Unitary four-body model*

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A field-theoretic model describing nonrelativistic four-body scattering processes is developed. The model is related to Bronzan's extended Lee model but the allowed interactions are restricted so that the resulting dynamical equations are as simple as possible, yet still exact. Two elementary particles n and a are introduced with the couplings $n+n \longleftrightarrow D$ and $a+a \longleftrightarrow C$. Three-particle processes are generated by the additional coupling $D+a \longleftrightarrow \alpha$, leading to the possible three-body reactions $D+a \rightarrow D+a$ and $D+a \rightarrow n+n+a$. The four-body sector then involves the $2\rightarrow 2$ reactions $a\alpha \rightarrow a\alpha$ and $a\alpha \rightarrow CD$, the $2\rightarrow 3$ reactions $a\alpha \rightarrow Daa$ and $a\alpha \rightarrow Cnn$, and the $2\rightarrow 4$ reaction $a\alpha \rightarrow nnaa$. Off-shell integral equations are obtained for the $2\rightarrow 2$ amplitudes, and from these, expressions for the $2\rightarrow 3$ and $2\rightarrow 4$ amplitudes are constructed. Possible applications and generalizations of the model are discussed.

I. INTRODUCTION

Over the past ten years there has been great interest and considerable progress in the study of three-body problems[1] in nuclear and particle physics. Much of this work has relied on the Faddeev equations as a framework for describing three-body bound-state and scattering systems. Some years ago an alternative field-theoretic description of the three-body problem was suggested by Amado[2] in analogy to V-θ scattering in the Lee model.[3] It is customary to work with on-shell dispersion methods in the Lee model, but Amado used instead an off-shell formulation together with nonrelativistic kinematics to yield soluble integral equations that have been used successfully in the three-nucleon system as well as in other problems.[4]

Recent attempts have been made to formulate integral equations that describe scattering in four-body systems,[5] but complete solutions of these equations are difficult owing to the expected complexities of the four-body problem. Perhaps some insight into four-body systems could be obtained again from the Lee model since there exists a generalized version of it that has the character of a four-body problem. This extended model has been solved by Bronzan[6] using on-shell dispersion methods. The usual Lee model involves the coupling $V \leftrightarrow N+\theta$ and the three-body sector includes a study of $V\theta \rightarrow V\theta$ and $V\theta \rightarrow N\theta\theta$. Bronzan has extended the model by introducing an additional particle, the U, with the coupling $U \leftrightarrow V+\theta$, and he has not only studied the resulting modifications in the $V\theta$ sector[7] but also analyzed the four-body sector[6] involving the processes $\theta U \rightarrow \theta U$, $\theta U \rightarrow V\theta\theta$, and $\theta U \rightarrow N\theta\theta\theta$.

In this paper we will study a simplified version of Bronzan's model in an off-shell formulation in which the model is reduced to a minimal set of couplings that will still produce four-body scattering. The main complication we want to avoid is the appearance of particle-exchange contributions to the three-body amplitude. In a nonstatic model this involves the contribution of several partial waves and, since we need the off-shell three-body amplitude as input to the four-body calculation, the resulting computational problems would be considerable. This elimination of particle exchange also rules out the appearance of dressed vertices, and these are also particularly hard to deal with numerically.

Our model is constructed so that in the three-particle sector only direct-channel contributions are allowed and in the resulting four-body sector we are able to obtain the amplitudes for $2\rightarrow 2$ processes as well as those for $2\rightarrow 3$ and $2\rightarrow 4$. Even though the resulting model is very simple, we find that is has quite a rich and complicated structure as compared to the three-body problem. Our approach also has the advantage of having a graphical representation that lends some aid in understanding the physical mechanisms that underlie the complicated amplitudes.

We introduce the two- and three-body amplitudes of our model in Sec. II. The resulting four-body equations are obtained in Sec. III for elastic and rearrangement scattering, and in Sec. IV for the breakup processes. Some conclusions, applications, and possible generalizations of the model are described in Sec. V and some technical calculations are worked out in the Appendixes.

II. TWO- AND THREE-BODY SCATTERING

We find it convenient to work with a system composed of two pairs of spinless identical particles that are named n and a so that the four-body state consists of $n+n+a+a$. In analogy to the Amado model we introduce two spinless quasiparticles D and C with the s-wave couplings $D \leftrightarrow n+n$ and $C \leftrightarrow a+a$ as depicted in Fig. 1. Two-body nn scattering then proceeds through the D and aa scattering through the C. There is no n-a interaction introduced at this point. The $D \leftrightarrow n+n$ process may be described in terms of a nonrelativistic unrenormalized Hamiltonian ($\hbar^2 = 2m_n = 1$, $m_a/m_n = m$)

$$H_D = \sum_{\vec{k}} \vec{k}^2 N^\dagger(\vec{k}) N(\vec{k}) + \sum_{\vec{k}} (-\epsilon_D^{(0)} + \tfrac{1}{2}\vec{k}^2) D^\dagger(\vec{k}) D(\vec{k}) + \left[\tfrac{1}{2}\gamma_D^{(u)} \sum_{\vec{q},\vec{Q}} f_D(\vec{q}) D^\dagger(\vec{Q}) N(\tfrac{1}{2}\vec{Q} - \vec{q}) N(\tfrac{1}{2}\vec{Q} + \vec{q}) + \text{H.c.}\right], \tag{1}$$

where N and D are annihilation operators suitable for bosons, $\vec{q}$ and $\vec{Q}$ are the relative and total momenta of the two interacting particles, and $\epsilon_D^{(0)}$ is the bare binding energy of the D.

It is assumed that the a's interact in a similar manner according to the Hamiltonian

$$H_C = \sum_{\vec{k}} (\vec{k}^2/m) A^\dagger(\vec{k}) A(\vec{k}) + \sum_{\vec{k}} (-\epsilon_C^{(0)} + \vec{k}^2/2m) C^\dagger(\vec{k}) C(\vec{k}) + \left[\tfrac{1}{2}\gamma_C^{(u)} \sum_{\vec{q},\vec{Q}} f_C(\vec{q}) C^\dagger(\vec{Q}) A(\tfrac{1}{2}\vec{Q} - \vec{q}) A(\tfrac{1}{2}\vec{Q} + \vec{q}) + \text{H.c.}\right]. \tag{2}$$

The two-body scattering amplitudes are now easily obtained and they have a simple separable form in momentum space

$$\langle \vec{k}' | T_{nn}(E) | \vec{k} \rangle = \tfrac{1}{2}\gamma_D^2 f_D(\vec{k}) \tau_D(E + \epsilon_D) f_D(\vec{k}') \tag{3}$$

and

$$\langle \vec{k}' | T_{aa}(E) | \vec{k} \rangle = \tfrac{1}{2}\gamma_C^2 f_C(\vec{k}) \tau_C(E + \epsilon_C) f_C(\vec{k}'). \tag{4}$$

Two-body scattering is depicted in Fig. 2 and Fig. 3, where the propagators τ_D and τ_C represent sums of self-energy bubbles. Each interaction is characterized by a coupling constant γ and a vertex function $f(\vec{k})$ as well as a wave-function renormalization constant Z that takes on the range of values $0 \le Z \le 1$. If, for example, $Z_D = 1$ the D is an elementary particle uncoupled to $n+n$, while if $Z_D = 0$, a separable-potential model is obtained in which the D is a bound state of two n's. The propagators are worked out in Appendix A.

At this point in the model we have only two pair interactions and therefore no interaction between any three particles of the system. We now introduce a spinless particle, analogous to the U particle of Bronzan, that we call the α, with the s-wave coupling $\alpha \leftrightarrow D+a$ as depicted in Fig. 4. In analogy to the previous cases we can generate the required process with the following interaction Hamiltonian:

$$H_\alpha^I = \gamma_\alpha^{(u)} \sum_{\vec{q},\vec{Q}} f_\alpha(\vec{q}) \alpha^\dagger(\vec{Q}) D\left(\vec{Q}\frac{2}{2+m} - \vec{q}\right) A\left(\vec{Q}\frac{m}{2+m} + \vec{q}\right) + \text{H.c.} \tag{5}$$

The process $D + a \to D + a$ now proceeds through the α giving only a direct-channel contribution to the three-particle amplitude that is graphically represented in Fig. 5(a) and the t matrix has the separable form in momentum space

$$\langle \vec{k}' | T_{Da}(E) | \vec{k} \rangle = \gamma_\alpha^2 f_\alpha(\vec{k}) \tau_\alpha(E + \epsilon_\alpha) f_\alpha(\vec{k}'), \tag{6}$$

where τ_α is the intermediate α propagator depicted in Fig. 5(b) and whose construction will be left for

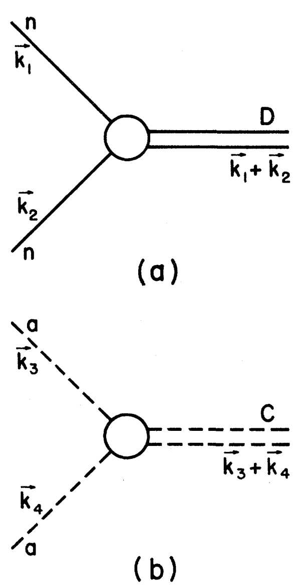

FIG. 1. Basic vertices for $D \longleftrightarrow n+n$ and $C \longleftrightarrow a+a$.

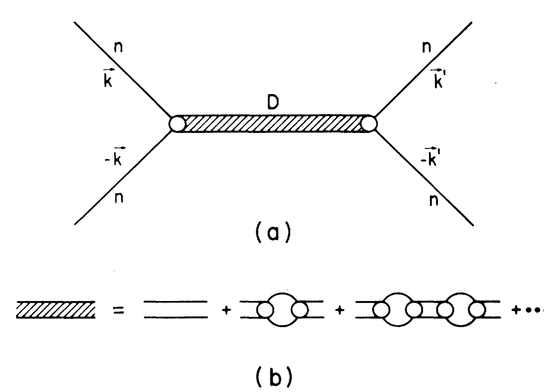

FIG. 2. (a) Graphical representation of the nn scattering amplitude. (b) First few terms in an expansion of the D-particle propagator.

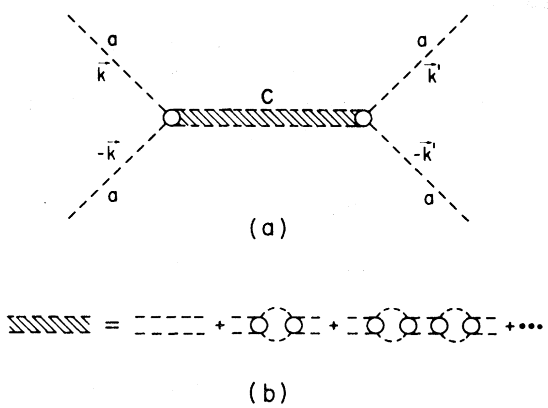

FIG. 3. (a) Graphical representation of the aa scattering amplitude. (b) First few terms in an expansion of the C-particle propagator.

Appendix B. As in two-body scattering, the three-body amplitude is characterized by a coupling constant γ_α and a vertex function $f_\alpha(\vec{k})$, as well as a wave-function renormalization constant Z_α. Since three-body intermediate states have been included in the propagator to all orders, our amplitude satisfies two- and three-body unitarity so that processes such as $aD \to ann$ are included. We could also disallow this possibility by taking $Z_D = 1$ (elementary D) with $Z_\alpha = 0$. We would then have a model involving an effective separable potential between a and D with no breakup possible.

III. TWO-TO-TWO AMPLITUDES

Proceeding to the four-body sector we encounter $a\alpha$ and CD as two-body channels, Daa and Cnn as three-body channels, and the single four-body channel $nnaa$. Our first aim is to obtain a set of integral equations that represents the contribution of all possible graphs to the elastic scattering process $a\alpha \to a\alpha$, as well as to the rearrangement reaction $a\alpha \to CD$. The basic driving mechanisms for these processes are D-particle exchange between two $a\alpha$ states and a-particle exchange leading to $a\alpha \to CD$ in lowest order. These Born terms are

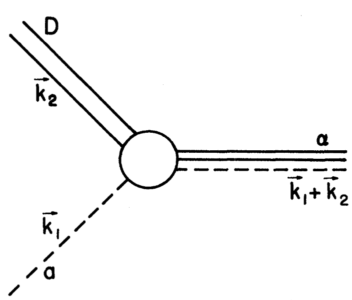

FIG. 4. Basic vertex for $\alpha \longleftrightarrow D + a$.

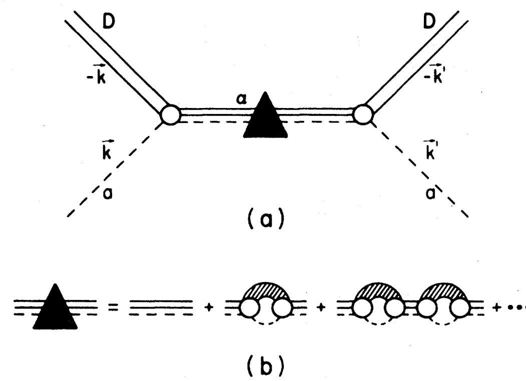

FIG. 5. (a) Graphical representation of the aD scattering amplitude. (b) Self-energy bubbles contributing to the α propagator.

depicted in Fig. 6(a) and Fig. 6(b).

If we first concentrate on $a\alpha$ elastic scattering, the two lowest-order contributions in a multiple-scattering expansion of the amplitude are the D-particle-exchange graph of Fig. 6(a) followed by the box diagram containing an intermediate CD state shown in Fig. 6(c). This intermediate state is a quasiparticle-quasiparticle state, and its structure is rather complicated since both C and D are fully dressed. In Fig. 7 we have indicated a few of the diagrams that contribute to Fig. 6(c).

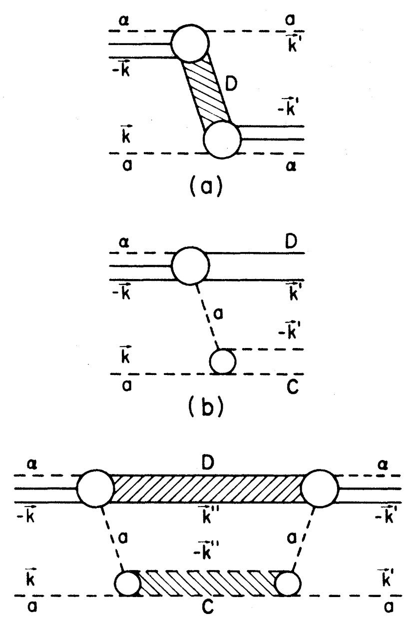

FIG. 6. (a) Graphical representation of the D-particle-exchange Born term connecting two $a\alpha$ states. (b) Graphical representation of the a-particle-exchange Born term connecting $a\alpha$ to CD. (c) Graphical representation of the box diagram having a fully dressed CD intermediate state.

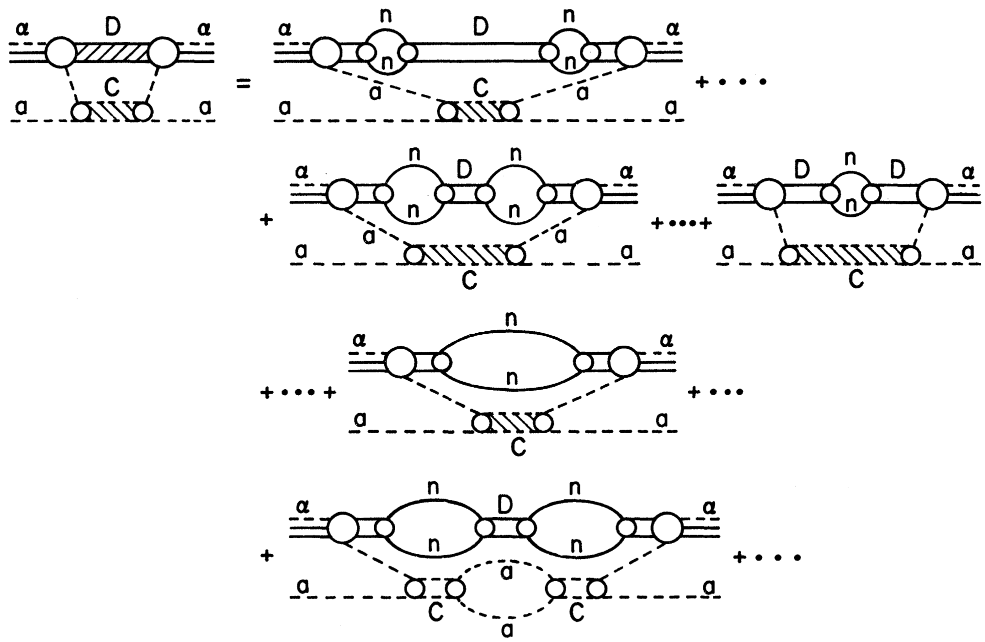

FIG. 7. A few of the contributing graphs to the box diagram.

Not only are there bubbles of the D before and after the lower vertices but also overlapping them. Special attention should be given to the last two graphs in Fig. 7, where we first show a bubble of the D that opens before the lower left vertex and then closes after the lower right vertex. The last graph shows a sequence of fully enchained bubbles, that is, each D bubble opens before the C particle is formed and closes after the C bubble has opened. The notable feature of these last two graphs is that no intermediate CD state is ever formed, and thus it is not possible to separate the box diagram into single-particle-exchange Born terms and intermediate propagators as is done in the three-body problem. We must therefore treat the box diagram as a fundamental element of the model and not attempt to obtain it by iteration of lower-order diagrams. To put this in different terms, we want to avoid dealing with quasiparticle-quasiparticle states as off-shell external lines. A single integral equation for the $\alpha\alpha \to \alpha\alpha$ amplitude may still be obtained and is illustrated in Fig. 8. The D-particle-exchange Born term and the box amplitude are both inhomogeneous terms in the equation and a Neumann iteration would generate all higher-order contributions of these processes. Letting T_1 represent the full $\alpha\alpha \to \alpha\alpha$ amplitude, the specific form of the integral equation is

$$\begin{aligned}
&\langle \vec{k}'|T_1(E)|\vec{k}\rangle = \langle \vec{k}'|B(E)|\vec{k}\rangle + \int \frac{d^3n}{(2\pi)^3}\langle \vec{k}'|B(E)|\vec{n}\rangle \tau_\alpha(E+\epsilon_\alpha - \vec{n}^2/\mu_{\alpha\alpha})\langle \vec{n}|T_1(E)|\vec{k}\rangle ,\\
&\langle \vec{k}'|B(E)|\vec{k}\rangle = \langle \vec{k}'|B_1(E)|\vec{k}\rangle + \langle \vec{k}'|\Box(E)|\vec{k}\rangle, \quad \mu_{\alpha\alpha} = \frac{m(m+2)}{2m+2} .
\end{aligned} \tag{7}$$

$B_1(E)$ corresponds to the D-particle-exchange Born term and $\Box(E)$ to the box diagram. The solution of Eq.

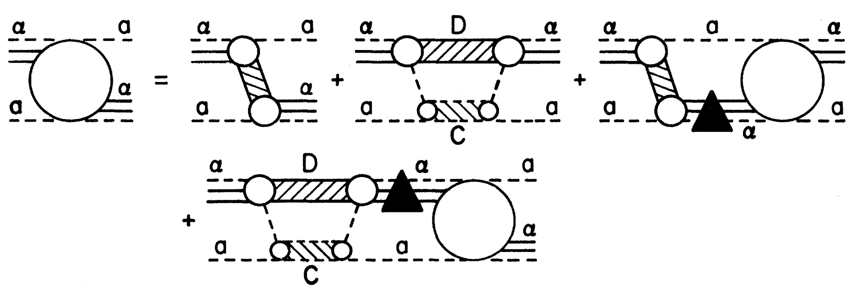

FIG. 8. Graphical representation of the integral equation for the $\alpha\alpha \to \alpha\alpha$ amplitude (circle).

(7) would also provide a means of obtaining the rearrangement amplitude for $a\alpha \to CD$. Graphically this amplitude is represented in Fig. 9, where we see that it may be written in terms of an integral over the half-off-shell elastic amplitude $T_1(E)$. Letting T_2 be the amplitude for $a\alpha \to CD$, the precise form of this relation is

$$\langle \vec{k}'|T_2(E)|\vec{k}\rangle = \langle \vec{k}'|B_2(E)|\vec{k}\rangle + \int \frac{d^3n}{(2\pi)^3}\langle \vec{k}'|B_2(E)|\vec{n}\rangle \tau_\alpha(E+\epsilon_\alpha - \vec{n}^2/\mu_{a\alpha})\langle \vec{n}|T_1(E)|\vec{k}\rangle. \tag{8}$$

The Born term B_2 has the CD side on-shell, and thus no C-D bubbles are allowed.

Having written a set of equations for the required $2 \to 2$ amplitudes, we must now calculate the Born terms $B_1(E)$ and $B_2(E)$ as well as the box amplitude $\Box(E)$. $B_2(E)$ is the simplest and we only quote the result

$$\langle \vec{k}'|B_2(E)|\vec{k}\rangle = \frac{\gamma_\alpha \gamma_C f_\alpha(\vec{k}' + \vec{k}2/(2+m)) f_C(\vec{k} + \frac{1}{2}\vec{k}')}{E+\epsilon_D - \frac{1}{2}\vec{k}'^2 - (\vec{k}+\vec{k}')^2/m - \vec{k}^2/m}, \tag{9}$$

where $\vec{k}$ and $\vec{k}'$ are the initial and final center-of-mass momenta, respectively. The calculation of the D-particle-exchange Born term is also straightforward, and the result is

$$\langle \vec{k}'|B_1(E)|\vec{k}\rangle = \gamma_\alpha^2 f_\alpha(\vec{k}' + \vec{k}m/(2+m))\tau_D(U) \times f_\alpha(\vec{k} + \vec{k}'m/(2+m)), \tag{10}$$

$$U = E + \epsilon_D - \vec{k}^2/m - \tfrac{1}{2}(\vec{k}+\vec{k}')^2 - k'^2/m,$$

where $\vec{k}$ and $\vec{k}'$ are the center-of-mass momenta of the initial and final states. τ_D is the full D-particle propagator, and thus B_1 contains both the D-particle-exchange pole as well as the n-n continuum contribution.

The remaining task is to construct the box amplitude $\Box(E)$, and this is more involved; its calculation is therefore presented in greater detail. The box represents an amplitude that is second order in the process $\alpha \to D + a$ but must contain all orders of $D \leftrightarrow n+n$ and $C \leftrightarrow a+a$. The amplitude of interest can be written as the matrix element

$$\langle \vec{k}'|\Box(E)|\vec{k}\rangle = \langle a(\vec{k}')\alpha(-\vec{k}')|H_\alpha^I(E-H_D-H_C)^{-1}H_\alpha^I|a(\vec{k})\alpha(-\vec{k})\rangle, \tag{11}$$

where H_C and H_D are the Hamiltonians given in Eqs. (1) and (2) and $\vec{k}$ and $\vec{k}'$ are the center-of-mass momenta of the initial and final states. Unless there is danger of confusion, we will omit the initial and final momentum labels to simplify the notation.

A new aspect of the four-body problem that does not arise in fewer-particle systems is the simultaneous propagation of two noninteracting quasiparticles represented by $(E-H_C-H_D)^{-1}$. Since H_C and H_D commute, we can simplify (11) by using the integral representation[8]

$$\frac{1}{E-H_C-H_D} = \frac{1}{2\pi i}\oint dZ\, \frac{1}{Z-H_D}\,\frac{1}{E-Z-H_C}, \tag{12}$$

where it is assumed that E has a small imaginary part. The contour over which the integral is carried out is illustrated in Fig. 10 and may enclose either the spectrum of H_D in a counterclockwise manner or the spectrum of H_C in a clockwise manner. We choose the former, and by introducing (12) into (11) together with complete sets of $|Daa\rangle$ states we obtain

$$\langle \vec{k}'|\Box(E)|\vec{k}\rangle = \sum_{\vec{q}_i,\vec{q}_i'} \frac{1}{2\pi i}\oint dZ\,\langle a\alpha|H_\alpha^I|D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)\rangle \times \left\langle D(\vec{q}_1)\left|\frac{1}{Z-H_D}\right|D(\vec{q}_1')\right\rangle \left\langle a(\vec{q}_2)a(\vec{q}_3)\left|\frac{1}{E-Z-H_C}\right|a(\vec{q}_2')a(\vec{q}_3')\right\rangle \langle D(\vec{q}_1')a(\vec{q}_2')a(\vec{q}_3')|H_\alpha^I|a\alpha\rangle. \tag{13}$$

The matrix elements of $(Z-H_D)^{-1}$ and $(E-Z-H_C)^{-1}$ can be easily related to the propagators for the D particle and the C particle. After excluding self-energy terms and the D-particle-exchange contribution from the result, we obtain

$$\langle \vec{k}'|\Box(E)|\vec{k}\rangle = \sum_{\vec{k}'',\vec{q}_i,\vec{q}_i'} \frac{1}{2\pi i}\oint dZ\langle a\alpha|H_\alpha^I|D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)\rangle \tau_D^{(u)}(Z+\epsilon_D^{(0)} - \tfrac{1}{2}\vec{q}_1^{\,2})\delta(\vec{q}_1 - \vec{q}_1') \times \frac{\langle a(\vec{q}_2)a(\vec{q}_3)|H_C^I|C(-\vec{k}'')\rangle}{E-Z-\vec{q}_2^{\,2}/m - \vec{q}_3^{\,2}/m}\tau_C^{(u)}(E-Z+\epsilon_C^{(0)} - \vec{k}''^2/2m) \times \frac{\langle C(-\vec{k}'')|H_C^I|a(\vec{q}_2')a(\vec{q}_3')\rangle}{E-Z-\vec{q}_2'^{\,2}/m - \vec{q}_3'^{\,2}/m}\langle D(\vec{q}_1')a(\vec{q}_2')a(\vec{q}_3')|H_\alpha^I|a\alpha\rangle. \tag{14}$$

We can renormalize the propagators $\tau_D^{(u)}$ and $\tau_C^{(u)}$ at this point according to the procedures of Appendix A. The remaining matrix elements are also easily evaluated leading to the result

$$\langle \vec{k}'|\Box(E)|\vec{k}\rangle = \int \frac{d^3k''}{(2\pi)^3}\,\gamma_\alpha\gamma_C f_\alpha(\vec{k}''+\vec{k}\,2/(2+m))f_C(\vec{k}+\tfrac{1}{2}\vec{k}'')$$
$$\times\frac{1}{2\pi i}\oint dZ\,\tau_D(Z+\epsilon_D-\tfrac{1}{2}\vec{k}''^2)\frac{\tau_C(E-Z+\epsilon_C-\vec{k}''^2/2m)}{[E-Z-(\vec{k}''+\vec{k})^2/m-\vec{k}^2/m][E-Z-(\vec{k}''+\vec{k}')^2/m-\vec{k}'^2/m]}$$
$$\times\gamma_\alpha\gamma_C f_\alpha(\vec{k}''+\vec{k}'2/(2+m))f_C(\vec{k}'+\tfrac{1}{2}\vec{k}''). \tag{15}$$

The contour integration around the singularities of τ_D is now easily performed if we keep in mind the analytic structure of the D-particle propagator. According to Appendix A, $\tau_D(\omega)$ has a pole at $\omega=0$ and a branch cut along the real axis for $\omega>\epsilon_D$. Using Cauchy's theorem and the Schwarz reflection principle we arrive at

$$\langle \vec{k}'|\Box(E)|\vec{k}\rangle = \int \frac{d^3k''}{(2\pi)^3}\frac{\gamma_\alpha\gamma_C f_\alpha(\vec{k}''+\vec{k}2/(2+m))f_C(\vec{k}+\tfrac{1}{2}\vec{k}'')}{E+\epsilon_D-\tfrac{1}{2}\vec{k}''^2-(\vec{k}''+\vec{k})^2/m-\vec{k}^2/m}$$
$$\times G(\vec{k},\vec{k}'',\vec{k}';E)\frac{\gamma_\alpha\gamma_C f_\alpha(\vec{k}''+\vec{k}'2/(2+m))f_C(\vec{k}'+\tfrac{1}{2}\vec{k}'')}{E+\epsilon_D-\tfrac{1}{2}\vec{k}''^2-(\vec{k}''+\vec{k}')^2/m-\vec{k}'^2/m}, \tag{16}$$

where

$$G(\vec{k},\vec{k}'',\vec{k}';E)=\tau_C(Y)-\frac{(Y-U)(Y-U')}{\pi}\int_0^\infty dx\,\frac{\mathrm{Im}[\tau_D(x+\epsilon_D)]\tau_C(Y-\epsilon_D-x)}{(Y-U-\epsilon_D-x)(Y-U'-\epsilon_D-x)}, \tag{17}$$

$$Y=E+\epsilon_D+\epsilon_C-\tfrac{1}{2}\vec{k}''^2-\vec{k}''^2/2m,$$
$$Y-U=E+\epsilon_D-\tfrac{1}{2}\vec{k}''^2-(\vec{k}''+\vec{k})^2/m-\vec{k}^2/m,$$
$$Y-U'=E+\epsilon_D-\tfrac{1}{2}\vec{k}''^2-(\vec{k}''+\vec{k}')^2/m-\vec{k}'^2/m.$$

The G propagator contains two-, three-, and four-body effects that lead to the following complicated analytic structure: a pole at $Y=0$ and five branch cuts for $Y>\epsilon_D$, $Y>\epsilon_C$, $Y>\epsilon_D+\epsilon_C$, $Y-U>\epsilon_D$, and $Y-U'>\epsilon_D$. The last two are due to four-body effects that have their origin in bubbles of the D particle before and after the C particle is formed.

At this point we have completed the construction of the dynamical equations for $2\to 2$ amplitudes in the model. Our integral equation contains one vector variable in intermediate states and will reduce to a single variable following partial-wave analysis. The resulting amplitudes are exact and unitary since all contributing graphs of the model are included. The propagators and Born terms are more complicated than in the three-body problem, but the structure of the equations is quite similar.

In all of the above discussion we have always considered $a\alpha$ as the incoming channel and we studied the amplitudes for all possible reactions initiated by the $a\alpha$ state. We could also study processes initiated by the CD state, which are $CD\to CD$, $CD\to a\alpha$, $CD\to Da a$, $CD\to Cnn$, and $CD\to nnaa$. Because the CD state is a quasiparticle-quasiparticle state we cannot establish an integral equation for the elastic scattering amplitude $CD\to CD$ as we did for $a\alpha\to a\alpha$ since this would require dealing with the quasiparticle-quasiparticle state as an off-shell external line. An integral equation for $CD\to a\alpha$ has to be solved first, and only then can the elastic amplitude be calculated by performing an extra integration over the off-

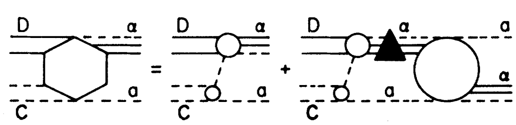

FIG. 9. Graphical representation of the integral relation expressing the $a\alpha\to CD$ amplitude (hexagon) in terms of the half-on-shell $a\alpha\to a\alpha$ amplitude (circle).

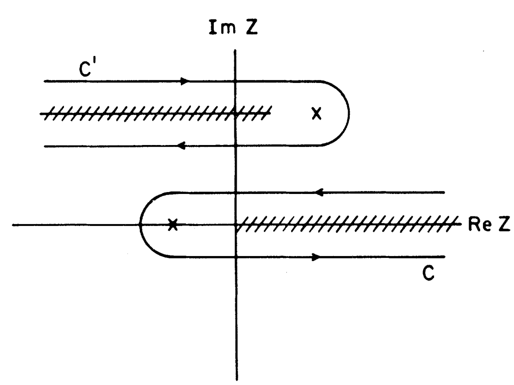

FIG. 10. Contour used in the calculation of the box diagram.

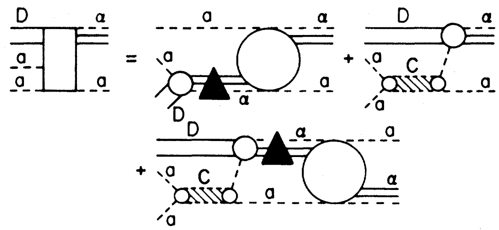

FIG. 11. Graphical representation of the integral relation expressing the $a\alpha \rightarrow Da a$ amplitude (rectangle) in terms of the half-on-shell $a\alpha \rightarrow a\alpha$ amplitude (circle).

shell momenta of the $a\alpha$ state. We will not deal with these processes in this paper.

It should also be noted that our model simplifies under some limiting circumstances. First, if we omit the coupling $C \rightarrow a + a$, and this can be accomplished by choosing $Z_C = 1$, the rearrangement process is no longer possible and the only remaining $2 \rightarrow 2$ reaction is $a\alpha \rightarrow a\alpha$ mediated by D exchange. In this limit we still have a genuine four-body problem involving the reactions $a\alpha \rightarrow a\alpha$, $a\alpha \rightarrow Daa$, and $a\alpha \rightarrow nnaa$. If instead we take the limit $Z_D = 1$, the D becomes an elementary particle with a "binding energy" of ϵ_D. The four-body problem then degenerates into a three-body problem involving $a\alpha \rightarrow a\alpha$ and $a\alpha \rightarrow Daa$ as mediated by the exchange of a structureless elementary D. Finally if we take $Z_\alpha = 1$, connected three- and four-body processes are no longer possible and we are left with a pair of independent two-body systems.

IV. BREAKUP AMPLITUDES

We find that $2 \rightarrow 3$ and $2 \rightarrow 4$ amplitudes in the present model follow a pattern similar to the breakup amplitudes in the three-nucleon problem,[9] although there are some significant differences. On the basis of the three-nucleon case one would expect that the calculation of the breakup would involve the half-off-shell $2 \rightarrow 2$ amplitudes T_1 and T_2, followed by the off-shell propagation of the final quasiparticles leading to their ultimate decay into three- or four-body states. This procedure must be modified in order that it be consistent with our previous warning that the CD state should not appear as an off-shell external line. This modification is easily carried out since we can always eliminate T_2 in favor of T_1 by using Eq. (8), as is illustrated in Fig. 9. We will use this procedure to construct the breakup amplitudes for $a\alpha \rightarrow Daa$, $a\alpha \rightarrow Cnn$, and $a\alpha \rightarrow nnaa$ in this section.

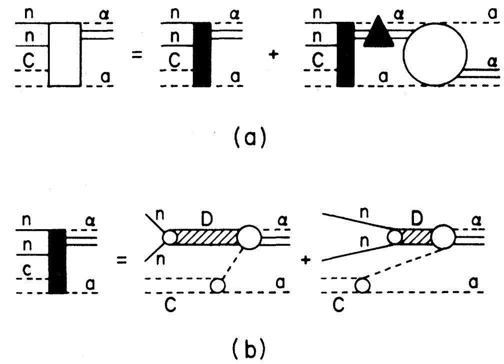

FIG. 12. Graphical representation of the integral relation expressing the $a\alpha \rightarrow Cnn$ amplitude (rectangle) in terms of the half-on-shell $a\alpha \rightarrow a\alpha$ amplitude (circle).

A. Two-to-three amplitudes

The simplest $2 \rightarrow 3$ reaction is $a\alpha \rightarrow Daa$ and the process is shown graphically in Fig. 11. The $2 \rightarrow 2$ amplitude T_1 appears twice because we have, in effect, used Eq. (8) to eliminate T_2. If we let $\mathcal{T}_1$ represent this breakup amplitude and choose the momenta as

$$a(\vec{k}) + \alpha(-\vec{k}) \rightarrow a(\vec{p}) + a(\vec{q}) + D(\vec{r}),$$

$\mathcal{T}_1$ satisfies the equation

$$\begin{aligned}\langle \vec{p}, \vec{q}, \vec{r} | \mathcal{T}_1(E) | \vec{k}\rangle = \{ [(\gamma_\alpha/\sqrt{2}) f_\alpha(\vec{r} + \vec{p}\,2/(2+m)) \tau_\alpha(E + \epsilon_\alpha - \vec{p}^{\,2}/\mu_{a\alpha}) \langle \vec{p} | T_1(E) | \vec{k}\rangle] + [\vec{p} \rightarrow \vec{q}] \}\, \delta(\vec{p} + \vec{q} + \vec{r}) \\ + \langle \vec{p}, \vec{q}, \vec{r} | \mathcal{B}_1(E) | \vec{k}\rangle + \int \frac{dn^3}{(2\pi)^3} \langle \vec{p}, \vec{q}, \vec{r} | \mathcal{B}_1(E) | \vec{n}\rangle \tau_\alpha(E + \epsilon_\alpha - \vec{n}^{\,2}/\mu_{a\alpha}) \langle \vec{n} | T_1(E) | \vec{k}\rangle . \end{aligned} \tag{18}$$

$\mathcal{B}_1$ represents the a-particle-exchange Born term B_1 followed by C-particle rescattering and final breakup. It should be noted that the D particle appears as an on-shell external line so that no D bubbles are necessary. The amplitude $\mathcal{B}_1$ is easily constructed and we quote only the result:

$$\langle \vec{p}, \vec{q}, \vec{r} | \mathcal{B}_1(E) | \vec{k}\rangle = (\gamma_C/\sqrt{2}) f_C(\tfrac{1}{2}\vec{p} - \tfrac{1}{2}\vec{q}) \tau_C(E + \epsilon_D + \epsilon_C - \vec{r}^{\,2}/\mu_{CD}) \frac{\gamma_\alpha \gamma_C f_\alpha(\vec{r} + \vec{k}\,2/(2+m)) f_C(\vec{k} + \frac{1}{2}\vec{r})}{E + \epsilon_D - \frac{1}{2}\vec{r}^{\,2} - (\vec{r} + \vec{k})^2/m - \vec{k}^2/m}\, \delta(\vec{p} + \vec{q} + \vec{r}), \tag{19}$$

$$\mu_{CD} = \frac{2 \times 2m}{2 + 2m}.$$

The other $2 \rightarrow 3$ process present in the model is $a(\vec{k}) + \alpha(-\vec{k}) \rightarrow n(\vec{p}) + n(\vec{q}) + C(-\vec{r})$ and its graphical representation is given in Fig. 12. The black box in this figure represents the final $\alpha \to D + a$ vertex followed by a-particle exchange and n-n rescattering. There are two possible contributions to the black box depending on whether the final breakup of the D particle occurs before or after the lower vertex. If we used T_2 instead of substituting it in favor of T_1 by using Eq. (8), the second contribution to the black box would have been omitted. Denoting the full breakup amplitude by $\mathcal{T}_2$ and the black box by $\mathcal{B}_2$, $\mathcal{T}_2$ satisfies the equation

$$\langle \vec{p}, \vec{q}, -\vec{r} | \mathcal{T}_2(E) | \vec{k} \rangle = \langle \vec{p}, \vec{q}, -\vec{r} | \mathcal{B}_2(E) | \vec{k} \rangle + \int \frac{d^3n}{(2\pi)^3} \langle \vec{p}, \vec{q}, -\vec{r} | \mathcal{B}_2(E) | \vec{n} \rangle \tau_\alpha (E + \epsilon_\alpha - \vec{n}^2/\mu_{a\alpha}) \langle \vec{n} | T_1(E) | \vec{k} \rangle . \tag{20}$$

Only the construction of the function $\mathcal{B}_2(E)$ is needed to evaluate $\mathcal{T}_2$. The method we use to construct $\mathcal{B}_2$ is similar to the one developed in Sec. III for the box diagram and also to the one we will present in the following subsection. Therefore, we only indicate the result:

$$\begin{aligned} \langle \vec{p}, \vec{q}, -\vec{r} | \mathcal{B}_2(E) | \vec{k} \rangle = (\gamma_D/\sqrt{2}) f_D(\tfrac{1}{2}\vec{p} - \tfrac{1}{2}\vec{q}) & \frac{\tau_D(E + \epsilon_D + \epsilon_C - \vec{r}^{\,2}/\mu_{CD})}{-\epsilon_C + \vec{r}^{\,2}/2m - (\vec{r} + \vec{k})^2/m - \vec{k}^2/m} \\ & \times \gamma_\alpha \gamma_C f_\alpha(\vec{r} + \vec{k}\, 2/(2+m)) f_C(\vec{k} + \tfrac{1}{2}\vec{r}) \delta(\vec{p} + \vec{q} - \vec{r}) . \end{aligned} \tag{21}$$

This completes the discussion of the $2 \to 3$ amplitudes.

B. Two-to-four amplitudes

The final process to be discussed is $a\alpha \to nnaa$ including all possible $n-n$ and $a-a$ final-state interactions. In Fig. 13(a) we show a graphical representation of the process in question, from which it is clear that two kinds of terms are present. The first involves the off-shell $2 \to 2$ amplitude T_1 followed by off-shell propagation and disassociation of the α and D quasiparticles; the last two terms involve again an amplitude denoted by a black box that represents the final α breakup followed by a-particle exchange and then all possible $n-n$ and $a-a$ rescatterings. As illustrated in Fig. 13(b), the black box contains three contributions according to whether the D-particle breakup occurs before or after the lower vertex and, if after, whether preceding or succeeding the C-particle final decay. If we let $\mathcal{B}_3$ denote the black box and $\mathcal{T}_3$ the full $2 \to 4$ amplitude, then $\mathcal{T}_3$ represents the amplitude for the process

$$a(\vec{k}) + \alpha(-\vec{k}) \to n(\vec{p}) + n(\vec{q}) + a(\vec{r}) + a(\vec{s}) ,$$

and is given by

$$\begin{aligned} \langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{T}_3(E) | \vec{k} \rangle = \{ [& (\gamma_D/\sqrt{2}) f_D(\tfrac{1}{2}\vec{p} - \tfrac{1}{2}\vec{q}) \tau_D(E + \epsilon_D - \tfrac{1}{2}(\vec{p} + \vec{q})^2 - \vec{r}^2/m - \vec{s}^2/m) \\ & \times (\gamma_\alpha/\sqrt{2}) f_\alpha(\vec{p} + \vec{q} + \vec{r}\, 2/(2+m)) \tau_\alpha(E + \epsilon_\alpha - \vec{r}^2/\mu_{a\alpha}) \langle \vec{r} | T_1(E) | \vec{k} \rangle] + [\vec{r} \to \vec{s}] \} \delta(\vec{p} + \vec{q} + \vec{r} + \vec{s}) \\ & + \langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{B}_3(E) | \vec{k} \rangle + \int \frac{d^3n}{(2\pi)^3} \langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{B}_3(E) | \vec{n} \rangle \tau_\alpha(E + \epsilon_\alpha - \vec{n}^2/\mu_{a\alpha}) \langle \vec{n} | T_1(E) | \vec{k} \rangle . \end{aligned} \tag{22}$$

The remaining task is to construct the black-box amplitude $\mathcal{B}_3$. The initial and final interactions in this amplitude are H_α^I and $(H_D^I + H_C^I)$, respectively, while we must include $D \longleftrightarrow n+n$ and $C \longleftrightarrow a+a$ interactions to all orders. We can therefore represent $\mathcal{B}_3$ by the matrix element

$$\langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{B}_3(E) | \vec{k} \rangle = \langle n(\vec{p}) n(\vec{q}) a(\vec{r}) a(\vec{s}) | (H_D^I + H_C^I)(E - H_D - H_C)^{-1} H_\alpha^I | a(\vec{k}) \alpha(-\vec{k}) \rangle . \tag{23}$$

As in Sec. III we can perform a convolution on $(E - H_D - H_C)^{-1}$; we also drop some momentum labels to simplify the notation leading to

$$\langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{B}_3(E) | \vec{k} \rangle = \frac{1}{2\pi i} \oint dZ \left\langle nnaa \left| (H_D^I + H_C^I) \frac{1}{Z - H_D} \frac{1}{E - Z - H_C} H_\alpha^I \right| a\alpha \right\rangle . \tag{24}$$

Introducing a complete set of $|Daa\rangle$ states to the left of H_α^I we are led to

$$\begin{aligned} \langle \vec{p}, \vec{q}, \vec{r}, \vec{s} | \mathcal{B}_3(E) | \vec{k} \rangle = \sum_{\vec{q}_i} \frac{1}{2\pi i} \oint dZ & \left\langle nnaa \left| (H_D^I + H_C^I) \frac{1}{Z - H_D} \frac{1}{E - Z - H_C} \right| D(\vec{q}_1) a(\vec{q}_2) a(\vec{q}_3) \right\rangle \\ & \times \langle D(\vec{q}_1) a(\vec{q}_2) a(\vec{q}_3) | H_\alpha^I | a\alpha \rangle . \end{aligned} \tag{25}$$

At this point we concentrate on the first matrix element in Eq. (25). It can be separated into two terms, the first one of which is given by

$$\langle n(\vec{p})n(\vec{q})a(\vec{r})a(\vec{s})|H_D^I(Z-H_D)^{-1}(E-Z-H_C)^{-1}|D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)\rangle$$
$$=\langle n(\vec{p})n(\vec{q})|H_D^I(Z-H_D)^{-1}|D(\vec{q}_1)\rangle\langle a(\vec{r})a(\vec{s})|(E-Z-H_C)^{-1}|a(\vec{q}_2)a(\vec{q}_3)\rangle\ . \quad (26)$$

As in Sec. III the matrix elements for $(Z-H_D)^{-1}$ and $(E-Z-H_C)^{-1}$ can be easily related to the D-particle and C-particle propagators. Equation (26) can therefore be rewritten as

$$\langle n(\vec{p})n(\vec{q})a(\vec{r})a(\vec{s})|H_D^I(Z-H_D)^{-1}(E-Z-H_C)^{-1}|D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)\rangle$$
$$=\langle n(\vec{p})n(\vec{q})|H_D^I|D(\vec{q}_1)\rangle\,\tau_D(Z+\epsilon_D-\tfrac{1}{2}\vec{q}_1^{\,2})\frac{\langle a(\vec{r})a(\vec{s})|H_C^I|C(\vec{r}+\vec{s})\rangle}{E-Z-\vec{r}^{\,2}/m-\vec{s}^{\,2}/m}$$
$$\times\tau_C(E-Z+\epsilon_C-(\vec{r}+\vec{s})^2/2m)\frac{\langle C(\vec{r}+\vec{s})|H_C^I|a(\vec{q}_2)a(\vec{q}_3)\rangle}{E-Z-\vec{q}_2^{\,2}/m-\vec{q}_3^{\,2}/m}\ , \quad (27)$$

after the propagators have been renormalized according tc the procedure of Appendix A. Following identical steps for the second term in the first matrix element of Eq. (25) we obtain

$$\langle n(\vec{p})n(\vec{q})a(\vec{r})a(\vec{s})|H_C^I(Z-H_D)^{-1}(E-Z-H_C)^{-1}|D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)\rangle$$
$$=\frac{\langle n(\vec{p})n(\vec{q})|H_D^I|D(\vec{q}_1)\rangle}{Z-\vec{p}^{\,2}-\vec{q}^{\,2}}\,\tau_D(Z+\epsilon_D-\tfrac{1}{2}\vec{q}_1^{\,2})\langle a(\vec{r})a(\vec{s})|H_C^I|C(\vec{r}+\vec{s})\rangle$$
$$\times\tau_C(E-Z+\epsilon_C-(\vec{r}+\vec{s})^2/2m)\frac{\langle C(\vec{r}+\vec{s})|H_C^I|a(\vec{q}_2)a(\vec{q}_3)\rangle}{E-Z-\vec{q}_2^{\,2}/m-\vec{q}_3^{\,2}/m}\ . \quad (28)$$

After substituting Eqs. (27) and (28) in Eq. (25), the Z integration can be performed using Cauchy's theorem as before. We finally obtain

$$\langle\vec{p},\vec{q},\vec{r}|\mathcal{B}_3(E)|\vec{k}\rangle=\sum_{q_i}\langle n(\vec{p})n(\vec{q})|H_D^I|D(\vec{q}_1)\rangle\langle a(\vec{r})a(\vec{s})|H_C^I|C(\vec{r}+\vec{s})\rangle$$
$$\times\,\mathcal{G}(E)\langle C(\vec{r}+\vec{s})|H_C^I|a(\vec{q}_2)a(\vec{q}_3)\rangle\langle D(\vec{q}_1)a(\vec{q}_2)a(\vec{q}_3)|H_\alpha^I|a(\vec{k})\alpha(-\vec{k})\rangle\ , \quad (29)$$

where

$$\mathcal{G}(E)=\frac{\tau_D(U')\tau_C(Y-U')}{Y-U-U'}$$
$$+(Y-U'-U'')\left\{\frac{-1}{U'}\,\frac{\tau_C(Y)}{(Y-U'')(Y-U)}-\frac{1}{\pi}\int_0^\infty dx\,\frac{\mathrm{Im}[\tau_D(x+\epsilon_D)]\tau_C(Y-\epsilon_D-x)}{(x+\epsilon_D-U')(Y-U-\epsilon_D-x)(Y-U''-\epsilon_D-x)}\right\}\ . \quad (30)$$

After evaluating all matrix elements left in Eq. (29) we obtain the following relations for Y, $Y-U$, $Y-U'$, and $Y-U''$:

$$\begin{aligned}
Y&=E+\epsilon_D+\epsilon_C-\tfrac{1}{2}(\vec{p}+\vec{q})^2-(\vec{r}+\vec{s})^2/2m\ ,\\
Y-U&=E+\epsilon_D-\tfrac{1}{2}(\vec{p}+\vec{q})^2-(\vec{k}+\vec{p}+\vec{q})^2/m-\vec{k}^2/m\ ,\\
Y-U'&=E+\epsilon_C-(\vec{r}+\vec{s})^2/2m-\vec{p}^{\,2}-\vec{q}^{\,2}\ ,\\
Y-U''&=E+\epsilon_D-\tfrac{1}{2}(\vec{p}+\vec{q})^2-\vec{r}^{\,2}/m-\vec{s}^{\,2}/m\ .
\end{aligned} \quad (31$$

Because the final four-body state is on the energy shell, that is, $E-\vec{p}^{\,2}-\vec{q}^{\,2}-\vec{r}^{\,2}/m-\vec{s}^{\,2}/m=0=Y-U'-U''$, the second term in $\mathcal{G}(E)$ is zero. Therefore, the final result for $\mathcal{B}_3$ to be used in Eq. (22) is given by

$$\langle\vec{p},\vec{q},\vec{r}|\mathcal{B}_3(q)|\vec{k}\rangle=(\gamma_D/\sqrt{2})f_D(\tfrac{1}{2}\vec{p}-\tfrac{1}{2}\vec{q})(\gamma_C/\sqrt{2})f_C(\tfrac{1}{2}\vec{r}-\tfrac{1}{2}\vec{s})$$
$$\times\frac{\tau_D(U')\tau_C(Y-U')}{Y-U-U'}\gamma_\alpha\gamma_C f_\alpha(\vec{p}+\vec{q}+\vec{k}2/(2+m))f_C(\vec{k}+\tfrac{1}{2}(\vec{p}+\vec{q}))\,\delta(\vec{p}+\vec{q}+\vec{r}+\vec{s})\ . \quad (32)$$

This completes the description of the $2\to4$ amplitude.

V. CONCLUSIONS

We have seen that some simplifying assumptions concerning two- and three-particle interactions lead to soluble integral equations for four-body processes. The resulting amplitudes are exact and possibly represent some kind of minimal set of four-body equations that are consistent with unitarity. This would be analogous to the three-body system, where it has been shown[10] that the

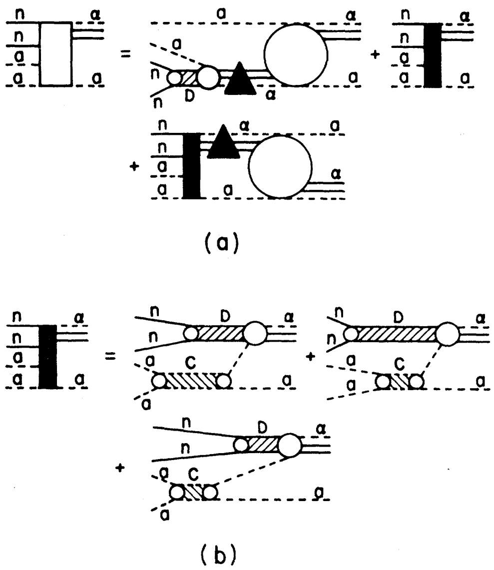

FIG. 13. Graphical representation of the integral relation expressing the $a\alpha \to nnaa$ amplitude (rectangle) in terms of the half-on-shell $a\alpha \to a\alpha$ amplitude (circle).

constraints of unitarity lead to a set of three-body equations very similar to the usual separable approximation to the Faddeev equations.

In spite of the simplicity of the model some interesting features emerge such as the correct exchange of composite particles and also the proper inclusion of the quasiparticle-quasiparticle state. We also see that breakup in the model, leading to three- and four-body final states, has complicated rescattering structure since two pairs of particles are allowed to participate in final-state interactions. This would add a note of pessimism to attempts to extract two-body information from a study of four-body final states.

Using simple analytic forms for the vertex functions we have already obtained numerical solutions to the proposed equations for elastic and rearrangement scattering. The results satisfy the requirements of unitarity above the two-, three-, and four-particle thresholds, which indicates the correctness of the model and its practical solvability. We can also construct more complicated models than are discussed in this paper by considering nonidentical particles. For example, we could take the four-body state to consist of $n+p+a+b$ with the pair of couplings $n+p \longleftrightarrow D$ and $a+b \longleftrightarrow C$. Four distinct quasiparticles could then be introduced into the three-body sector as illustrated in Table I. Each particle of one pair is coupled to the quasiparticle of the other pair in such a way that particle-exchange contributions in the three-body sector are still absent. The resulting four-body scattering equations are indicated in Table I and the necessary integral equations to describe these processes would be straightforward although complicated generalizations of the equations presented above.

A common process contributing to many reactions both in nuclear and particle physics is the exchange of a pair of correlated particles. It is customary to approximate or to neglect completely the two-particle continuum contribution to such processes. We plan to investigate the range of validity of such approximations and its dependence on the strength of the two-particle interaction. We are also at present applying these methods to the four-nucleon system as well as to two-nucleon-exchange processes in heavy-ion reactions.

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APPENDIX A: $\tau_D(X), \tau_C(X)$

The two-particle propagator is discussed in detail elsewhere[2] so that here we just list the results:

$$\tau_D(X) = \frac{S_D(X)}{X}\,, \tag{A1}$$

TABLE I. Possible four-body reactions resulting from specified two- and three-body interactions.

Two-body sector	Three-body sector	Four-body reactions
		$b\alpha \to b\alpha$
	$D+a \leftrightarrow \alpha$	$\to a\beta$
$n+p \leftrightarrow D$		$\to CD$
	$D+b \leftrightarrow \beta$	$\to n\eta$
		$\to p\delta$
	$C+n \leftrightarrow \delta$	$\to abD$
$a+b \leftrightarrow C$		$\to npC$
	$C+p \leftrightarrow \eta$	$\to npab$

$$[S_D(X)]^{-1}=Z_D-\tfrac{1}{2}\gamma_D{}^2\frac{1}{(2\pi)^3}\times\int d^3n\,\frac{f_D{}^2(\vec{n})}{(2\vec{n}^2+\epsilon_D)(X-\epsilon_D-2\vec{n}^2)}\,,\quad (A2)$$

$$Z_D=1-\tfrac{1}{2}\gamma_D{}^2\frac{1}{(2\pi)^3}\int d^3n\,\frac{f_D{}^2(\vec{n})}{(2\vec{n}^2+\epsilon_D)^2}\,.\quad (A3)$$

The corresponding expressions for τ_C have identical form.

APPENDIX B: $\tau_\alpha(X)$

We construct the α propagator by summing the series of self-energy bubbles.[2] Graphically the propagator is shown in Fig. 5(b) and the corresponding unrenormalized amplitude is given by

$$\tau_\alpha^{(u)}=\frac{1}{E+\epsilon_\alpha^{(0)}}+\frac{1}{E+\epsilon_\alpha^{(0)}}\,I(E)\frac{1}{E+\epsilon_\alpha^{(0)}}+\cdots\,.\quad (B1)$$

We take the bare energy of the α at rest to be $-\epsilon_\alpha^{(0)}$. $I(E)$ represents a bubble involving an a and a fully dressed D, and is given by

$$I(E)=\gamma_\alpha^{(u)2}\int\frac{d^3n}{(2\pi)^3}\,\frac{f_\alpha{}^2(\vec{n})S_D(E+\epsilon_D-\vec{n}^2/\mu_\alpha)}{E+\epsilon_D-\vec{n}^2/\mu_\alpha}\,,\quad (B2)$$

where

$$\mu_\alpha=2m/(2+m)\,,$$

$$m=m_a/m_n\,.$$

We have taken the D-particle propagator to be already renormalized. Requiring that $\tau_\alpha^{(u)}$ have a pole at the physical α energy $E=-\epsilon_\alpha$ gives the relation

$$\epsilon_\alpha^{(0)}=\epsilon_\alpha-(\gamma_\alpha^{(u)})^2\int\frac{d^3n}{(2\pi)^3}\,f_\alpha{}^2(\vec{n})\,\frac{S_D(\epsilon_D-\epsilon_\alpha-\vec{n}^2/\mu_\alpha)}{\epsilon_D-\epsilon_\alpha-\vec{n}^2/\mu_\alpha}\,.\quad (B3)$$

Assigning to the renormalized propagator $\tau_\alpha=\tau_\alpha^{(u)}/Z_\alpha$ a unit residue at the pole gives

$$\tau_\alpha(X)=S_\alpha(X)/X\,,\quad (B4)$$

where $S_\alpha(X)$ has the form

$$[S_\alpha(X)]^{-1}=Z_\alpha-\frac{\gamma_\alpha{}^2}{X}\int\frac{d^3n}{(2\pi)^3}[\tau_D(X+Y)-\tau_D(Y)]f_\alpha{}^2(\vec{n})\,,\quad (B5)$$

$$X=E+\epsilon_\alpha\,,$$

$$Y=\epsilon_D-\epsilon_\alpha-\vec{n}^2/\mu_\alpha\,.$$

The wave-function renormalization constant Z_α is given by

$$Z_\alpha=1+\gamma_\alpha{}^2\int\frac{d^3n}{(2\pi)^3}\,\{[YS_D'(Y)-S_D(Y)]/Y^2\}f_\alpha{}^2(\vec{n})\,,\quad (B6)$$

where $S_D'(Y)$ indicates the derivative of $S_D(Y)$. γ_α is the renormalized coupling constant given by

$$\gamma_\alpha{}^2=Z_\alpha(\gamma_\alpha^{(u)})^2\,.$$

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[1]See, for example, *Few Body Problems in Nuclear and Particle Physics*, Proceedings of the International Conference, Laval University, Quebec City, Canada, 1974, edited by R. J. Slabodrian, B. Cujec, and K. Ramavataram (Les Presses de l'Université Laval, Quebec, 1975).

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