Quantum corrections to nonlinear wave solutions*

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We investigate the second-order corrections to quantum-soliton solutions in the scheme proposed by Goldstone and Jackiw and expose the systematics as well as the self-consistency of their approach. We find exact agreement between our results and the corresponding predictions of the collective-coordinate method provided we take into account the modifications found by Tomboulis to the original results of Gervais and Sakita and of Callan and Gross.

I. INTRODUCTION

A considerable amount of interest has recently been paid to the quantization of certain field theories which, as has been known for a long time, possess exact nonlinear wave solutions. The main interest that these theories present to particle theorists is the possibility that such solutions (which are usually called solitary waves or solitons) might, in the quantized theory, correspond to extended hadrons.

There have been several different approaches to this problem. In particular, Goldstone and Jackiw[1] used two techniques: a variational calculation using the effective action formalism[2] and the so-called Kerman-Klein method. Gervais and Sakita[3] and Callan and Gross[4] used the method of collective coordinates to quantize the classical theory by a functional integral. Subsequently, Tomboulis[5] showed that their results could be obtained by means of a canonical transformation[6] and noted that, owing to operator orderings, certain terms appear in the quantized theory that seem to be absent in the functional approach.

To lowest order, calculations using the Kerman-Klein method agree with the corresponding collective-coordinate results[5] and, at least formally, also with the static effective action results of Ref. 1 (there are certain difficulties in this approach, discussed in the text, which make certain results ill defined.)

In the present note we extend these investigations by including higher-order terms in the coupling-constant expansion. Our main goal is to establish the self-consistency of the Kerman-Klein method as well as its compatibility with "orthodox" perturbative techniques. To this end, we compute the rest energy of the quantum soliton (baryon) in three different ways: (i) a variational calculation with the static effective action, (ii) ordinary perturbation treatment of the effective Hamiltonian of Ref. 5, and (iii) the Kerman-Klein method developed in Ref. 1.

We work in a class of theories in one space dimension and one time dimension described by the Lagrangian

$$\mathcal{L}[\lambda;\Phi]=\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-U[\lambda;\Phi] \tag{1.1}$$

with $\Phi(x,t)$ a real, spinless field and $U[\lambda;\Phi]$ scaling with respect to λ as

$$U[\lambda;\Phi]=\frac{1}{\lambda}U[1;\lambda^{1/2}\Phi]. \tag{1.2}$$

Although our results do not depend in any important way on the explicit functional form of $U[\lambda;\Phi]$, we shall often carry out our computations in a $\lambda\Phi^4$ theory with

$$U[\lambda;\Phi]=\frac{1}{4!\lambda}(6m^2-\lambda\Phi^2)^2 . \tag{1.3}$$

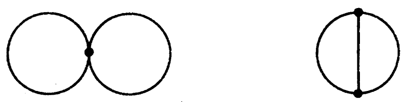
This breaks the symmetry $\Phi\to-\Phi$ in the vacuum states $\Phi_{1,2}=\pm(6/\lambda)^{1/2}m$.

To third order in λ $[O(\lambda^1)]$ we find that the static effective-action approach does not give the full result (which is not a surprise, since the first kinematic corrections appear at this order).

The results obtained by the Kerman-Klein and collective-coordinate methods are shown to be equivalent to this order by making use of consistency sum rules derived from the equations of motion and the field commutation relations. We point out that the fact that the Kerman-Klein method reproduces the "ordering terms" found by Tomboulis to be essential for the Lorentz covariance of the theory provides another proof of the correctness of the relativistic interpretation of Ref. 1.

II. THE STATIC EFFECTIVE-ACTION APPROACH

The use of the effective action as a means to determine the ground state of a theory consists, as is well known, in finding the expectation value of



(a) (b)

FIG. 1. Graphs that contribute to Γ to $O(\lambda)$. The quartic vertex is proportional to λ and the cubic vertex is proportional to $\lambda\phi$.

the Hamiltonian

$$E[\phi, G] = \langle s|H|s\rangle$$

between normalized states constrained by

$$\langle s|\Phi(x,t)|s\rangle = \phi(x) \tag{2.1}$$

and

$$\langle s|\Phi(x,t)\Phi(y,t')|s\rangle_{t=t'} = \phi(x)\phi(y) + G(x,y). \tag{2.2}$$

Then, the minimization condition for the effective action

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x,t)} = 0 \tag{2.3}$$

is equivalent to[2]

$$\frac{\delta E[\phi,G]}{\delta\phi(x)} = \frac{\delta E[\phi,G]}{\delta G(x,y)} = 0. \tag{2.4}$$

The loop expansion for $E[\phi, G]$ is an expansion in powers of λ. The terms of order λ^{-1} and λ^0 have been previously computed.[1,2] The graphs that contribute to the effective action (and hence to $E[\phi,G]$) to order λ are shown in Fig. 1. Of these, the graph in Fig. 1(a) has also been given. To complete the calculation to $O(\lambda)$, we now include the contribution from the graph in Fig. 1(b). This is given by

$$\Gamma_\lambda^{(b)} = \frac{i\lambda^2}{12}\int d^2x\, d^2y\, \phi(x)\phi(y) G(x_0 - y_0; x, y) \times G(x_0 - y_0; x,y) G(x_0 - y_0; x, y), \tag{2.5}$$

where

$$G(x_0 - y_0; x, y) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(x_0 - y_0)} \tilde{G}(\omega; x, y) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega(x_0-y_0)}}{\omega^2 - \frac{1}{4}G^{-2}}(x,y)\ , \tag{2.6}$$

with

$$G(x,y) \equiv G(0; x, y).$$

To compute $\Gamma_\lambda^{(b)}$ in terms of $G(x,y)$, we use a complete, orthonormal set of functions $\{\psi_n\}$ (which we choose, without any loss of generality, to be real) in terms of which G^{-1} has the spectral form

$$G^{-1}(x,y) = \sum_n (2\omega_n)^1 \psi_n(x)\psi_n(y), \tag{2.7}$$

hence,

$$\tilde{G}(\omega; x, y) = \int dx_0 e^{-i\omega x_0} G(x_0; x, y) = i\sum_n \psi_n(x)\frac{1}{\omega^2 - \omega_n^2 + i\epsilon}\psi_n(y). \tag{2.8}$$

Using (2.8) in (2.5) we obtain

$$\Gamma_\lambda^{(b)} = \frac{\lambda^2}{48}\int dx_0 \int dx\, dy\, \phi(x)\phi(y) \sum_{nlm} \psi_n(x)\psi_l(x)\psi_m(x) \times \frac{\psi_m(y)\psi_l(y)\psi_n(y)}{\omega_l\omega_m\omega_n}\left(\frac{1}{\omega_n + \omega_m + \omega_l}\right). \tag{2.9}$$

Therefore, the contribution to the energy functional from this graph is given by

$$E_\lambda^{(b)}[\phi, G] = -\tfrac{1}{48}\sum_{lmn}\frac{(a_{lmn}[\phi,\psi])^2}{\omega_l\omega_m\omega_n}\left(\frac{1}{\omega_l + \omega_m + \omega_n}\right), \tag{2.10}$$

where[7]

$$a_{lmn} \equiv \int dx\, U'''(\phi(x))\psi_l(x)\psi_m(x)\psi_n(x). \tag{2.11}$$

The full expression for $E[\phi, G]$ to $O(\lambda)$ is thus

$$E[\phi,G] = E_c[\phi] + E_0[\phi, G] + E_\lambda[\phi, G], \tag{2.12}$$

where[1,2]

$$E_c[\phi] = \int dx[\tfrac{1}{2}(\phi')^2 + U(\phi)] = \int dx (\phi')^2, \tag{2.13a}$$

$$E_0[\phi, G] = \tfrac{1}{8}\int dx\, G^{-1}(x,x) + \tfrac{1}{2}\int dx\, dy\, G(x,y)\left[\frac{-d^2}{dx^2} + U''(\phi)\right]\delta(x-y), \tag{2.13b}$$

and

$$E_\lambda[\phi, G] = E_\lambda^{(a)} + E_\lambda^{(b)}, = \tfrac{1}{8}\int dx\, U''''(\phi(x)) G(x,x) G(x,x) - \tfrac{1}{48}\sum_{lmn}\frac{b_{lmn}}{\omega_l + \omega_m + \omega_n}, \tag{2.13c}$$

$$b_{lmn} = \frac{(a_{lmn})^2}{\omega_l\omega_m\omega_n}.$$

To obtain the energy of the physical state $|s\rangle$ from Eqs. (2.13), we must impose the stability conditions, Eqs. (2.3), on the energy functional.

Expanding $E[\phi, G]$ in powers of λ with

$$\phi=\phi_c+\sum_i \phi_i, \quad G=G_0+\sum_i G_i, \tag{2.14}$$

and $\phi_c \sim O(\lambda^{1/2})$, $G_0 \sim O(\lambda^0)$, the condition $\delta E/\delta\phi=0$ implies

$$-\phi''+U'(\phi)+\tfrac{1}{2}U'''(\phi)G(x,x)=\frac{1}{48}\frac{\delta}{\delta\phi}\sum_{lmn}\frac{b_{lmn}[\phi,\psi]}{\omega_l+\omega_m+\omega_n}. \tag{2.15}$$

Expanding and keeping terms to order $\lambda^{3/2}$ and using $\delta E_c[\phi_c]/\delta\phi=0$ we obtain

$$-\phi_c''+U'(\phi_c)=0, \tag{2.16a}$$

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\phi_1=-\tfrac{1}{2}U'''(\phi_c)G_0(x,x), \tag{2.16b}$$

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\phi_1=-\tfrac{1}{2}U'''(\phi_c)G_1(x,x)-\tfrac{1}{2}\phi_1{}^2U'''(\phi_c) -\frac{1}{48}\sum_{lmn}\frac{(\delta/\delta\phi)b_{lmn}[\phi_c,\psi]}{\omega_l+\omega_m+\omega_n} -\tfrac{1}{2}\phi_1U''''(\phi_c)G_0(x,x). \tag{2.16c}$$

Varying now with respect to G, we obtain

$$\tfrac{1}{4}G^{-2}(x,y)=\left[-\frac{d^2}{dx^2}+U''(\phi)+\tfrac{1}{2}U''''(\phi)G(x,x)\right]\delta(x-y) +2\frac{\delta}{\delta G(x,y)}E_\lambda^{(b)}[\phi,G]. \tag{2.17}$$

However, since, to order λ, neither ϕ_2 nor G_1 appear in $E[\phi, G]$,[8] we need determine only G_0. This is given by

$$\tfrac{1}{4}G_0{}^{-2}(x,y)=\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\delta(x-y). \tag{2.18}$$

This implies that the set $\{\psi_n\}$ is formed by the solutions of

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\psi_n=\omega_n{}^2\psi_n. \tag{2.19}$$

We have, until now, said nothing about whether G^{-2} can be inverted or not. As is well known, the formalism, as it stands, leads to an unavoidable infrared singularity in G and hence in $E[\phi, G]$. Indeed, from Eq. (2.16a), it follows that

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\phi_c'=0, \tag{2.20}$$

so that Eq. (2.19) has an eigensolution $\psi_0=N\phi_c'$ with eigenvalue $\omega=0$. This is a consequence of translational invariance and is therefore referred to as the translation mode.[1] We shall, for the time being, forget about this problem and proceed to calculate the energy. Using Eqs. (2.16)–(2.18), we obtain

$$E[\phi,G]=E_c[\phi_c]+\tfrac{1}{8}\int dz dz'\,\phi_1(z)G_0{}^{-2}(z,z')\phi_1(z') +E_0[\phi_c,G_0]+\tfrac{1}{2}\int dz\,\phi_1(z)G_0(z,z)U'''(\phi_c) +E_\lambda[\phi_c,G_0]+O(\lambda^2). \tag{2.21}$$

Writing ϕ_1 in terms of ϕ_c and simplifying we find for the mass of the state $|s\rangle$ to $O(\lambda)$

$$\phi_1(x)=-2\int dy\,U'''(\phi_c(y))G_0{}^2(x,y)G_0(y,y), \tag{2.22}$$

$$M=M_c+\Delta M_0+\Delta M_1+O(\lambda^2), \tag{2.23}$$

where ϕ_c and G_0 satisfy, respectively, Eqs. (2.16a) and (2.20); $M_c=E_c[\phi_c]=\int dx\,(\phi_c')^2$ is the classical mass, and

$$\Delta M_0=\tfrac{1}{4}\int dx\,G_0{}^{-1}(x,x) \sim O(\lambda^0), \tag{2.24a}$$

$$\Delta M_1=\tfrac{1}{8}\int dx\,U''''(\phi_c(x))G_0(x,x)G_0(x,x) -\tfrac{1}{2}\int dxdy\,U'''(\phi_c(x))G_0(x,x)G_0{}^2(x,y)G_0(y,y) \times U'''(\phi_c(y))-\tfrac{1}{48}\sum_{lmn}\frac{b_{lmn}[\phi_c,\psi]}{\omega_l+\omega_m+\omega_n} \sim O(\lambda^1). \tag{2.24b}$$

III. PERTURBATION THEORY

In this section we compute the baryon energy to $O(\lambda)$ as well as the matrix elements of Φ between one-baryon–one-meson states and between one-baryon–two-meson states to lowest order $[O(\lambda^{1/2})]$, by ordinary perturbation treatment of the effective Hamiltonian of Ref. 5.

The method for obtaining an effective Hamiltonian for the theory given by Eq. (1.1) consists in separating the original Hamiltonian into a free meson and baryon part plus a meson self-interaction part and a meson-baryon interaction. This separation can be accomplished by means of a canonical transformation on the original fields Φ and Π_0.[5]

The transformation is given by

$$\Phi(x,t)\to\phi_c(x-X)+\chi(x-X,t), \tag{3.1a}$$

$$\Pi_0(x,t)\to\pi(x-X,t)-\frac{p+\int dx\chi'\pi}{M_c(1+\xi/M_c)}, \tag{3.1b}$$

where χ and π represent the meson degrees of freedom of the original fields and $\xi=\int dx\,\chi'\phi_c'$. Here $X(t)$ stands for a new dynamical variable (the center-of-mass coordinate) conjugate to the new momentum variable $p(t)$.

The transformation is subject to the constraints

$$\int dx\, \chi(x,t)\phi_c{}'(x)=0 \tag{3.2a}$$

and

$$\int dx\, \pi(x,t)\phi_c{}'(x)=0. \tag{3.2b}$$

The resulting Hamiltonian is given by

$$H=M_c+\int dx[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\chi')^2+\tfrac{1}{2}\chi^2U''(\phi_c)]+\int dx\left(\frac{\lambda}{6}\phi_c\chi^3+\frac{\lambda}{4!}\chi^4\right)+\frac{1}{2M_c}\left[\left(p+\int dx\chi'\pi\right),\frac{1}{2(1+\xi/M_c)}\right]_+^2$$
$$-\frac{1}{8M_c^2}\left(1+\frac{\xi}{M_c}\right)\int dx(\phi_c'')^2$$
$$\equiv H_0+H_I, \tag{3.3}$$

where the fields χ and π are quantized in the standard way

$$\chi(x,t)=\sum_n{}'\frac{1}{(2\omega_n)^{1/2}}(a_n\psi_ne^{-i\omega_nt}+\text{H.c.}), \tag{3.4a}$$

$$\pi(x,t)=\sum_n{}'\left(\frac{\omega_n}{2}\right)^{1/2}i(a_n^\dagger\psi_ne^{i\omega_nt}-\text{H.c.}), \tag{3.4b}$$

with

$$[a_n,a_m]=[a_n^\dagger,a_m^\dagger]=0;$$
$$[a_n,a_m^\dagger]=\delta_{nm}.$$

Here the expansion is made in terms of the complete set $\{\psi_n\}$ of solutions to the free-field equation [compare Eq. (2.19)]

$$-\chi''+U''(\phi_c)\chi=\omega^2\chi. \tag{3.5}$$

This set includes the eigensolution ψ_0 for $\omega=0$, which must be excluded from the expansions in Eqs. (3.4) because of the constraint equations, (3.2a) and (3.2b). Whether the expansion is made in terms of "in" states or "out" states is of no consequence for our purpose.

Expanding H_I to $O(\lambda)$, we obtain

$$H_I=H_I^1+H_I^2,$$

where

$$H_I^1=\frac{\lambda}{6}\int dx\,\phi_c\chi^3 \tag{3.6a}$$

and

$$H_I^2=\frac{p^2}{2M_c}+\int dx\left(\frac{\lambda}{4!}\chi^4+\frac{p}{M_c}\chi'\pi\right)+\frac{1}{2M_c}\left(\int dx\chi'\pi\right)^2-\frac{1}{8M_c^2}\int dx(\phi_c'')^2 \tag{3.6b}$$

[clearly, $H_I^1\sim O(\lambda^{1/2})$ and $H_I^2\sim O(\lambda)$].

To order λ^0, the energy of the one-baryon state $|p\rangle$ is given by[5]

$$\langle p'|H_0|p\rangle=E_0(p)\delta(p-p'),$$
$$E_0(p)=M_c+\tfrac{1}{4}\int dx\,G_f^{-1}(x,x), \tag{3.7}$$

where

$$G_f=\sum_n\psi_n\frac{1}{2\omega_n}\psi_n$$

is the infrared-finite part of G_0 [see Eq. (2.7)]. The terms that contribute to $E(p)$ to $O(\lambda)$ are obtained by a first-order calculation with H_I^2 and a second-order calculation with H_I^1. The second-order term is given by

$$E_\lambda^1(p)\delta(p-p')=\int\frac{dq}{2\pi}\sum_{\{l\}}\frac{\langle p|H_I^1|q;\{l\}\rangle\langle q;\{l\}|H_I^1|p'\rangle}{E(p')-E(p;\{l\})}. \tag{3.8}$$

The only nonvanishing terms are those with one and three mesons in the intermediate states, Explicitly,

$$E_\lambda^1(p)=-\frac{\lambda^2}{36}\int dx\,dy\,\phi_c(x)\phi_c(y)\sum_n\frac{\langle 0|\chi^3(x)|n\rangle\langle n|\chi^3(y)|0\rangle}{\omega_n}-\frac{\lambda^2}{36}\int dx\,dy\,\phi_c(x)\phi_c(y)\sum_{lmn}\frac{\langle 0|\chi^3(x)|lmn\rangle\langle lmn|\chi^3(y)|0\rangle}{\omega_l+\omega_m+\omega_n}.$$

This gives

$$E_\lambda^1(p)=-\tfrac{1}{2}\int dx\,dy\,U'''(\phi_c(x))G_f(x,x)G_f^2(x,y)G_f(y,y)U'''(\phi_c(y))-\tfrac{1}{48}\sum_{lmn}\frac{b_{lmn}[\phi_c,\psi]}{\omega_l+\omega_m+\omega_n}. \tag{3.9a}$$

The first-order calculation with H_I^2 gives

$$E_\lambda^2(p)=\frac{p^2}{2M_c}+\tfrac{1}{8}\int dx\,U''''(\phi_c(x))G_f(x,x)G_f(x,x)+\frac{1}{8M_c^2}\int dx(\phi_c'')^2+\frac{1}{8M_c}\int dx\,dy\,G_f^{-1}(x,y)\partial_x\partial_yG_f(x,y), \tag{3.9b}$$

and $E(p)=E_0(p)+E_\lambda^1(p)+E_\lambda^2(p)$. Setting $p=0$ we obtain the baryon mass, M, to order λ,

$$M=M_c+\Delta M_0+\Delta M_1+\frac{1}{8M_c^2}\int dx(\phi_c'')^2 +\frac{1}{8M_c}\int dx\,dy\,G_f^{-1}(x,y)\partial_x\partial_y G_f(x,y), \qquad (3.10)$$

with ΔM_0 and ΔM_1 defined in Eqs. (2.24).

We observe that, aside from the fact that here M is given only in terms of the infrared-finite G_f, Eq. (3.10) differs from our previous result, Eq. (2.23), by the appearance of two new terms. This discrepancy might be explained from the viewpoint that kinematical corrections, which cannot be assessed in the direct static variational calculation of the past section, enter at this point.

One can also trivially compute the energy of the state $|p;n\rangle$. Using Eq. (3.3), we immediately obtain

$$E_0(p;n)\delta(p-p')\delta_{nn'}=\langle p;n|H_0|p';n'\rangle,$$
$$E_0(p;n)=M_c+\tfrac{1}{4}\int dx\,G_f^{-1}(x,x)+\omega_n. \qquad (3.11)$$

This result *defines* ω_n to this order, and as we shall show in the following section, the determination of ω_n to $O(\lambda^0)$ suffices to determine M to $O(\lambda)$.

For completeness, and for further comparison, we now compute the two-meson matrix elements of Φ. The no-meson-to-two-meson matrix element is clearly given by

$$\langle p|\Phi(x)|p';nm\rangle =\int\frac{dq}{2\pi}\sum_{\{l\}}\frac{\langle p|\Phi(x)|q;\{l\}\rangle\langle q;\{l\}|H_I^1|p';nm\rangle}{E(p';nm)-E(q;\{l\})} +\int\frac{dq}{2\pi}\sum_{\{l\}}\frac{\langle p|H_I^1|q;\{l\}\rangle\langle q;\{l\}|\Phi(x)|p';nm\rangle}{E(p)-E(q;\{l\})}. \qquad (3.12)$$

Only one-meson intermediate states contribute to the first sum, and three-meson intermediate states to the second. After a straightforward calculation, we obtain,

$$\langle p|\Phi(x)|p'nm\rangle=\int dx_0 e^{i(p'-p)x_0}\frac{\theta_{nm}(x-x_0)}{(2\omega_n 2\omega_m)^{1/2}},$$

where

$$\theta_{nm}(x)=\frac{1}{\sqrt{2}}\sum_l{}'\frac{\psi_l(x)}{2\omega_l}\int dy\,U'''(\phi_c)\psi_n(y)\psi_m(y)\psi_l(y) \times\left(\frac{1}{\omega_n+\omega_m-\omega_l}-\frac{1}{\omega_n+\omega_m+\omega_l}\right). \qquad (3.13)$$

In much the same way, one finds for the one-meson to one-meson matrix element,

$$\langle p;n|\Phi(x)|p';m\rangle_c=\int dx_0 e^{i(p'-p)x_0}\frac{\theta_n^m(x-x_0)}{(2\omega_n 2\omega_m)^{1/2}},$$

with

$$\theta_n^m(x)=\sum_l{}'\frac{\psi_l(x)}{2\omega_l}\int dy\,U'''(\phi_c)\psi_n(y)\psi_m(y)\psi_l(y) \times\left(\frac{1}{\omega_m-\omega_n-\omega_l}-\frac{1}{\omega_m-\omega_n+\omega_l}\right). \qquad (3.14)$$

Of course, for an explicit evaluation of our formulas in a given model, one must still perform a vacuum energy subtraction as well as a meson-mass renormalization to render M finite. This in fact removes all the divergences from the theory.[3,11,12]

IV. THE KERMAN-KLEIN METHOD

In this section we shall extend the computational scheme of Ref. 1 beyond the static approximation to compute all the $O(\lambda)$ corrections to the baryon.

The main goal of this technique is to compute all the matrix elements of the quantum field Φ between single-baryon–multimeson states. Such states we shall label by the total momentum, p, and a set of labels, $\{n\}$, one for each meson, required to specify the state uniquely. The state $|p;n_1n_2\cdots n_l\rangle$ has an energy $E(p;\{n\}) =[(M_c+\sum\omega_n)^2+p^2]^{1/2}$ with M_c the classical baryon mass and $\omega_n(k)=(\mu^2+k_n^2)^{1/2}$ the meson energy. Requiring the states to be energy and momentum eigenstates, from the quantum equations of motion for Φ, one finds a set of equations for the matrix elements the field in these states. In Ref. 1, an approximation procedure for these equations was developed. Here we shall first show that such an approximation is self-consistent and then proceed to extend the scheme to include kinematical corrections.

Defining first

$$\langle p;n_1n_2\cdots|\Phi(x)|p';m_1m_2\cdots\rangle_c =\frac{F_{n_1n_2\cdots}^{m_1m_2\cdots}(p,p';x)}{(2\omega_{n_1}\cdots 2\omega_{m_1}\cdots)^{1/2}}$$

(the normalization factors are for later convenience; the subscript "c" stands for connected) we find

$$\{[E(p;\{m\})]^2-(p-p')^2\}\frac{F_{\{n\}}^{\{m\}}(p,p';x)}{(2\omega_{n_1}\cdots 2\omega_{m_1}\cdots)^{1/2}} =\langle p;\{n\}|U'(\Phi)|p';\{m\}\rangle_c. \qquad (4.1)$$

We shall now derive and solve all the equations that result from Eq. (4.1) to order $\lambda^{3/2}$.

A. Lowest-order equations

To lowest order, Galilean invariance for F is exact. (This amounts to the static approximation of Ref. 1.) Then, defining the Fourier transforms

$$F_{0\{n\}}^{\{m\}}(p'-p;x)=\int dx_0\, e^{i(p'-p)x_0}\tilde{F}_{0\{n\}}^{\{m\}}(x-x_0), \quad (4.2)$$

one gets (at first explicitly for the Φ^4 theory)

$$\left[-\frac{d^2}{dx^2}+\left(\sum_{\{n\}}\omega_n-\sum_{\{m\}}\omega_m\right)^2+m^2\right]\tilde{F}_{0\{n\}}^{\{m\}}(x)=\frac{\lambda}{6}\sum_{\{k,l\}}\tilde{F}_{0\{k\}}^{\{m\}}\tilde{F}_{0\{l\}}^{\{k\}}\tilde{F}_{0\{n\}}^{\{l\}}, \quad (4.3)$$

so that the no-meson function obeys the classical equation,

$$-\tilde{F}_0''+U'(F_0)=0, \quad (4.4a)$$

with solution $\tilde{F}_0=\phi_c$. For the one-meson function we have

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\tilde{F}_{0n}=\omega_n{}^2\tilde{F}_{0n}. \quad (4.4b)$$

This is Eq.(3.5) once more. We choose the solutions of this equation to be the members of the set $\{\psi_n\}$ used in the past sections. Here, however, this set does not include ψ_0, the zero-frequency solution, since the state with $\omega=0$ does not correspond to a physical state.[9] The completeness relation for this set then reads

$$\sum_l{}'\psi_l(x)\psi_l(y)=\delta(x-y)-\psi_0(x)\psi_0(y), \quad (4.5)$$

with

$$\psi_0=(M_c)^{-1/2}\phi_c{}'. \quad (4.6)$$

For the two-meson functions we have

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\tilde{F}_n^m=(\omega_m-\omega_n)^2\tilde{F}_n^m-U'''(\phi_c)\tilde{F}^m\tilde{F}_n, \quad (4.7a)$$

and

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\tilde{F}_{nm}=(\omega_n+\omega_m)^2\tilde{F}_{nm}-\frac{1}{\sqrt{2}}U'''(\phi_c)\tilde{F}_n\tilde{F}_m. \quad (4.7b)$$

Using the completeness and orthonormality of the set $\{\psi_n\}$, one can easily solve Eqs. (4.7a) and (4.7b) in terms of ψ_n and ϕ_c. The solutions are

$$\tilde{F}_{nm}(x)=\sum_l{}'\frac{1}{\sqrt{2}}\frac{\psi_l(x)}{2\omega_l}\int dy\,U'''(\phi_c)\psi_l(y)\psi_m(y)\psi_n(y)\times\left(\frac{1}{\omega_m+\omega_n-\omega_l}-\frac{1}{\omega_l+\omega_m+\omega_n}\right) \quad (4.8a)$$

and

$$\tilde{F}_n^m(x)=\sum_l{}'\frac{\psi_l(x)}{2\omega_l}\int dy\,U'''(\phi_c)\psi_l(y)\psi_m(y)\psi_n(y)\times\left(\frac{1}{\omega_m-\omega_n-\omega_l}-\frac{1}{\omega_m-\omega_n+\omega_l}\right). \quad (4.8b)$$

Thus the functions $F_n^m(x)$ and $F_{nm}(x)$ coincide, respectively, with the functions θ_n^m and θ_{nm} of the past section [see Eqs. (3.13) and (3.14)].

B. Higher-order results

The first correction to $\tilde{F}$ is of order $\lambda^{1/2}$ and is given by[1]

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\phi_1(x)=-\tfrac{1}{2}U'''(\phi_c)G_f(x,x), \quad (4.9a)$$

with solution

$$\phi_1(x)=2\int dy\,G_f{}^2(x,y)G_f(y,y)U'''(\phi_c) \quad (4.9b)$$

[compare Eqs. (2.16b) and (2.22)]. In this case, Galilean invariance is still an exact statement for F, since the first deviations [coming from the energy differences in Eq. (4.1)] would be of the form $(\Delta E)^2F_0$, which is of order $\lambda^{3/2}$.

The equations for $\tilde{F}_n$ to order λ are slightly more complicated than the previous examples due to the fact that here there will be a term $[(p^2-p'^2)/M_c]\omega_nF_{0n}$ in the left-hand side of (4.1) that spoils Galilean invariance. However, demanding Lorentz invariance for F_n, it is not difficult to see that it must be of the form

$$\delta_1\mathfrak{F}_n(p'-p;x)+(p'+p)\delta_2\mathfrak{F}_n(p'-p;x), \quad (4.10)$$

to order λ. Using this result, one finds

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)-\omega_n{}^2\right]\delta_2\psi_n{}'=-\frac{2p}{M_c}\omega_n\psi_n{}' \quad (4.11a)$$

and

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\delta_p\psi_n=J_n, \quad (4.11b)$$

where

$$J_n=\frac{\omega_n}{M_c}\psi_n''-\tfrac{1}{2}U''''(\phi_c)G_f\psi_n-U'''(\phi_c)\phi_1\psi_n+2\omega_n\delta\omega_n\psi_n-U'''(\phi_c)\sum_l{}'\frac{\psi_l}{2\omega_l}\left(\tfrac{1}{2}\theta_n^l+\frac{1}{\sqrt{2}}\theta_{ln}\right), \quad (4.11c)$$

where we have written $\delta\omega_n\,[O(\lambda)]$ as the first (and as yet, unknown) quantum correction to the meson energy. It is easy to show, however, that $\tilde{F}_{1n}$ is in fact independent of $\delta\omega_n$.[13] The full correction can thus be written as

$$\langle p; n|\Phi(x)|p'\rangle^{(1)} = \frac{1}{(2\omega_n)^{1/2}}\int dx_0 e^{i(p'-p)x_0}\Psi_{1n}(x-x_0), \tag{4.12}$$

with

$$\delta_p \mathfrak{F}_n(q;x) = \delta_1\mathfrak{F}_n(q;x) + 2p\,\delta_2\mathfrak{F}_n(q;x) = \int dx_0 e^{iqx_0}\delta_p\psi_n(x-x_0) \tag{4.13a}$$

and

$$\Psi_{1n}(x) = \delta_p\psi_n(x) - i\delta_2\psi_n'(x). \tag{4.13b}$$

To solve these equations we again use the spectral form for the operator $[-d^2/dx^2 + U''(\phi_c)]\delta(x-y)$. Eq. (4.11b) gives

$$(\omega_l^2 - \omega_n^2)(\psi_l, \delta_p\psi_n) = (\psi_l, J_n) \tag{4.14}$$

and, for $l \neq n$,

$$\delta_p\psi_n(x) = \sum_l c_{nl}\psi_l(x), \tag{4.15}$$

with $c_{nl} = (\psi_l, J_n)/(\omega_l^2 - \omega_n^2)$. However, since, for any constant α, $(\delta_p\psi_n) \equiv \delta_p\psi_n + \alpha\psi_n$ is also a solution of Eq. (4.11b), we can always choose α such that $((\delta_p\psi_n), \psi_n) = 0$. With this choice (which only affects the normalization of $\delta_p\psi_n$), one has $c_{nn} = 0$. Using a similar argument for $\delta_2\psi_n'$, we obtain

$$\Psi_{1n}(x;p) = \sum_l{}' \frac{\psi_l(x)}{\omega_l^2 - \omega_n^2}\left(\psi_l, J_n + \frac{2pi}{M_c}\psi_n'\right), \tag{4.16a}$$

with

$$(\Psi_{1n}, \psi_n) = 0. \tag{4.16b}$$

This implies the necessary condition

$$(\psi_n, J_n) = 0. \tag{4.17}$$

For the kinematical analysis of the second correction to the no-meson function, notice that Lorentz invariance demands that it be a function (to all orders in λ) of the difference $\nu - \nu'$ where ν is the rapidity associated with the momentum p. This suggests an elegant and practical way of deriving the equations of motion[10] (this result can also be obtained, although in a somewhat more cumbersome way, by using the same method we used for F_{1n}). These are obtained from the general equation

$$M_c^2\left[\phi\left(x + \frac{i}{M_c}\right) + \phi\left(x - \frac{i}{M_c}\right) - 2\phi(x)\right] = -U'(\phi) + \Lambda, \tag{4.18}$$

where Λ is the part of $\langle p|U'(\Phi)|p'\rangle$ that includes meson intermediate states. The equations for ϕ to a given order in λ are then obtained by expanding the left-hand side of Eq. (4.18) in powers of $1/M_c$ [which, as we have seen, is $O(\lambda)$] and then expanding ϕ as usual. To $O(\lambda^{3/2})$ after using Eqs. (4.5a) and (4.9a), we get

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\phi_2 = T, \tag{4.19}$$

where

$$-T = \frac{1}{12M_c^2}\phi_c'''' + \tfrac{1}{2}U''''(\phi_c)\phi_1 G_f + \tfrac{1}{2}\phi_1^2 U'''(\phi_c) + \tfrac{1}{2}U'''(\phi_c)\left[\sum_l{}'\frac{\delta\omega_l}{(2\omega_l)^2}\psi_l\psi_l + \sum_l{}'\frac{1}{2\omega_l}(\psi_l\Psi_{1l}^* + \text{c.c.}) + \sum_{ln}{}''\frac{\theta_{ln}\theta_{ln}}{2\omega_l 2\omega_n}\right] + \frac{1}{24}U''''(\phi_c)\sum_{ln}{}''\frac{\psi_l\psi_n}{\omega_l\omega_n}\left(\theta_n^l + \frac{4}{\sqrt{2}}\theta_{nl}\right). \tag{4.20}$$

The term in square brackets corresponds to the order-λ term in the expansion of $\langle p|\Phi\Phi|p\rangle - \phi\phi$, that is, to the infrared-finite part of the function we called G_1 in Sec. II [see Eq. (2.2)]. Using the explicit form for the two-meson functions, θ_n^0 and θ_{ln}, given by Eqs. (4.8a) and (4.8b), the last term in the above equation can easily be shown to be equal to

$$-\frac{1}{48}\sum_{lmn}\frac{(\delta/\delta\phi)b_{lmn}[\phi_c, \psi]}{\omega_l + \omega_m + \omega_n}.$$

So that, apart from the first term in Eq. (4.20), Eq. (4.19) agrees with our previous equation for ϕ_2, Eq. (2.16c). This discrepancy is of the same nature as the one encountered in the mass calculation of the previous section and may be explained by the same argument about kinematics which we used in that case. Thus, provided that $(\psi_0, T) = 0$.

$$\phi_2(x) = \sum_l{}'\frac{\psi_l(x)}{\omega_l^2}(\psi_l, T). \tag{4.21}$$

With these results, we can now compute the baryon energy

$$E(p) = \langle p|\mathcal{H}(0)|p\rangle,$$

where, in our Φ^4 theory,

$$\mathcal{H}=\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\Phi')^2-\frac{m^2}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4+\frac{3}{2}\frac{m^4}{\lambda}\,,\qquad \pi=\frac{\partial\Phi}{\partial t}\,. \tag{4.22}$$

Expanding as before with single-baryon–multi-meson states, we can express $E(p)$ in terms of the functions we have calculated above. For the quadratic parts one needs up to two-meson intermediate states; for the quartic term we need one- and two-meson matrix elements as well as the disconnected parts of one-to-two-meson matrix elements. For example,

$$\tfrac{1}{2}\langle p|\pi^2(0)|p\rangle=\frac{p^2}{2M_c^2}\int(\phi_c')^2dx-\frac{1}{8M_c^2}\int(\phi_c'')^2dx+\tfrac{1}{8}\int G_f^{-1}(x,x)dx+\tfrac{1}{4}\sum_n{}'\,\omega_n\int dx(\psi_n\Psi_{1n}^*+\text{c.c.})$$
$$+\tfrac{1}{4}\sum_n{}'\,\delta\omega_n(\psi_n,\psi_n)+\tfrac{1}{8}\sum_{ln}{}''\,\frac{(\omega_n+\omega_l)^2}{\omega_n\omega_l}(\theta_{nl},\theta_{nl})\,. \tag{4.23}$$

The other quadratic terms are also trivially obtained. For the quartic term, a straightforward but somewhat lengthy calculation is needed. Collecting terms and using the equations of motion as well as Eqs. (4.16) and (4.17), one sees that the terms proportional to ϕ_2, ψ_{1n} and $\delta\omega_n$ do not contribute to this order. Notice that the fact that $E(p)$ is independent of the corrections to ω_n of the same order is an obviously essential self-consistency requirement, for we would have no way, within the formalism, to calculate $\delta\omega_n$ without knowing $E(p)$ as well as $E(p;n)$ to the same order. It is expected that such a result will appear at each new order in the calculation.

After some rearranging and with the aid of Eqs. (4.8a), (4.8b) and (4.9b), we arrive at the following result for the baryon mass:

$$M=M_c+\Delta M_0+\Delta M_1+\frac{1}{8M_c^2}\int dx(\phi_c'')^2-\frac{1}{8}\sum_{lmn}\frac{\omega_l(\omega_m+\omega_n)}{\omega_n\omega_m}(\psi_l,\theta_{mn})^2\,, \tag{4.24}$$

with ΔM_0 and ΔM_1 defined as before [see Eq. (2.24)]. Equation (4.24) will be shown to coincide with our previous result, Eq. (3.10), by using a consistency sum rule derived from the canonical equal-times commutator:

$$\left\langle p\left|\left[\Phi(x,t),\frac{\partial\Phi(y,t)}{\partial t}\right]\right|p'\right\rangle_{t=0}=2\pi i\delta(x-y)\delta(p-p')\,. \tag{4.25}$$

This relation has been computed within the formalism to lowest order.[1] The $O(\lambda)$ corrections come from four places: (i) $O(\lambda^{1/2})$ corrections to the no-meson matrix elements; these give

$$\frac{1}{M_c}\int dx_0e^{i(p'-p)x_0}[\phi_c'(x-x_0)\phi_1'(y-x_0)+(x\longleftrightarrow y)]\,; \tag{4.26a}$$

(ii) $O(\lambda)$ kinematical corrections to the one-meson terms,

$$\frac{1}{M_c}\int dx_0e^{i(p'-p)x_0}\partial_x\partial_yG_f(x-x_0,y-x_0)\,; \tag{4.26b}$$

(iii) $O(\lambda)$ corrections to the one-meson matrix elements,

$$\tfrac{1}{2}\int dx_0e^{i(p'-p)x_0}\sum_n[\psi_n(x-x_0)\Psi_{1n}^*(y-x_0;p')+\psi_n(y-x_0)\Psi_{1n}(x-x_0;p)+(x\longleftrightarrow y)]\,; \tag{4.26c}$$

(iv) two-meson terms coming from the lowest-order ($\lambda^{1/2}$) two-meson matrix elements,

$$\tfrac{1}{2}\int dx_0e^{i(p'-p)x_0}\sum_{nm}{}''\,\frac{(\omega_n+\omega_m)}{\omega_n\omega_m}[\theta_{nm}(x-x_0)\theta_{nm}(y-x_0)]\,. \tag{4.26d}$$

However, since the left-hand side of Eq. (4.25) is of order λ^0, the sum of these terms must vanish. After multiplying by $\psi_l(x)\psi_l(y)$ and integrating over x and y (shifting by x_0), this sum rule leads to

$$\frac{1}{M_c}\int dxdy[\partial_x\partial_yG_f(x,y)]G_f^{-1}(x,y)+\sum_{lmn}{}'''\,\frac{\omega_l(\omega_n+\omega_m)}{\omega_n\omega_m}(\psi_l,\theta_{nm})^2=0\,. \tag{4.27}$$

Using Eq. (4.27) in Eq. (4.24), we reproduce Eq. (3.10), thus completing our proof.

V. CONCLUSION

The main goal of this calculation has been to establish the self-consistency and systematics of the Kerman-Klein method developed in Ref. 1 to $O(\lambda)$. This we have accomplished by comparing with the corresponding results obtained by collective-coordinate methods. We found, moreover, that the last term in Eq. (3.3) is crucial for the equivalence. Although we were unable to obtain this term from the functional integral directly, we believe that its absence can be explained along the lines of the discussion of Rajaraman and Weinberg,[14] who find that a naive change of variables in the path integral can lead to erroneous results.

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[1]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[2]See also J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974); and K. Cahill, Phys. Lett. 53B, 174 (1974).

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[5]E. Tomboulis, Phys. Rev. D 12, 1678 (1975).

[6]N. H. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).

[7]Here we use the notation of Ref. 1 unless noted otherwise. Throughout the paper, U' refers to a derivative of $U(\phi)$ with respect to $\phi(x)$; $F'(x)$ stands for dF/dx. We also often use the notation (f,g) for $\int dx\, f(x)g(x)$.

[8]The term involving G_1 is

$$\int dz dz' G_1(z,z')\delta E_0[\phi_c, G_0]/\delta G(z,z')\,,$$

but $G_1 \sim O(\lambda)$, so we need $\delta E/\delta G$ to $O(\lambda^0)$, which is zero. The situation for ϕ_2 is analogous.

[9]This is proven in Ref. 1 by saturating the canonical commutator to lowest order.

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[11]S. Coleman, R. Jackiw, and H. Politzer, Phys. Rev. D 10, 2491 (1974).

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[13]This is trivially seen by noting that $J_n \to J_n + 2\omega_n \delta\omega_n \psi_n$ implies

$$\Psi_{1n} \to \Psi_{1n} + 2\omega_n\delta\omega_n \sum_{l\neq n} \frac{\psi_l}{\omega_l^2 - \omega_n^2}(\psi_l, \psi_n) = \Psi_{1n}\,.$$

[14]R. Rajaraman and E. Weinberg, Phys. Rev. D 11, 2950 (1975), Sec. II.