Lattice gauge theory calculations in 1 + 1 dimensions and the approach to the continuum limit

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(Received 17 December 1975)

Lattice perturbation theory for gauge theories of strong interactions is studied for (1 + 1)-dimensional Abelian gauge theories. Strong-coupling expansions for the theory's mass spectrum are carried out to order $1/g^{16}a^{16}$ (a = lattice spacing). These expansions are extrapolated to the continuum limit using Padé approximants. These high-order results are compared to $1/g^8a^8$ expansions reported previously, and convergence to known continuum theory results is noted. In many cases, agreement between the lattice calculations and continuum theory results exists to 2–3%. These calculational methods generalize to (3 + 1)-dimensional theories.

I. INTRODUCTION

This paper continues our study of lattice gauge theories.[1,2] In the long run, we want to develop practical approximate calculational methods which will yield the mass spectrum of color gauge theories of quarks and gluons. Our program is well underway, and some interesting four-dimensional results will be reported soon.[3] Our intent here is somewhat less ambitious. In a recent article[4] by two of the authors in collaboration with Banks, we formulated $(1+1)$-dimensional Abelian gauge theories on a lattice and calculated various quantities in strong-coupling perturbation theory. Then, using Padé approximants, the series expansions for these quantities were extrapolated to zero lattice spacing. The results of this procedure, similar to those used in statistical mechanics, were encouraging—good agreement with known results were found in several cases. It is the purpose of this article to present higher-order calculations of the same type and confirm that the strong-coupling perturbation theory improved with Padé approximants converges to the continuum theory. In many cases we shall find that the approximate calculations lie within 2–3% of various exact and approximate calculations in the continuum theory. We feel that our numerical work confirms in a practical fashion the claim that the continuum limits of lattice gauge theories exist and agree with the continuum theories—at least for our $(1+1)$-dimensional theories. Furthermore, the calculational methods are simple and can be applied in more realistic settings.

This paper is a continuation of the study begun in Ref. 4 and is not meant to be self-contained. The reader should consult Ref. 4 for the fundamentals.

This article is organized into six sections. To begin, we review briefly the (massive) Schwinger model formulated on a lattice. Strong-coupling perturbation theory in the dimensionless parameter $x = 1/g^2a^2$ is briefly described. In Sec. III the eighth-order series expansions for masses of particles in the theory are recorded. In the next section we present the Padé extrapolations for several quantities whose series expansions can be unambiguously extrapolated to the continuum limit, $x\to\infty$. These include the vacuum energy, $\partial M_V/\partial m$ (M_V = mass of the vector particle of the Schwinger model, m = fermion bare mass), M_S/M_V (M_S = mass of a scalar state), and M_V itself. We also compare the mass of the vector particle determined "statically" (as the mass gap at zero momentum) to its mass determined "kinetically" (as the curvature of its energy-momentum relation). Good agreement between the two methods is found. This supports our claim that our calculational methods retrieve a relativistic realistic spectrum of the continuum theory. This is a particularly nontrivial test for the theory to pass since lattice Hamiltonian methods do not have manifest Lorentz invariance. In Sec. V various plots of the Padé approximants are presented for finite x. It is argued that the absolute mass scale of the theory (in units of g) can also be inferred by studying these graphs. Considerable improvements in the eighth-order calculations relative to the fourth-order calculations are noted. A brief section of conclusions and prospects follows.

II. MASSIVE SCHWINGER MODEL ON A LATTICE

We shall collect here the most useful constructions and formulas from Ref. 4. Recall that we place fermion fields on a lattice by assigning a one-component Fermi field $\phi(n)$ to each site $\{n\}$. The continuum two-component field $\chi(x)$ is reconstructed by the equality

$$\chi_{\text{upper}} = \frac{1}{\sqrt{a}}\phi(n), \quad n = \text{even}$$
$$\chi_{\text{lower}} = \frac{1}{\sqrt{a}}\phi(n), \quad n = \text{odd}. \tag{2.1}$$

The gauge potential $A(n)$ enters the theory through unitary operators

$$U(n, n+1) \equiv \exp[iagA(n)] \equiv \exp[i\theta(n)]. \tag{2.2}$$

We always work in the class of gauges $A^0 = 0$ for which a simple Hamiltonian form of the theory exists. The Fermi field $\phi(n)$ satisfies the dimensionless anticommutation relations

$$\{\phi^\dagger(n), \phi(m)\} = \delta_{n,m}, \tag{2.3}$$

while the gauge variables satisfy

$$[\theta(n), L(m)] = i\delta_{n,m}, \tag{2.4}$$

where $\theta(n) = agA(n)$, and $L(n) = (1/g)E(n) = (1/g)\dot{A}(n)$. $L(n)$ is an angular momentum operator whose spectrum consists of the integers. The Hamiltonian, scaled by a factor $2/ag^2$ for convenience, then reads

$$\begin{aligned} W &= \frac{2}{ag^2}H \\ &= \sum_n L^2(n) + \mu\sum_n(-1)^n\phi^\dagger(n)\phi(n) \\ &\quad + ix\sum_n[\phi^\dagger(n)U(n, n+1)\phi(n+1) - \text{H.c.}], \end{aligned} \tag{2.5}$$

where

$$x = \frac{1}{g^2a^2},$$
$$\mu = \frac{2m}{g^2a} = \frac{2m}{g}\sqrt{x},$$
$$m = \text{fermion mass}.$$

The physical spectrum of W is now obtained in perturbation theory. We identify the zeroth-order piece of W, W_0, as the first two terms of Eq. (2.5). W_0 leaves different lattice sites uncoupled. The ground state of W_0 is then a fluxless state $[L(n) = E(n) = 0$ for all $n]$ with $\langle\phi^\dagger(n)\phi(n)\rangle_0 = +1$ for n odd and 0 for n even (we recognize this as the static limit of the Dirac sea). The determination of the ground state for $m = 0$ is more subtle—one argues using second-order perturbation theory in the third term in Eq. (2.5) that discrete γ_5 invariance is spontaneously broken. Then the $m \to 0$ limit of the vacuum of the massive model connects smoothly to that of the massless model. Given the zeroth-order ground state one can calculate the masses of the particles in the theory. Of particular interest will be the vector state which is generated in the continuum theory by applying the spatial integral of the vector flux to the vacuum. On the lattice this state becomes

$$|V\rangle = \sum_n[\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) + \text{H.c.}]|0\rangle. \tag{2.6}$$

A scalar state can be made by using the kinetic-energy density,

$$|S\rangle = i\sum_n[\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) - \text{H.c.}]|0\rangle, \tag{2.7}$$

as discussed in Ref. 4. States of nonzero momentum will also be considered. The vector particle at momentum p is

$$|V(p)\rangle = \sum_n[e^{ik\cdot n}\phi^\dagger(n)e^{i\theta(n)}\phi(n+1) + \text{H.c.}]|0\rangle, \tag{2.8}$$

where

$$p = k/a.$$

Our calculational methods were explained in detail in Ref. 4. In brief, we simply treat the third term in Eq. (2.5) as a perturbation and develop the physical quantities as power series in x. Simple illustrative calculations can be found in Ref. 4 along with detailed graphical rules.

III. EIGHTH-ORDER EXPANSIONS

The calculations of Ref. 4 have been extended to eighth order in $x = 1/g^2a^2$. This was accomplished by computer using a finite lattice and programs employing abstract algebra manipulations (SNOBOL). Many of the calculations, which are simple but tedious, were also done by hand. The eighth-order vacuum energy, sixth-order vector-particle mass, and fourth-order energy-momentum relations were checked by hand. The higher-order computer calculations were checked by repeating them on lattices of various sizes, etc.

Let us list the expansions. It is convenient to define

$$\alpha = 1 + 2\mu,$$
$$\beta = 3 + 2\mu, \tag{3.1}$$
$$\gamma = 1 + \mu.$$

Then the ground-state energy per site, E_0/N, is

$$\frac{E_0}{N}=-\frac{1}{\alpha}x^2+\frac{3}{\alpha^3}x^4-\frac{(58+40\mu)}{\alpha^5\beta}x^6+\frac{(5772+13802\mu+10940\mu^2+3000\mu^3+80\mu^4)}{4\alpha^7\beta^2\gamma}x^8. \quad (3.2)$$

The vector-particle mass is given by

$$2\sqrt{x}\,\frac{M_V}{g}=1+2\mu+\frac{2}{\alpha}x^2-\frac{(10+4\mu)}{\alpha^3}x^4+\frac{(236+272\mu+96\mu^2+16\mu^3)}{\alpha^5\beta}x^6 -\frac{(6626+18738\mu+19980\mu^2+10244\mu^3+2808\mu^4+464\mu^5+32\mu^6)}{\alpha^7\beta^2\gamma}x^8. \quad (3.3)$$

The scalar-particle mass is given by

$$2\sqrt{x}\,\frac{M_S}{g}=1+2\mu+\frac{6}{\alpha}x^2-\frac{(26+4\mu)}{\alpha^3}x^4+\frac{(572+448\mu+16\mu^2-16\mu^3)}{\alpha^5\beta}x^6 -\frac{(15810+32188\mu+32284\mu^2+8964\mu^3-262\mu^4-176\mu^5+32\mu^6)}{\alpha^7\beta^2\gamma}x^8. \quad (3.4)$$

The energy-momentum relation for the vector particle is

$$2\sqrt{x}\,\frac{E_V(p)}{g}=2\sqrt{x}\,\frac{E_V(0)}{g}+\frac{2(1-\cos k)}{\alpha}x^2+\left[\frac{(1-\cos 2k)}{\alpha^2}-\frac{4(1-\cos k)^2}{\alpha^3}\right]x^4 +\frac{1}{\alpha^2}\left\{\frac{1}{\alpha}[(\cos k-\tfrac{1}{2})\cos 2k+\cos k-\tfrac{3}{2}]+\frac{1}{\alpha^2}[(1+\cos 2k)\cos k+4\cos 2k-16)]-\left(\frac{4}{\alpha\beta}+\frac{1}{2\beta}+\frac{6}{\alpha^2\beta}\right)(\cos 2k-1) +\frac{1}{\alpha^3}[14\cos k+28\cos k\cos 2k-16(1+\cos k)^3+22\cos 2k+64]\right\}x^6. \quad (3.5)$$

For the massless Schwinger model these quantities reduce to

$$E_0/N=-x^2+3x^4-\tfrac{58}{3}x^6+\tfrac{481}{3}x^8, \quad (3.6a)$$

$$2\sqrt{x}\,\frac{M_V}{g}=1+2x^2-10x^4+\tfrac{236}{3}x^6-\tfrac{6626}{9}x^8, \quad (3.6b)$$

$$2\sqrt{x}\,\frac{M_S}{g}=1+6x^2-26x^4+\tfrac{572}{3}x^6-\tfrac{15\,810}{9}x^8, \quad (3.6c)$$

$$2\sqrt{x}\,\frac{E_V(p)}{g}=2\sqrt{x}\,\frac{E_V(0)}{g}+2(1-\cos k)x^2+[(1-\cos 2k)-4(1-\cos k)^2]x^4+2(1-\cos k)(5-26\cos k-27\cos^2 k)x^6. \quad (3.6d)$$

IV. EXTRAPOLATION TO THE CONTINUUM LIMIT

Now we begin to extract physics from these results. We first discuss calculations which can be extrapolated to the continuum limit with no ambiguity. First, however, recall the nature of the general problem. In Eq. (3.6), for example, we have expansions derived assuming $x=1/g^2a^2\ll 1$, i.e., large, fixed lattice spacing. In our field theory examples we want to allow a^2 to go to zero, so physics lies at the $x\to\infty$ limit. Clearly the series in Eq. (3.6) cannot be continued to large x without a systematic extrapolation procedure which reflects the character of the Taylor series (note how quickly the coefficients grow with the order of perturbation theory). At this point we will follow the methods of statistical mechanics and employ Padé approximants.[5]

Before considering field theories we shall use our calculation of E_0/N to find the ground-state energy of the x-y antiferromagnetic spin lattice. In this case the lattice spacing is physical and fixed but g must be taken to zero. By consulting Eq. (3.10) of Ref. 4 we see that setting $g=0$ for fixed a^2 reduces the Schwinger model to the spin lattice of interest. Exact calculations of the spin-lattice problem give[6]

$$E_0/N=-0.637. \quad (4.1)$$

From Eq. (3.6a) we form the [2, 2] Padé approximant, using the variable $y=x^2$ for convenience,

$$E_0/N+1=1-y+3y^2-\tfrac{58}{3}y^3+\tfrac{481}{3}y^4 =\frac{1+8.903y+3.473y^2}{1+9.903y+10.376y^2}. \quad (4.2)$$

So

$$E_0/N \xrightarrow[y\to\infty]{} 0.335 - 1 = -0.665\,, \tag{4.3}$$

which differs by 4.4% from the exact result. Observe that the ground-state energy obtained from the first two terms in Eq. (3.60) is not nearly as good,

$$E_0/N + 1 = 1 - y + 3y^2$$

$$= \frac{1+2y}{1+3y} \xrightarrow[y\to\infty]{} \frac{2}{3}\,, \tag{4.4}$$

or

$$E_0/N \approx -0.333\,. \tag{4.5}$$

It is encouraging that our eighth-order extrapolation retrieves a free-field result ($g=0$), although it begins from a static zeroth-order approximation ($g=\infty$).

Now we turn to Schwinger models as theories of hadrons made from confined quarks. Recall from Ref. 4 that there is a composite scalar particle in the massive Schwinger model. It is bound by an energy proportional to the square of the fermion mass. The continuum-theory formula, obtained in the Appendix, is

$$1 - \frac{M_S}{2M_V} = 5.505 f^2 + O(f^4)\,, \tag{4.6}$$

where $f \equiv 4m/g$ is a dimensionless measure of the fermion mass. Furthermore, as $f \to 0$ and M_S approaches $2M_V$, the scalar particle decouples from the theory. Now consider M_S/M_V in the lattice theory. For the massless ($m=0$) model

$$\frac{M_S}{M_V} = \frac{1 + 6y - 26y^2 + \frac{572}{3}y^3 - \frac{15\,810}{9}y^4}{1 + 2y - 10y^2 + \frac{236}{3}y^3 - \frac{6626}{9}y^4}\,. \tag{4.7}$$

To extrapolate this quantity to $y \to \infty$, we expand it in a power series,

$$\frac{M_S}{M_V} = 1 + 4y - 24y^2 + 200y^3 - 1975.11y^4\,, \tag{4.8}$$

form the [2, 2] Padé approximant,

$$\frac{M_S}{M_V} = \frac{1 + 17.84y + 64.41y^2}{1 + 13.84y + 33.05y^2}\,, \tag{4.9}$$

then let $y \to \infty$,

$$\frac{M_S}{M_V} = 1.95\,, \tag{4.10}$$

which lies within 2.5% of the exact answer. Recall that we found $M_S/M_V = 1.67$ in the fourth-order calculation, so the convergence of the lattice calculation to the true result appears ensured.

In Fig. 1 we plot the [1, 1]- and [2, 2]-order Padé approximants of M_S/M_V as functions of $f = 4m/g$.

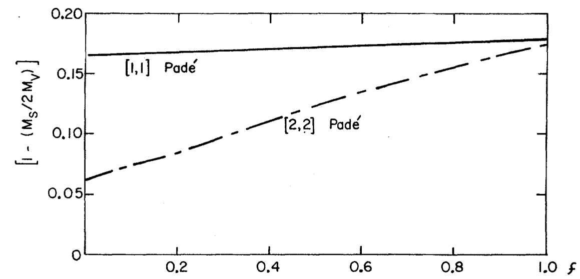

FIG. 1. Binding in the scalar channel as a function of $4m/g$. The higher-order calculation (eighth order in y) intersects the ordinate well below the lower-order calculation.

Our calculations are not sensitive enough to confirm the delicate quadratic behavior of Eq. (4.6)—that behavior cannot be expected from an approximate calculation until the intercept of the curves decreases closer to the origin. However, the magnitude of $(1 - M_S/2M_V)$ at $f \sim 1.0$ has been checked against an approximate continuum calculation in which q-$\bar{q}$ bound states were constructed in the infinite-momentum frame and a Schrödinger-type equation was solved numerically.[7] Agreement between the two methods was found through two significant figures.

An interesting quantity which can be computed exactly in the massive Schwinger model is the rate of change of the vector-particle mass M_V as the fermion mass m is turned on. A continuum calculation gives[4]

$$\frac{\partial M_V}{\partial m} = e^{\gamma} = 1.781\,. \tag{4.11}$$

Previously this quantity was calculated to fourth order with the result

$$\left.\frac{\partial M_V}{\partial M}\right|_{m=0} = 2\left(\frac{1+12y}{1+14y}\right)$$

$$\xrightarrow[y\to\infty]{} 2(1 - \tfrac{1}{7}) = 1.71\,, \tag{4.12}$$

which lies 3.8% below the exact answer. The same manipulations applied to Eq. (3.3) give a [2, 2] Padé approximant

$$\left.\frac{\partial M_V}{\partial m}\right|_{m=0} = 2(1 - 2y + 28y^2 - 374.444y^3 + 4971.481y^4)$$

$$= 2\left(\frac{1 + 13.054y + 21.538y^2}{1 + 15.054y + 23.646y^2}\right)$$

$$\xrightarrow[y\to\infty]{} 1.822\,, \tag{4.13}$$

which lies 2.3% above the exact answer. The convergence of this quantity, which is sensitive to the free-field character of the continuum theory at short distances, is very nice.

To this point we have found energy levels as the mass gaps in zero-momentum states. Since the lattice theory does not possess translation symmetry and is completely static in zeroth order, its hadrons will not have precise relativistic energy-momentum relations. Only by computing to high orders and passing to the continuum limit can we hope to retrieve relativity. To see how this works we computed the energy of the vector particle as a function of its momentum as described in Eq. (2.8) and Eq. (3.6d). Then the mass of the state was identified through the coefficient of p^2 in

$$(m^2+p^2)^{1/2}\approx m+p^2/2m\,. \tag{4.14}$$

From Eq. (3.6d) we have for the massless Schwinger model

$$2\sqrt{x}\,\frac{E_V(p)}{g}=2\sqrt{x}\,\frac{E_V(0)}{g}+(x^2+4x^4-48x^6)k^2\,. \tag{4.15}$$

From Eq. (3.14) we identify

$$2\sqrt{x}\,\frac{M_V^{\text{kin}}}{g}=\frac{2}{1+4x^2-48x^4}\,, \tag{4.16}$$

where the superscript "kin" reflects the fact that the mass is determined from the curvature of the energy momentum relation. To compare with the mass determined statically we form the ratio

$$\frac{M_V}{M_V^{\text{kin}}}=\tfrac{1}{2}(1+2y-10y^2)(1+4y-48y^2)\,, \tag{4.17}$$

where we have inserted Eq. (3.6b) to the appropriate order. Expanding Eq. (4.17) into a Taylor series to $O(y^2)$, forming the $[1,1]$ Padé approximant, and letting $y\to\infty$ we have

$$\frac{M_V}{M_V^{\text{kin}}}=\tfrac{1}{2}(1+6y-50y^2)=\frac{1}{2}\,\frac{1+\frac{43}{3}y}{1+\frac{25}{3}y}\to\tfrac{1}{2}\left(\tfrac{43}{25}\right)=0.86\,. \tag{4.18}$$

Since this calculation used only the fourth-order calculation of M_V, the 14% error appears quite reasonable. In the next section a more detailed comparison of static and kinetic masses will be made. Once the kinetic mass is calculated through two more orders, all the terms in Eq. (3.6b) can be used in calculating the ratio.

Next we calculate M_V/g using the sixth-order expansion. Because of the presence of the $\sqrt{x}$ in Eq. (3.6b), we cannot employ a diagonal Padé approximant and let $x\to\infty$ in a completely straightforward way. As will be discussed in the next section much can be learned from diagonal Padé approximants at finite but large y. However, first consider the fourth power of M_V/g,

$$\begin{aligned}\left(2y^{1/4}\frac{M_V}{g}\right)^4&=(1+2y-10y^2+\tfrac{236}{3}y^3)^4\,,\\ 16y\left(\frac{M_V}{g}\right)^4&=1+8y-16y^2+\tfrac{320}{3}y^3\,.\end{aligned} \tag{4.19}$$

Forming the $[2,1]$ Padé approximant of this series, we obtain an expression for $(M_V/g)^4$ which has a finite limit as $y\to\infty$,

$$\left(\frac{M_V}{g}\right)^4=\frac{1}{16y}\,\frac{1+\frac{44}{3}y+\frac{112}{3}y^2}{1+\frac{20}{3}y}\xrightarrow[y\to\infty]{}\tfrac{7}{20}\,. \tag{4.20}$$

So

$$\frac{M_V}{g}\cong 0.769\,. \tag{4.21}$$

In the exact theory $M_V/g=1/\sqrt{\pi}=0.564$, so the calculation here lies 36% high. This represents considerable improvement over the calculation which would proceed from Eq. (4.19) if we knew only the first two terms in the power series. Then we have

$$16y\left(\frac{M_V}{g}\right)^4=1+8y\,. \tag{4.22}$$

So

$$\frac{M_V}{g}\xrightarrow[y\to\infty]{}\left(\tfrac{1}{2}\right)^{1/4}=0.841\,, \tag{4.23}$$

which lies 49% above the exact answer. The improvement of Eq. (4.21) beyond Eq. (4.23) is encouraging, but we learn that it is more difficult to obtain accurate masses rather than mass ratios by our methods. Clearly some detailed information in the original power series of Eq. (3.6b) is lost when its fourth power is taken and the series is truncated after several terms.

V. PADÉ APPROXIMANTS AT FINITE y

Now we return to Eq. (3.6), and discuss M_V/g and M_S/g using $[2,2]$ Padé approximants. Recall from Ref. 4 that in the massless ($m=0$) model the $[1,1]$ Padé approximants for these quantities are

$$\frac{M_V}{g}=\frac{1}{2y^{1/4}}\left(\frac{1+7y}{1+5y}\right), \tag{5.1a}$$

$$\frac{M_S}{g}=\frac{1}{2y^{1/4}}\left(\frac{3+31y}{3+13y}\right). \tag{5.1b}$$

We are interested in comparing these functions to their eighth-order counterparts

$$\frac{M_V}{g}=\frac{1}{2y^{1/4}}\,\frac{1+13.96y+34.40y^2}{1+11.90y+20.47y^2}\,, \tag{5.2a}$$

$$\frac{M_S}{g}=\frac{1}{2y^{1/4}}\,\frac{1+17.93y+65.49y^2}{1+11.93y+19.91y^2}\,. \tag{5.2b}$$

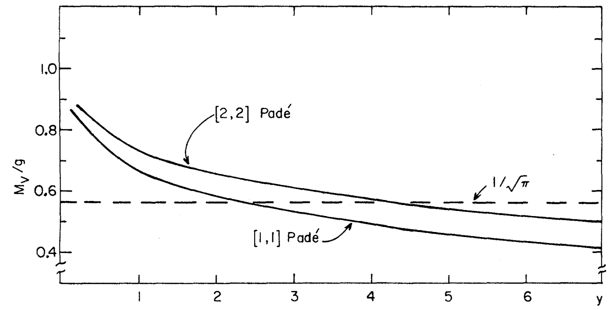

FIG. 2. Equations (5.1a) and (5.2a) as a function of y. The horizontal line $1/\sqrt{\pi}$ is the continuum-theory result.

These functions are plotted in Figs. 2 and 3. Although these functions cannot be extrapolated to $y\rightarrow\infty$, they are useful nonetheless. The tendency of the eighth-order calculations to better approximate the continuum theory at finite y than the fourth-order calculations is also clear. For example, note that the [2, 2] Padé approximant for M_V/g is within ± 0.05 of the exact answer $1/\sqrt{\pi}$ for any y between 3 and 7, while the [1, 1] Padé approximant is in this range only for y between 1.5 and 3. A similar comment can be made concerning Fig. 3. The [2, 2] Padé approximant for $2M_V/g$ is also plotted in this figure to show that the eighth-order calculation lies closer to the continuum result $M_S = 2M_V$ than the fourth-order calculation.

The fact that the [2, 2] Padé approximants pick out the values $1/\sqrt{\pi}$ for M_V/g and M_S/g, respectively, better than the [1, 1] Padé approximants can be stated in the following way. We suspect that the expansions for $y^{1/4}M_V/g$ are actually building up the function $y^{1/4}/\sqrt{\pi}$. To test this hypothesis, consider the $y\rightarrow\infty$ limit of the Padé approximant for $y(1/M_V)dM_V/dy$. This would be $\frac{1}{4}$ if the lattice theory is a success and if we could calculate to very high orders. Using just the fourth-order expansions we have

$$\frac{d}{dy}\left(2y^{1/4}\frac{M_V}{9}\right)=\frac{d}{dy}(1-2y-10y^2)$$
$$=2(1-10y)\,. \tag{5.3}$$

So

$$y\frac{1}{M_V}\frac{dM_V}{dy}=2y\left(\frac{1-10y}{1+2y}\right)$$
$$=2y(1-12y)$$
$$=\frac{2y}{1+12y}$$
$$\xrightarrow[y\to\infty]{}\tfrac{1}{6}=0.167\,, \tag{5.4}$$

which falls 33% short of the desired 0.25. Using the eighth-order calculation yields

$$y\frac{1}{M_V}\frac{dM_V}{dy}=2y\,\frac{1-10y+118y^2-1472.89y^3}{1+2y-10y^2+78.67y^3}$$
$$=2y(1-12y+152y^2-1975.56y^3)$$
$$=2y\frac{1+6.94y}{1+18.94y+75.33y^2}$$
$$\xrightarrow[y\to\infty]{}2\left(\frac{6.94}{75.33}\right)=0.184\,, \tag{5.5}$$

which is better than the fourth-order calculation, but still 26% below the exact power law. We learn that convergence to the desired power will require many orders of perturbation theory. The analogous calculation in statistical mechanics is the determination of critical indices using high-temperature expansions.[5] Relatively slow convergence to the exact answers occurs in that field also.

Now consider the Padé approximants for M_V/g and M_S/g for nonzero fermion mass.[8] Beginning with Eqs. (3.3) and (3.4), the diagonal Padé approximants for various choices of $f=4m/g$ were

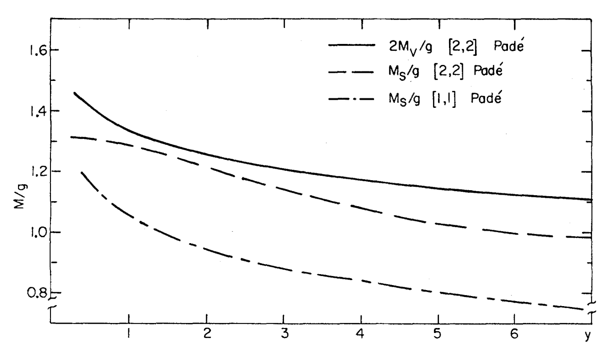

FIG. 3. Equations (5.1b) and (5.2b) as a function of y. Also shown is Eq. (5.2a) multiplied by 2. The two upper curves should coalesce at $2/\sqrt{\pi}$ in a higher-order calculation.

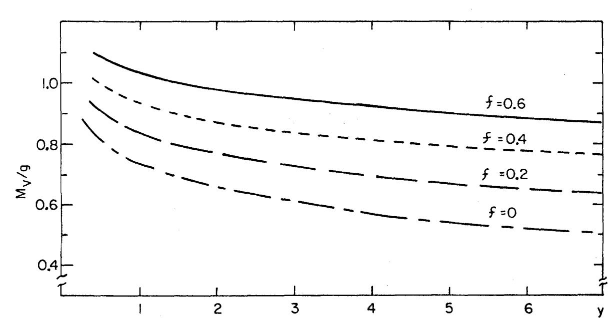

FIG. 4. The dependence of vector-particle mass on $f=4m/g$ in eighth order. The diagonal Padé approximant is used.

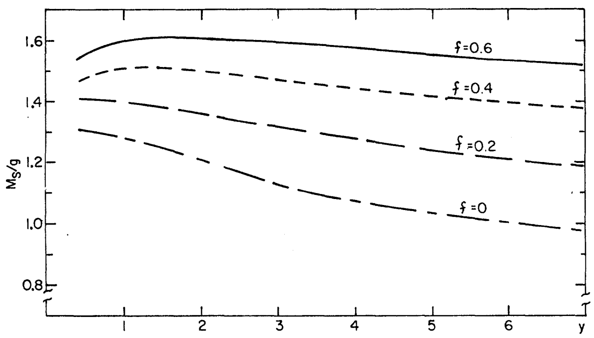

FIG. 5. Same as Fig. 4 but for the scalar particle.

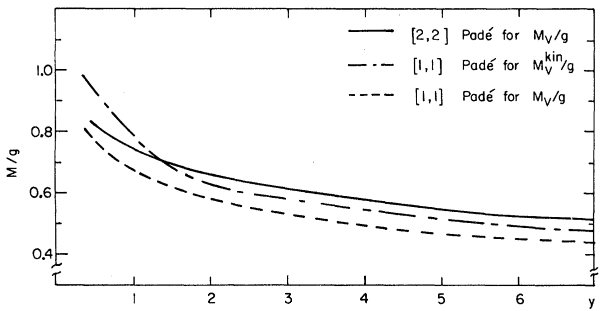

FIG. 6. The "kinetic" mass of the vector (a sixth-order calculation) compared to the [1, 1] and [2, 2] Padé approximants for the "static" definition of the mass.

computed numerically and plotted in Figs. 4 and 5. Useful information can be extracted from these figures, although they are meaningful only for finite y. For example, choosing y anywhere in the flat region of the curves (where $M_V/g = 0.55 \pm 0.05$ for $f=0$), we infer the dependence of the vector mass on f,

$$\frac{M_V(f)}{M_V(0)} - 1 \simeq (0.81 - 0.85)f\,. \tag{5.6}$$

This should be compared with the continuum-theory prediction which follows from Eq. (4.11),

$$\begin{aligned}\frac{M_V(f)}{M_V(0)} - 1 &= \left(\frac{\sqrt{\pi}}{4}\, e^{\gamma}\right) f + O(f^2) \\ &= 0.79f + \cdots\,,\end{aligned} \tag{5.7}$$

which is within several percent of our lattice theory estimate. A similar consideration of Fig. 5 shows that M_S/g grows more slowly with f than M_V/g in agreement with Fig. 1.

Finally, we reconsider the quantity $2\sqrt{x}\, M_V^{\text{kin}}/g$ at finite y,

$$y^{\frac{1}{2}}\frac{M_V^{\text{kin}}}{g} = \frac{1}{1 + 4y - 48y^2}\,. \tag{5.8}$$

Its [1, 1] Padé approximant reads

$$\frac{M_V^{\text{kin}}}{g} = \frac{1}{y^{\frac{1}{2}}}\left(\frac{1 + 12y}{1 + 16y}\right), \tag{5.9}$$

which is plotted in Fig. 6 along with the fourth- and eighth-order static expressions for M_V. M_V^{kin} lies between the two other curves. This detailed agreement is a very nontrivial success of our lattice methods.

We learn from these studies that choosing finite $y \gtrsim 4$ gives good agreement with the continuum theory. In other words, as long as the lattice spacing can be chosen a factor of 2 or 3 smaller than the size of the physical state being considered, a good approximation to its energy results. This is not very surprising since, if the wave function of the bound state extends over several lattice spacings, it is well determined by knowing its values only on the lattice sites. Of course, the energies of highly excited states will not be obtained equally well because their wave functions vary relatively rapidly over the lattice.

VI. CONCLUSIONS

These calculations strongly suggest that lattice calculations will be a helpful tool in extracting information out of field theories. In those cases presented here where eighth-order series were used in full and extrapolations to the continuum limit were unambiguous, our lattice-theory answers lay within several percent of the known continuum-theory results. When only lesser orders in the series were exploited, less detailed agreement was found, but no serious discrepancies existed. Since the calculational method is simple, programmable, and works in any number of dimensions, we are using it to explore non-Abelian gauge theories in 3 + 1 dimensions.[3] Our calculations are proceeding both analytically[9] and by machine.[10] We are hopeful that they shall shed light on previously intractable problems in field theory.

ACKNOWLEDGMENT

Most of this work was carried out when two of the authors (D.K.S. and L.S.) were summer visitors at Cornell University. They thank the Laboratory of Nuclear Studies for hospitality, partial financial support, and an environment conducive to making lengthy calculations. D.K.S. also thanks the Physics Department of Northwestern University for financial support.

APPENDIX

We sketch here the derivation of Eq. (4.6) in the massive Schwinger model. We begin with the equivalent-boson formulation of the theory[11]

$$H=\frac{1}{2}\int\left[\dot{\phi}^2+(\partial_1\phi)^2+\frac{g^2}{\pi}\phi^2-2K\cos(2\sqrt{\pi}\,\phi)\right]dx, \tag{A1}$$

where

$$K=\frac{mg}{2\pi\sqrt{\pi}}e^{\gamma}\quad(\gamma=0.577\ldots)\ . \tag{A2}$$

for small m ($m\ll g$) we can expand the $\cos(2\sqrt{\pi}\,\phi)$ and keep only terms through ϕ^4,

$$H\approx\frac{1}{2}\int[\dot{\phi}^2+(\partial_1\phi)^2+(g/\sqrt{\pi}+me^{\gamma})^2\phi^2 - \tfrac{2}{3}\sqrt{\pi}\,mge^{\gamma}\phi^4]\,dx\,. \tag{A3}$$

For $m\ll g$, this H describes a heavy quantum interacting with itself through a weak, attractive ϕ^4 interaction. Alternatively, we can write a Schrödinger equation for the heavy quantum interacting through a weak, attractive δ-function potential

$$\left[-\frac{1}{2M_V}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)\,, \tag{A4}$$

where $M_V=g/\sqrt{\pi}+me^{\gamma}$, and

$$V(x)=c\delta(x)\,. \tag{A5}$$

The coefficient of the δ function is proportional to the coefficient of the ϕ^4 term in Eq. (A3). A nonrelativistic reduction of the ϕ^4 term gives

$$c=-\frac{2\sqrt{\pi}\,mge^{\gamma}}{3}\,\frac{12}{(2M_V)^2}\,. \tag{A6}$$

Now the bound state of Eq. (A4) can be found. Working in momentum space with the ansatz

$$\psi(p)=\frac{\lambda}{E-p^2/2M_V} \tag{A7}$$

leads to the eigenvalue condition

$$-\frac{c}{\pi}\int_0^\infty\frac{dl}{|E|+l^2/2M_V}=1\,. \tag{A8}$$

By scaling variables, $y=l/\sqrt{|E|}$, an explicit equation for the binding energy E follows,

$$E=\frac{c}{\pi}\int\frac{dy}{(1+y^2/2M_V)}\,, \tag{A9}$$

which gives

$$E=2\pi^2e^{2\gamma}\frac{m^2}{M_V^2}\ . \tag{A10}$$

Therefore, M_S, the mass of the scalar bound state, is

$$M_S=2M_V-2\pi^2e^{2\gamma}\frac{m^2}{M_V}\,. \tag{A11}$$

Using the notation of the text

$$1-\frac{M_S}{2M_V}\simeq\left(\frac{\pi^3}{4^2}e^{2\gamma}\right)f^2+\text{higher orders in } f \simeq(5.51)f^2\ . \tag{A12}$$

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*Work supported in part by the National Science Foundation.

†Work supported in part by NSF Grant No. GP-38863.

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[8]Since the parameter $\mu=\frac{1}{2}fy^{1/4}$ in W and in Eq. (3.3) has weak y dependence, we must expand Eq. (3.3) in powers of $y^{1/4}=z$ through $O(z^{16})$ and make Padé approximants in the variable z. When f is very small this prodigious algebraic exercise can be avoided by treating μ as a numerical constant when forming a [2, 2] Padé approximant in y. Then, when the final expression M_V/g is being considered, one treats μ exactly. If f is small this process is numerically in detailed agreement with a lengthy honest treatment done in the variable z.

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