Extended-hadron model based on non-Abelian superconductivity

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(Received 15 December 1975)

A non-Abelian, relativistic Ginzburg-Landau-type Lagrangian is derived from Heisenberg-Nambu nonlinear spinor models. In this Lagrangian, for self-consistent mass fields, two gauge fields are coupled to a complex scalar field through a special type of covariant derivative.

Nielsen and Olesen noted[1] that the Higgs-type Lagrangian is essentially a relativistic generalization of the Ginzburg-Landau Lagrangian used in superconductivity and proposed an analogy between dual strings and Abrikosov vortices.

Eguchi and Sugawara (ES)[2] have succeeded in *deriving* an Abelian Higgs-type Lagrangian from a fundamental Lagrangian comprising only massless fermions. They generalize the ideas of Nambu-Jona-Lasinio[3,4] by letting the self-consistent mass be space-time-dependent. Their Lagrangian is

$$\mathcal{L}(x) = i\bar{\psi}(x)\gamma\cdot\partial\psi(x) + g[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] - g'[(\bar{\psi}\gamma_\mu\psi)^2 + (\bar{\psi}\gamma_5\gamma_\mu\psi)^2]\,. \tag{1}$$

Here the gauge bosons do not have to be introduced externally. They arise as collective excitations of quasiparticle pairs just as the scalar and pseudoscalar densities do. This provides a dynamical basis for the Higgs-type Lagrangian.

The following question arises: Can one use a non-Abelian nonlinear spinor model, as a source, to extract a corresponding Lagrangian, providing, possibly, a more adequate model for hadrons? Thus we are led to extend the ES formalism to the non-Abelian case.

In this paper we derive a non-Abelian Ginzburg-Landau-type Lagrangian, which exhibits an interesting group-theoretical structure. Physical implications of this model will be studied in ensuing publications.

In order to derive our results in a fairly general form, let us take $SU(n)\otimes U(1)$ as the internal-symmetry group and let us consider the following (chiral-invariant) Lagrangian:

$$\begin{aligned}\mathcal{L}(x) = {} & i\bar{\psi}(x)\gamma\cdot\partial\psi(x) \\ & + g_1\{[\bar{\psi}(x)C_0\psi(x)]^2 + [\bar{\psi}(x)C_0 i\gamma_5\psi(x)]^2\} \\ & + g_2\{[\bar{\psi}(x)C_i\psi(x)]^2 + [\bar{\psi}(x)C_i i\gamma_5\psi(x)]^2\} \\ & + g_3\{[\bar{\psi}(x)C_0\gamma_\mu\psi(x)]^2 + [\bar{\psi}(x)C_0\gamma_5\gamma^\mu\psi(x)]^2\} \\ & + g_4\{[\bar{\psi}C_i\gamma_\mu\psi(x)]^2 + [\bar{\psi}(x)C_i\gamma_5\gamma^\mu\psi(x)]^2\}\,, \end{aligned} \tag{2}$$

where C_0, C_i are the generators ($n\times n$ Hermitian matrices) satisfying

$$\begin{aligned} & C_0 = (2/n)^{1/2} I_{(n)}\,, \\ & \mathrm{Tr}C_i = 0, \quad \mathrm{Tr}(C_iC_j) = 2\delta_{ij} \quad (i,j = 1,2,\ldots,n^2-1) \\ & C_iC_j = (2/n)^{1/2}C_0 + (d_{ijk} + if_{ijk})C_k\,. \end{aligned} \tag{3}$$

The d's and the f's are the usual symmetric and antisymmetric structure constants for SU(n).

For certain purposes one might prefer to introduce a chiral-symmetry-breaking mass term and even an explicit symmetry-breaking term[1,5] by adding to (2) a mass term

$$-\bar{\psi}(x)\mu\psi(x)\,.$$

Thus for $n=3$, one may take

$$\mu = [\mathrm{diag}(m_p, m_n, m_l)]\otimes I_{(4)}\,.$$

It will be seen that the structure of our Lagrangian remains unaffected except for the mass terms in it.

Let us note that

$$\begin{aligned} & \delta_{\alpha\beta}\delta_{\gamma\delta} = \tfrac{1}{2}C^i_{\alpha\delta}C^i_{\gamma\beta} + \frac{1}{n}\delta_{\alpha\delta}\delta_{\gamma\beta}\,, \\ & C^i_{\alpha\beta}C^i_{\gamma\delta} = -\frac{1}{n}C^i_{\alpha\delta}C^i_{\gamma\beta} + \left(2-\frac{2}{n^2}\right)\delta_{\alpha\delta}\delta_{\gamma\beta} \\ & \qquad (i = 1,2,\ldots,n^2-1)\,. \end{aligned} \tag{4}$$

Combining these results with the corresponding well-known results for the γ matrices, the Fierz transform of the interaction Lagrangian is found to be

$$f_3'[\bar{\psi}(x)C_0\gamma_\mu\psi(x)]^2 + h_3'[\bar{\psi}(x)C_0\gamma_5\gamma^\mu\psi(x)]^2 + f_4'[\bar{\psi}(x)C_i\gamma_\mu\psi(x)]^2 + h_4'[\bar{\psi}(x)C_i\gamma_5\gamma^\mu\psi(x)]^2\,, \tag{5}$$

where

$$\begin{aligned} & f_3' = -\frac{1}{2n}g_1 - \frac{n^2-1}{2n}g_2 + \frac{1}{n}g_3 + \frac{n^2-1}{n}g_4\,, \\ & h_3' = \frac{1}{2n}g_1 + \frac{n^2-1}{2n}g_2 + \frac{1}{n}g_3 + \frac{n^2-1}{n}g_4\,, \\ & f_4' = -\frac{1}{2n}g_1 + \frac{1}{2n}g_2 + \frac{1}{n}g_3 - \frac{1}{n}g_4\,, \\ & h_4' = \frac{1}{2n}g_1 - \frac{1}{2n}g_2 + \frac{1}{n}g_3 - \frac{1}{n}g_4\,. \end{aligned} \tag{6}$$

Thus it is seen that in our model the tensor terms do not enter through the Fierz transformation. Further simplification may be obtained, if so desired, by introducing, for example, restrictions of the type

$$g_1=g_2\,,\quad g_3=g_4\,, \tag{7}$$

when the total U(n) symmetry is evident in (2).

Let us define

$$f_3=f_3'+g_3,\quad h_3=h_3'+g_3,\quad f_4=f_4'+g_4,\quad h_4=h_4'+g_4. \tag{8}$$

Using (2) and (6) along with definition (8), the self-consistent mass is given by the $(n\times n)\otimes(4\times4)$ matrix

$$\begin{aligned} m(x)=&(i2g_1)[C_0\mathrm{Tr}(C_0S_F(x,x))+i\gamma_5C_0\mathrm{Tr}(C_0i\gamma_5S_F(x,x))]\\ &+(i2g_2)[C_i\mathrm{Tr}(C_iS_F(x,x))+i\gamma_5C_i\mathrm{Tr}(C_ii\gamma_5S_F(x,x))]\\ &+(i2f_3)C_0\gamma^\mu\mathrm{Tr}(C_0\gamma_\mu S_F(x,x))+(i2h_3)C_0\gamma_5\gamma^\mu\mathrm{Tr}(C_0\gamma_5\gamma_\mu S_F(x,x))\\ &+(i2f_4)C_i\gamma^\mu\mathrm{Tr}(C_i\gamma_\mu S_F(x,x))+(i2h_4)C_i\gamma_5\gamma^\mu\mathrm{Tr}(C_i\gamma_5\gamma_\mu S_F(x,x))\\ &+\mu\\ \equiv\,& m^S(x)+i\gamma_5m^P(x)+\gamma^\mu m_\mu^V(x)+\gamma_5\gamma^\mu m_\mu^A(x)\,, \end{aligned} \tag{9}$$

where $m^S(x)$, etc., are general $(n\times n)$ matrices and Tr in (9) implies traces of the γ's as well as those of the C's. The term μ has been included in $m^S(x)$. Henceforth, we will implicitly assume that, at least for a suitably restricted set of coupling constants, the masses are real, i.e., the mass matrices are Hermitian. And we will first derive the equations of motion for these matrices without separating from the beginning the SU(n) and U(1) components.

The Nambu mass (denoted by m_∞ by ES) is given by (denoting the corresponding propagator by S_F^∞)

$$\begin{aligned} m_\infty=\mu&+[g_1C_0\mathrm{Tr}(C_0S_F^\infty(0))]\otimes I_{(4)}\\ &+[g_2C_i\mathrm{Tr}(C_iS_F^\infty(0))]\otimes I_{(4)}. \end{aligned} \tag{10}$$

Thus for SU(3) setting $\mu=[\mathrm{diag}(m_p,m_n,m_l)]\otimes I_{(4)}$

$$\begin{aligned} m_\infty=\mu+\frac{2}{\sqrt3}\Big[&g_1\lambda_0(I_p+I_n+I_l)\\ &+g_2\frac{1}{\sqrt2}\lambda_8(I_p+I_n-2I_l)\\ &+g_2(\tfrac32)^{1/2}\lambda_3(I_p-I_l)\Big]\otimes I_{(4)}, \end{aligned} \tag{11}$$

where

$$I_p=(i8)\int^\Lambda\frac{d^4p}{(2\pi)^4}\,\frac{m_{(p)\infty}}{p^2-m_{(p)\infty}{}^2+i\epsilon}, \tag{12}$$

and so on, where Λ is some cutoff.

In particular, for $g_1=g_2=g$

$$\begin{aligned} m_\infty&=[\mathrm{diag}((m_p+2gI_p),(m_n+2gI_n),(m_l+2gI_l))]\otimes I_{(4)}\\ &\equiv[\mathrm{diag}(m_{(p)\infty},m_{(n)\infty},m_{(l)\infty})]\otimes I_{(4)}. \end{aligned} \tag{13}$$

These give Nambu-type consistency conditions. The mass matrix $m(x)$ is, of course, not in general diagonal.

Now we can write the ES perturbation series for the propagator in the matrix form as

$$\begin{aligned} S_F(x,x)=&S_F^\infty(x,x)\\ &+\int S_F^\infty(x,x')[m(x')-m_\infty]S_F^\infty(x',x)dx'\\ &+\cdots. \end{aligned} \tag{14}$$

Our procedure will be as follows. We will calculate $\mathrm{Tr}_{(\gamma)}(S_F(x,x))$, $\mathrm{Tr}_{(\gamma)}(i\gamma_5S_F(x,x))$, and so on ($\mathrm{Tr}_{(\gamma)}$ meaning the traces over the γ matrices only) to obtain the equations for the $(n\times n)$ matrices $m^S(x), m^P(x), \ldots$. Like ES we will collect the coefficients of the singular terms arising from the one-loop integration. Owing to noncommutativity of the matrices our calculations are even more lengthy than those of ES. But the final result can again be exhibited in a condensed and elegant form. It is this final form that we will present. But let us start by giving a few definitions and by fixing our notations.

When symmetry breaking is explicitly introduced through μ, S_F^∞ is a diagonal matrix corresponding to propagators of different masses. In this case it turns out to be most convenient to develop the divergent integrals in powers of the masses retaining only the ultraviolet-divergent parts (see Appendix).

Let us now come to the necessary definitions concerning the fields. Let

$$\begin{aligned} &\phi(x)\equiv(m^S(x)+im^P(x))\,,\\ &\phi^\dagger(x)\equiv(m^S(x)-im^P(x))\,,\\ \text{and}\\ &B_\epsilon^\mu(x)\equiv(m^{V\mu}(x)+\epsilon m^{A\mu}(x))\quad(\epsilon=\pm1)\,. \end{aligned} \tag{15}$$

Let us also define

$$(\mathfrak{D}^\mu\phi)\equiv\partial^\mu\phi+i(B_+^\mu\phi-\phi B_-^\mu), \quad (16)$$

$$(D_\epsilon^\mu B_\epsilon^\nu)\equiv\partial^\mu B_\epsilon^\nu+i[B_\epsilon^\mu, B_\epsilon^\nu], \quad (17)$$

and

$$F_\epsilon^{\mu\nu}\equiv\partial^\mu B_\epsilon^\nu-\partial^\nu B_\epsilon^\mu+i[B_\epsilon^\mu, B_\epsilon^\nu] \quad (\epsilon=\pm). \quad (18)$$

The self-consistent equations for the masses $(m^S(x), m^P(x), m^{V\mu}(x), m^{A\mu}(x))$ can now be written as

$$(2I_2)^{-1}[\mathrm{Tr}_{(\gamma)}((1-\gamma_5)S_F(x,x))-4I\phi] = \mathfrak{D}^\mu\mathfrak{D}_\mu\phi+2(\phi\phi^\dagger\phi) \quad (19)$$

(with a corresponding equation for $\phi^\dagger$),

$$(2I_2)^{-1}[\mathrm{Tr}_{(\gamma)}((1+\gamma_5)\gamma^\mu S_F(x,x))+2IB_+^\mu] = \tfrac{2}{3}D_{+\nu}F_+^{\mu\nu}+i[(\mathfrak{D}^\mu\phi)\phi^\dagger-\phi(\mathfrak{D}^\mu\phi)^\dagger], \quad (20)$$

and

$$(2I_2)^{-1}[\mathrm{Tr}_{(\gamma)}((1-\gamma_5)\gamma^\mu S_F(x,x))+2IB_-^\mu] = \tfrac{2}{3}D_{-\nu}F_-^{\mu\nu}+i[(\mathfrak{D}^\mu\phi)^\dagger\phi-\phi^\dagger(\mathfrak{D}^\mu\phi)] \quad (21)$$

(I_2 and I are defined in the Appendix).

Thus it is seen that though we have considered the broken-symmetry case, all the explicit dependence on m_∞ has disappeared from the right-hand-side members of (19)–(21). (Some m_∞ dependence is implicit in the left-hand-side members, as will be seen later on.) This is a consequence of perturbing precisely about the point m_∞, but shows up in this straightforward calculation only on adding together terms of all orders. When we have only U(1) symmetry the ES equations are recovered, taking care to introduce the Abelian Fierz-transformation coefficients.

Let us first consider the right-hand sides of Eqs. (19), (20), and (21), ignoring (for the time being) the left-hand sides. These equations, written in the form

(right-hand side) = 0,

can all be obtained from the Lagrangian

$$L(x)=-\tfrac{2}{3}\times\tfrac{1}{8}\mathrm{Tr}(F_{+\mu\nu}F_+^{\mu\nu}+F_{-\mu\nu}F_-^{\mu\nu}) +\tfrac{1}{2}\mathrm{Tr}((\mathfrak{D}_\mu\phi)^\dagger(\mathfrak{D}^\mu\phi))-\tfrac{1}{2}\mathrm{Tr}((\phi^\dagger\phi)^2) \quad (22)$$

(considering ϕ, $\phi^\dagger$, B_+, B_- as independent fields). [The factor $\frac{2}{3}$ can be absorbed by rescaling $\phi \to (\frac{2}{3})^{1/2}\phi$ and introducing a scalar quartic coupling constant $\lambda=\frac{3}{2}$ in the last term.]

This Lagrangian is invariant under the following local gauge transformation laws. Let us consider the local chiral transformation group

$$U_+(n)\otimes U_-(n).$$

The transformation laws are taken to be

$$\phi(x)\to U_+(x)\phi(x)U_-(x)^{-1}, \quad (23)$$

$$B_\epsilon^\mu(x)\to U_\epsilon(x)B_\epsilon^\mu(x)U_\epsilon(x)^{-1} + i[\partial^\mu U_\epsilon(x)]U_\epsilon(x)^{-1} \quad (\epsilon=\pm). \quad (24)$$

Thus ϕ transforms as the representation $(n,\bar{n})$, while B_ϵ^μ has the usual vector-gauge-fields transformation law under $U_\epsilon(n)$. Thus D_ϵ^μ and $F_\epsilon^{\mu\nu}$ have the usual covariance properties, and one obtains, as required,

$$(\mathfrak{D}_\mu\phi(x))\to U_+(x)(\mathfrak{D}_\mu\phi(x))U_-(x)^{-1}. \quad (25)$$

Now that the vector-mass field is no longer decoupled for the non-Abelian case, the correspondence between the charge-conjugate pairs and chiral pairs is fully exhibited.

We have the situation that two otherwise independent gauge fields ($B_\pm^\mu$) are coupled to a non-Hermitian scalar field (ϕ) through the "chiral-covariant derivative" ($\mathfrak{D}_\mu\phi$). Let us note also that the last term in this is $\mathrm{Tr}((\phi^\dagger\phi)^2)$ and not the more frequently introduced quartic term $(\mathrm{Tr}(\phi^\dagger\phi))^2$.

The contributions of the left-hand side in each case provide possible mass terms. These terms depend on the nature of the symmetry breaking introduced and also in general depend on the cutoff or some other regularization scheme adopted.

In their case ES suppress the terms ($IB_\pm^\mu$) on the left-hand side by appealing to gauge invariance. It will be noted that this has the same consequence as the convention used in dimensional regularization [Eq. (A5) of the Appendix]. As for (19) the use of the Nambu–Jona-Lasinio–type consistency condition reduces the left-hand side for the Abelian case to $(2m_\infty{}^2\phi)$. The Abelian gauge fields exhibit cutoff-dependent masses.

For the non-Abelian case we will not attempt any systematic exploration of the various possibilities for the left-hand-side members. As an example, let us consider the simple case without explicit symmetry breaking and set

$$\mu=0, \quad g_1=g_2=g, \quad g_3=g_4=g'. \quad (26)$$

In this case we obtain

$$\mathrm{Tr}_{(\gamma)}((1-\gamma_5)S_F(x,x))=(i4g)^{-1}\phi, \quad (27)$$

and from (13)

$$(1-i16gI_1)=0.$$

Thus the left-hand side of (19) reduces (as for the Abelian case) to

$$2m_\infty{}^2\phi \quad (28)$$

and leads to the same type of mass term in the Lagrangian $(+m_\infty{}^2\mathrm{Tr}(\phi^\dagger\phi))$.

As regards the gauge fields we obtain

$$(i4g')\mathrm{Tr}_{(\gamma)}(\gamma^\mu S_F(x,x)) = m^{V\mu}(x) - in(2g'-g)C_0\mathrm{Tr}(C_0\gamma^\mu S_F(x,x))$$

and (29)

$$(i4g')\mathrm{Tr}_{(\gamma)}(\gamma_5\gamma^\mu S_F(x,x)) = m^{A\mu}(x) + in(2g'+g)C_0\mathrm{Tr}(C_0\gamma^\mu\gamma_5 S_F(x,x)) .$$

Thus, again, not only do we get masses dependent on the regularization parameter (as in the Abelian case), but the U(1) components play a special role and complicate the chiral picture.

One possible point of view would be to use the spinor model to extract this Lagrangian and then use the gauge-invariant terms (neglecting the cutoff-dependent masses) as a new starting point. That is, at this point one may choose to consider a Lagrangian of the form[6]

$$L(x) = -\tfrac{1}{8}\mathrm{Tr}\left(\sum_{\epsilon=\pm} F_{\epsilon\mu\nu}F_\epsilon^{\mu\nu}\right) + \tfrac{1}{2}\mathrm{Tr}((\mathfrak{D}_\mu\phi)^\dagger(\mathfrak{D}^\mu\phi)) + m^2\mathrm{Tr}(\phi^\dagger\phi) - \tfrac{3}{4}\mathrm{Tr}((\phi^\dagger\phi)^2) .$$

A Higgs shift of the scalar fields can then be introduced to give the gauge fields masses leading to a suitable spectrum generalizing, if necessary, the scalar interaction terms.

Nielsen and Olesen[1] introduced (for the non-Abelian case) two scalar fields coupled to a single gauge field. Here we have obtained a different picture—two gauge fields coupled to a single complex scalar field. In a different context the possibility of introducing two Yang-Mills fields has been mentioned by Mandelstam.[7]

In this paper we have extracted the mass field Lagrangian. In works to follow we intend to study classical solutions, possibilities of quark confinement, particle spectra, the introduction of magnetic monopoles, quantization, and the reformulation of the model using the "boson method."[8] We will also study the situation in two dimensions.

Meanwhile one can contemplate the fact that had one been completely ignorant of the Yang-Mills formalism, one could have extracted it (in a generalized form) from the nonlinear spinor models, which have so many other interesting features.

It is a pleasure to acknowledge interesting discussions with Professor E. Brezin, Professor Z. F. Ezawa, and Professor L. O'Raifeartaigh.

APPENDIX

Defining

$$I_\alpha(m_i) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m_i^2+i\epsilon)^\alpha}, \quad \text{(A1)}$$

we may write, developing the integral in powers of the mass,

$$I_1 = I + m_i^2 I_2 , \quad \text{(A2)}$$

where

$$I \equiv \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2} , \quad \text{(A3)}$$

and we have dropped the argument (m_i) in I_2, since the singularity is now independent of m_i, from this point of view.

In the dimensional-regularization procedure[9] one has (with $\epsilon = 4-n$, where n is the number of dimensions)

$$I_1(m_i) = \frac{im_i^2}{8\pi^2\epsilon} = m_i^2 I_2 , \quad \text{(A4)}$$

and as a consistency condition one has to set

$$I = 0 . \quad \text{(A5)}$$

However, we will keep I in our formulas in order to make contact with other regularization procedures involving cutoffs. (The nature of the regularization procedure will be left implicit in the symbols I_1, I_2, I.) In the matrix notation we will write

$$I_1(m_\infty) = I + (m_\infty^2)I_2 . \quad \text{(A6)}$$

This may be considered as the definition of I, and in the final formulas I_2 can be considered as a (divergent) number without confusion.

We have taken the trouble to state these points in detail in order to avoid misunderstanding and since this point of view is particularly suitable for the broken-symmetry case. Thus, for example, in calculating the second-order perturbation terms one obtains an integral (writing m_i for $m_{i\infty}$ and so on)

$$\int \frac{d^4p}{(2\pi)^4}\frac{p\cdot(p-q)+m_im_j}{(p^2-m_i^2)[(p-q)^2-m_j^2]} = I_1(m_i^2) + (m_j^2+m_im_j-\tfrac{1}{2}q^2)I_2 + \text{nonsingular terms} \quad \text{(A7)}$$

$$= I + (m_i^2+m_j^2+m_im_j-\tfrac{1}{2}q^2)I_2 + \cdots . \quad \text{(A8)}$$

In the form (22) the mass dependences are displayed in a symmetric form and the total final results can be expressed compactly in terms of commutators and anticommutators.

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*Equipe de Recherche du CNRS 174.

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