauge theories, (b) the derivation of equations of motion in superspace from an action principle without averaging over group parameters, (c) a natural extension of Born-Infeld theory to curved space-time and the corresponding gravitational field equations, and (d) the construction of supersymmetric strings based on superspaces in which the vertical parameters and the connection forms are anticommuting objects.

Once the parallelism between gravitation and local gauge theories is established, one can turn the argument around and attempt to construct a gauge theory of gravitation. The hope in constructing such a theory is to use the renormalizable non-Abelian gauge theories as models for arriving at a renormalizable quantum theory of gravity. However, we note an essential difference between this approach to gravity and the more conventional non-Abelian gauge theories: For gravity the motions in horizontal and vertical directions are no longer independent. To ensure this, the bases in the horizontal and vertical tangent spaces must be "interlocked."

The formalism can also be easily extended to local supersymmetries. Since the supersymmetry algebra contains a Poincaré subalgebra,[37] a local supergauge theory necessarily involves local space-time symmetry and thus some sort of gravitation. Because of this the horizontal and vertical bases must again be interlocked. The details of this are in preparation and will be given elsewhere.

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