Path-integral solutions of the Dirac equation*

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The feasibility of applying the path-integral formalism for solving the Dirac equation is shown in the case of a free particle for which the Dirac propagator is obtained by evaluating an appropriate path integral, directly constructed from the Dirac equation. Furthermore, the propagator for a Dirac electron in a constant magnetic field is indirectly obtained from the propagator of an auxiliary wave equation by evaluating a world-line (space-time path) integral. The spectrum of the Dirac equation is also, in this case, extracted from an auxiliary propagator.

I. INTRODUCTION

The path-integral method developed by Feynman[1] has hitherto been extremely successful in tackling nonrelativistic quantum-mechanical problems. Indeed, on many occasions it was possible to yield solutions that were difficult to obtain by other methods, e.g., when time-dependent external potentials are involved.

Although, in principle, there is nothing to stop us from applying the method for obtaining a formal expression for the propagator of the Dirac equation (see Feynman,[2] Morette,[3] Schulman,[4] and Hamilton and Schulman[5]), in practice a variety of mathematical difficulties has prevented direct actual evaluations.

The purpose of the present paper is to make progress toward deriving explicit solutions. Thus, we have obtained the propagator of the Dirac equation in the case of the free particle through a path integral, directly constructed from the Dirac equation. However, further applications have not been concluded in this direct manner. Nevertheless, the free-particle solution, thus obtained, was in a certain transparent form, which enabled us to see through the structure of the Dirac propagator in the general case. This structure we have further exploited by means of operator manipulations, and developed an auxiliary equation (whose propagator, in a certain sense, can be cast into path-integral form) from which the Dirac propagator can be extracted.

In Sec. II we evaluate the free-particle Dirac propagator by a path integral pertaining to the Dirac equation. In Sec. III we obtain the propagator of the Dirac electron in a constant magnetic field by evaluating a path integral giving the propagator (with respect to a new parameter) of an appropriate auxiliary equation involving this parameter. Finally, from our procedure we extract the spectrum of the Dirac electron subject to a constant magnetic field.

II. THE FREE PARTICLE

Let us initially proceed a little more generally. If $\underline{H}$ is the Dirac Hamiltonian, then the propagator of the Dirac equation

$$\left(i\hbar\frac{\partial}{\partial t}-\underline{H}\right)\underline{\Psi}=\underline{0} \tag{2.1}$$

for $\underline{H}$ time-independent is given by

$$\underline{K}(\vec{x}t|\vec{x}'0)=\exp\left(-\frac{i}{\hbar}\underline{H}t\right)\underline{I}\delta(\vec{x}-\vec{x}'), \tag{2.2}$$

where $\underline{I}$ is the (4×4) unit matrix.

Clearly (2.2) satisfies the Dirac equation (2.1) and as $t\to0$ it goes to the (4×4) δ-function diagonal matrix and so propagates a given 4-component spinor $\underline{\Psi}$, evolving via the Dirac equation.

Next, the finite-time propagator (2.2) can be generated, as is well known, through approximate short-time propagators, which need only be correct to first order in the short time, via the composition law[6]

$$\underline{K}(\vec{x}t|\vec{x}'0)=\lim_{(\max\Delta t_j)\to0}\int\cdots\int\underline{K}'(\vec{x}t|\vec{x}_{N-1}t_{N-1})\underline{K}'(\vec{x}_{N-1}t_{N-1}|\vec{x}_{N-2}t_{N-2})\cdots\underline{K}'(\vec{x}_2t_2|\vec{x}_1t_1)\underline{K}'(\vec{x}_1t_1|\vec{x}'0)\prod_{j=1}^{N-1}d\vec{x}_j, \tag{2.3}$$

where the integral is a $3(N-1)$-fold integral and where a prime on a $\underline{K}$ indicates an approximate short-time propagator correct to order Δt ($\Delta t_j=t_{j+1}-t_j$).

Equation (2.3) provides essentially the required path integral. It is quite clear that in this form the path integral allows all imaginable paths, including those with velocities greater than that of light. Quite correctly it seems that in composing our Dirac propagator we made use of path-violating principles of relativistic mechanics. However, the situation bears some relation to the case of nonrelativistic quantum mechanics in which we allow paths not conforming with the actual trajectory consistent with classical dynamics.

It is significant to observe the order of the various $\underline{K}'$ in (2.3); they are noncommuting matrices and their time ordering relates to the way a given spinor evolves via the Dirac dynamics.

The Dirac equation, being of first order in time, enables one to write a typical short-time propagator $\underline{K}'$ in terms of the infinitesimal generator $[\underline{I}-(i/\hbar)\Delta t\underline{H}]$ of the Dirac equation as

$$\underline{K}'(\vec{x}t'+\Delta t'|\vec{x}'t')=\left(\underline{I}-\frac{i}{\hbar}\Delta t'\underline{H}\right)\delta(\vec{x}-\vec{x}'), \qquad (2.4)$$

where $\underline{H}$ operates on the $\vec{x}$ coordinates. The expression (2.4) can also be obtained by expanding the exact expression for the propagator to first order in time.

We wish now to restrict ourselves to the free Dirac Hamiltonian

$$\underline{H}=c\sum_{r=1}^{3}\underline{\alpha}_r\frac{\hbar}{i}\frac{\partial}{\partial x_r}+mc^2\underline{\beta}, \qquad (2.5)$$

where the matrices $\underline{\alpha}_r$ ($r=1,2,3$) and $\underline{\beta}$ have their usual meaning and satisfy the well-known anticommutation relations.

Decomposing the δ function in (2.4) into plane waves we have the following expression for the short-time propagator:

$$\underline{K}'(\vec{x}t'+\Delta t'|\vec{x}'t')=\int\left[\underline{I}-\frac{i}{\hbar}\Delta t'\left(\sum_{r=1}^{3}\underline{\alpha}_r c\hbar k_r+mc^2\underline{\beta}\right)\right]\exp[i\vec{k}\cdot(\vec{x}-\vec{x}')]\frac{d\vec{k}}{(2\pi)^3}. \qquad (2.6)$$

In what follows we shall make use of isomeric time partitions, i.e., all Δt of the Nth partition will be taken equal to t/N.

Inserting (2.6) into (2.3) we obtain for the free-particle finite-time propagator the expression

$$\underline{K}(\vec{x}t|\vec{x}'0)=\lim_{N\to\infty}\int\cdots\int\underline{\mathcal{K}}_N\underline{\mathcal{K}}_{N-1}\cdots\underline{\mathcal{K}}_2\underline{\mathcal{K}}_1\left\{\prod_{j=1}^{N-1}\frac{d\vec{k}_j}{(2\pi)^3}\exp[i\vec{k}_j\cdot(\vec{x}_j-\vec{x}_{j-1})]\right\}\prod_{j=1}^{N-1}d\vec{x}_j, \qquad (2.7a)$$

where $\vec{x}_0=\vec{x}'$ and where $\underline{\mathcal{K}}_j$ is a (4×4) matrix given by

$$\underline{\mathcal{K}}_j=\underline{I}-\frac{i}{\hbar}\frac{t}{N}\left(\sum_{r=1}^{3}\underline{\alpha}_r c\hbar k_{jr}+mc^2\underline{\beta}\right), \qquad (2.7b)$$

where k_{jr} is the rth component of the vector $\vec{k}_j$.

Again it should be noted that as far as formula (2.7a) is concerned the order of the $\underline{\mathcal{K}}_j$ matrices is significant. Equation (2.7a) is already a form of the path integral giving the propagator of the free Dirac particle. The summation over all paths starting from $\vec{x}'$ at time 0 and ending at position $\vec{x}$ at time t is attained through the infinitely multiple process of integration.

The integrations over the various $\vec{k}_j$'s are essentially path summations in momentum space ($\hbar\vec{k}=\vec{p}$), and one would have liked to be left with pure summation over paths in configuration space. This is easily done in the following manner: As pointed out earlier, the short-time propagators entering the process of multiple integration need only be taken correct to first order in the short time Δt $(=t/N)$. If we then replace $-ick_{jr}t/N$ in (2.7b) by $[\exp(-ick_{jr}t/N)-1]$ and put the resulting $\underline{\mathcal{K}}_j$ in (2.7a), the limit as $N\to\infty$ will not be affected, for we are using short-time propagators correct to order Δt. With the above replacement we are able to perform the integrations, and the resulting path integral in terms of configuration paths alone is given by

$$\underline{K}(\vec{x}t|\vec{x}'0)=\lim_{N\to\infty}\int\cdots\int\left(\prod_{j=1}^{N}\left\{\left[\left(\underline{I}-\sum_{r=1}^{3}\underline{\alpha}_r\right)-\frac{i}{\hbar}mc^2\underline{\beta}\frac{t}{N}\right]\delta(\vec{x}_j-\vec{x}_{j-1})+\sum_{r=1}^{3}\underline{\alpha}_r\delta\left(\vec{x}_j-\vec{x}_{j-1}-\vec{n}_r c\frac{t}{N}\right)\right\}\right)\prod_{j=1}^{N-1}d\vec{x}_j \qquad (2.8)$$

with $\vec{x}_0=\vec{x}'$, $\vec{x}_N=\vec{x}$. The $\vec{n}_r$ ($r=1,2,3$) in (2.8) are the unit vectors in the x,y,z directions. The product of the various short-time propagators in (2.8) is ordered from right to left in increasing order of time.

For the purpose of evaluating the path integral (2.8) there is nothing to stop us from taking the Fourier transform of the δ functions of the integrand (the sequence of the integrands defines the functional under path integration). If we then expand each short-time propagator in power series of Δt and retain from each of them only a small portion, up to order Δt, we are brought back to (2.7a), which we proceed to evaluate as follows: We perform first the integrations over all $\vec{x}_j$ and thus the various exponentials produce the product of δ functions and two plane waves

$$\delta(\vec{k}_N-\vec{k}_{N-1})\delta(\vec{k}_{N-1}-\vec{k}_{N-2})\cdots\delta(\vec{k}_3-\vec{k}_2)\delta(\vec{k}_2-\vec{k}_1)\exp(i\vec{k}_N\cdot\vec{x}-i\vec{k}_1\cdot\vec{x}'),$$

with the aid of which, after integrating over the $\vec{k}_j$ ($j=1,2,\ldots,N-1$) and setting $\vec{k}_N=\vec{k}$, (2.7a) reduces to

$$\underline{K}(\vec{x}t|\vec{x}'0)=\lim_{N\to\infty}\int\left[\underline{I}-\frac{i}{\hbar}\left(\sum_{r=1}^{3}\underline{\alpha}_r c\hbar k_r+mc^2\underline{\beta}\right)\frac{t}{N}\right]^N\exp[i\vec{k}\cdot(\vec{x}-\vec{x}')]\frac{d\vec{k}}{(2\pi)^3}. \tag{2.9}$$

The limit as $N\to\infty$ can now be taken and leads to

$$\underline{K}(\vec{x}t|\vec{x}'0)=\int\exp\left[-\frac{i}{\hbar}\left(\sum_{r=1}^{3}\underline{\alpha}_r c\hbar k_r+mc^2\underline{\beta}\right)t\right]\exp[i\vec{k}\cdot(\vec{x}-\vec{x}')]\frac{d\vec{k}}{(2\pi)^3}. \tag{2.10}$$

It should be noted here that (2.10) could have been derived by a non-path-integral method, but that was not our object.

The argument of the first exponential in (2.10) is $(-i/\hbar)t$ times the matrix obtained from the free Dirac Hamiltonian by replacing the operator $\vec{p}$ $(=-i\hbar\partial/\partial\vec{x})$ by the vector $\hbar\vec{k}$. We shall then make use of the notation

$$\underline{H}(\vec{k})=\sum_{r=1}^{3}\underline{\alpha}_r c\hbar k_r+mc^2\underline{\beta}. \tag{2.11}$$

With the aid of the anticommutation relations of the matrices $\underline{\alpha}_r$ and $\underline{\beta}$, it becomes an easy matter to establish

$$\underline{H}^2(\vec{k})=[(c\hbar\vec{k})^2+(mc^2)^2]\underline{I}=\epsilon^2(\vec{k})\underline{I}, \tag{2.12}$$

where $\epsilon(\vec{k})$ will denote the positive root as

$$\epsilon(\vec{k})=+[(c\hbar\vec{k})^2+(mc^2)^2]^{1/2}. \tag{2.13}$$

Next, with the aid of (2.12) we are able to write

$$\exp\left[-\frac{i}{\hbar}\underline{H}(\vec{k})t\right]=\cos\left[\frac{i}{\hbar}\epsilon(\vec{k})t\right]\underline{I}-i\sin\left[\frac{i}{\hbar}\epsilon(\vec{k})t\right]\frac{\underline{H}(\vec{k})}{\epsilon(\vec{k})}. \tag{2.14}$$

Equation (2.14) inserted into (2.10) gives an expression for the required propagator

$$\underline{K}(\vec{x}t|\vec{x}'0)=\int\left\{\cos\left[\frac{i}{\hbar}\epsilon(\vec{k})t\right]\underline{I}-i\sin\left[\frac{i}{\hbar}\epsilon(\vec{k})t\right]\frac{\underline{H}(\vec{k})}{\epsilon(\vec{k})}\right\}\exp[i\vec{k}\cdot(\vec{x}-\vec{x}')]\frac{d\vec{k}}{(2\pi)^3}. \tag{2.15a}$$

Clearly, (2.15a) satisfies the free-particle Dirac equation and at the same time as $t\to 0$ it goes to $\underline{I}\,\delta(\vec{x}-\vec{x}')$. It can therefore propagate any spinor given at $t=0$, and evolving via the free-particle Dirac equation. As regards the matrix structure of the propagator this is in a substantially simple form, since it only depends linearly on the off-diagonal matrix $\underline{H}(\vec{k})$.

We shall now put (2.15a) into a more transparent form, as far as the energy spectrum is concerned. To this end we just decompose the trigonometric functions into their combinations of exponentials. We have

$$\underline{K}(\vec{x}t|\vec{x}'0)=\int\frac{1}{2}\left[\underline{I}+\frac{1}{\epsilon(\vec{k})}\underline{H}(\vec{k})\right]\exp\left[i\vec{k}\cdot(\vec{x}-\vec{x}')-\frac{i}{\hbar}\epsilon(\vec{k})t\right]\frac{d\vec{k}}{(2\pi)^3}$$
$$+\int\frac{1}{2}\left[\underline{I}-\frac{1}{\epsilon(\vec{k})}\underline{H}(\vec{k})\right]\exp\left[i\vec{k}\cdot(\vec{x}-\vec{x}')+\frac{i}{\hbar}\epsilon(\vec{k})t\right]\frac{d\vec{k}}{(2\pi)^3}. \tag{2.15b}$$

The first term in (2.15b) contains all the states with positive energy, $\epsilon(\vec{k})$, while the second comprises the states of negative energy, $-\epsilon(\vec{k})$. Of course, for the propagation of a given spinor both types of states enter the procedure, in general.

As a final remark the above form of the propagator, with appropriate replacements, is quite general, and we shall discuss this point further in the next section.

The integrations over $\vec{k}$ in (2.15b) can be performed as a matter of routine, and our propagator is expressed, with the aid of the function $\varphi(x^0,\lambda)$ appearing in Bogoliubov and Shirkov,[7] as

$$\underline{K}(\vec{x}t|\vec{x}'0)=\frac{1}{4\pi}\left(\underline{I}\frac{\partial}{\partial x^0}-\underline{\alpha}_3\frac{\partial}{\partial r}-im'\underline{\beta}\right)\frac{1}{r}\frac{\partial}{\partial r}[\varphi(x^0,\lambda)+\varphi^*(x^0,\lambda)], \tag{2.16}$$

where φ is given in terms of standard special functions through

$$\varphi(x^0,\lambda)=\begin{cases}\frac{1}{2i}N_0(m'\sqrt{\lambda})-\frac{1}{2}\epsilon(x^0)J_0(m'\sqrt{\lambda}) & \text{for } \lambda>0\\ \frac{i}{\pi}K_0(m'\sqrt{-\lambda}) & \text{for } \lambda<0,\end{cases}$$

with

$$m'=mc/\hbar,\quad x^0=ct,\quad r=|\vec{x}-\vec{x}'|,\ \text{and}\ \lambda=(x^0)^2-r^2.$$

III. ELECTRON IN A MAGNETIC FIELD

We wish to evaluate the propagator for a Dirac electron in a constant magnetic field along the z direction. We have not been able to do this by the previous method, but we proceed via an indirect approach. Let us begin with a more general Dirac Hamiltonian having an energy spectrum.

Although in the literature one may find justifications for the final product of our procedure,[8] here we shall sketch an alternative procedure which touches upon aspects of structure of the propagator of the Dirac equation.

The required propagator obeys the equation $\underline{\mathfrak{D}}\underline{K}=0$, where $\underline{\mathfrak{D}}=i\hbar\underline{I}\partial/\partial t-\underline{H}$ (= the Dirac operator). It would then be possible to reveal $\underline{K}$ by undoing the effect of the operator $\underline{\mathfrak{D}}$ on $\underline{K}$, which gives identically zero, i.e., operating by $\underline{\mathfrak{D}}^{-1}$ on $\underline{\mathfrak{D}}\underline{K}$. Of course, this requires $\underline{K}$ itself and the inverse Dirac operator, $\underline{\mathfrak{D}}^{-1}$. But suppose we consider the expression $\underline{B}\underline{J}$, with $\underline{B}$ another operator and $\underline{J}$ a solution of the equation $\underline{B}\underline{J}=0$. Then $(\underline{\mathfrak{D}}^{-1}\underline{B})\underline{J}$ is a solution of the Dirac equation. To obtain the propagator $\underline{K}$ we need a particular solution of an appropriate equation $\underline{B}\underline{J}=0$.

Let us, then, exploit the operator identity

$$\left(i\hbar\underline{I}\frac{\partial}{\partial t}-\underline{H}\right)^{-1}=\left(i\hbar\underline{I}\frac{\partial}{\partial t}+\underline{H}\right)\left(-\hbar^2\underline{I}\frac{\partial^2}{\partial t^2}-\underline{H}^2\right)^{-1}. \tag{3.1}$$

According to the preceding considerations, by taking $\underline{B}=\hbar^2\underline{I}\partial^2/\partial t^2+\underline{H}^2$ we can obtain the propagator of the Dirac equation through a special solution of the wave equation

$$\left(\hbar^2\frac{\partial^2}{\partial t^2}+\underline{H}^2\right)\underline{J}=\underline{0}. \tag{3.2}$$

This solution is given by

$$\underline{J}(\vec{x}t|\vec{x}'t')=-\frac{i}{\underline{H}}\sin\left[\frac{1}{\hbar}\underline{H}(t-t')\right]\delta(\vec{x}-\vec{x}') \tag{3.3}$$

since it is quite easy to deduce that

$$\begin{aligned}\underline{K}(\vec{x}t|\vec{x}'t')&=\left(i\hbar\frac{\partial}{\partial t}+\underline{H}\right)\underline{J}(\vec{x}t|\vec{x}'t')\\&=\exp\left[-\frac{i}{\hbar}\underline{H}(t-t')\right]\delta(\vec{x}-\vec{x}')\end{aligned} \tag{3.4}$$

is the propagator of the Dirac equation.

Clearly, the special solution (3.3) has a power series in the operator $\underline{H}^2$ and so we can expect to obtain an eigenfunction expansion of $\underline{J}(\vec{x}t|\vec{x}'t)$ in terms of the eigenvectors of the operator $\underline{H}^2$.

Since $\underline{H}^2$ has in general the form of a (4×4) matrix, its eigenvectors will be 4-component column vector functions (spinors, for short) $\underline{\Phi}_{n_\mu}(\vec{x})$ $(\mu=1,2,3,4)$, where n_μ stands collectively for the quantum numbers of the nth spinor. The set of all these spinors is complete in the sense

$$\sum_{\{n_\mu\}}\underline{\Phi}_{n_\mu}(\vec{x})\underline{\Phi}^{*T}_{n_\mu}(\vec{x}')=\underline{I}\delta(\vec{x}-\vec{x}'). \tag{3.5}$$

The eigenvalue equation obeyed by the $\underline{\Phi}_{n_\mu}$'s is

$$\underline{H}^2\underline{\Phi}_{n_\mu}(\vec{x})=\epsilon_{n_\mu}{}^2\underline{\Phi}_{n_\mu}(\vec{x}). \tag{3.6}$$

With the aid of (3.5) and (3.6), (3.3) takes the form

$$\begin{aligned}\underline{J}(\vec{x}t|\vec{x}'t')=-i\sum_{\{n_\mu\}}\frac{1}{\epsilon_{n_\mu}}\sin\left[\frac{i}{\hbar}\epsilon_{n_\mu}(t-t')\right]\\ \times\underline{\Phi}_{n_\mu}(\vec{x})\underline{\Phi}^{*T}_{n_\mu}(\vec{x}').\end{aligned} \tag{3.7}$$

The function $(1/\epsilon)\sin[\epsilon(t-t')/\hbar]$ is even with respect to ϵ, and so it is immaterial which sign we choose for the root of $\epsilon_{n_\mu}{}^2$, which is positive due to the large term mc^2, which it contains. We shall make use of the convention $\epsilon_{n_\mu}=+\sqrt{\epsilon_{n_\mu}{}^2}$.

The particular solution $\underline{J}(\vec{x}t|\vec{x}'t')$ contains the positive and negative eigenvalues of the Dirac equation, through the sine function. Not only does it contain the spectrum of the Dirac equation, but in addition through (3.4) it provides the propagator in the form

$$\underline{K}(\vec{x}t|\vec{x}'t')=\sum_{\{n_\mu\}}\frac{1}{2}\left(\underline{I}+\frac{\underline{H}}{\epsilon_{n_\mu}}\right)\underline{\Phi}_{n_\mu}(\vec{x})\underline{\Phi}^{*T}_{n_\mu}(\vec{x}')\exp\left[-\frac{i}{\hbar}\epsilon_{n_\mu}(t-t')\right]+\sum_{\{n_\mu\}}\frac{1}{2}\left(\underline{I}-\frac{\underline{H}}{\epsilon_{n_\mu}}\right)\underline{\Phi}_{n_\mu}(\vec{x})\underline{\Phi}^{*T}_{n_\mu}(\vec{x}')\exp\left[+\frac{i}{\hbar}\epsilon_{n_\mu}(t-t')\right]. \tag{3.8}$$

So, really, what is needed in order to obtain the Dirac propagator is the effect of the Dirac Hamiltonian on the eigenvectors of its square. This effect is in general an off-diagonal matrix function. It should be noted here that the form (3.8) of the Dirac propagator is essentially the same as that obtained for the free particle. As a matter of fact the free-particle solution served as a guiding tool for our further considerations.

Next, we notice that the particular solution (3.7) of the wave equation (3.2) can be written in the form

$$\underline{J}(\vec{x}t|\vec{x}'t') = \sum_{\{n_\mu\}} i \int_{-\infty}^{+\infty} d\omega\, \delta\left(\omega^2 - \left(\frac{\epsilon_{n_\mu}}{\hbar}\right)^2\right) \underline{\Phi}_{n_\mu}(\vec{x}) \underline{\Phi}^{*T}_{n_\mu}(\vec{x}') \exp[-i\omega(t-t')]. \tag{3.9a}$$

This can be seen by writing the δ function of ω^2 in (3.9a) as a linear combination of two δ functions of ω. Then upon Fourier-transforming the δ function in (3.9a), we immediately see that $\underline{J}$ takes the form

$$\underline{J}(\vec{x}t|\vec{x}'t') = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \int du\, d\omega \sum_{\{n_\mu\}} \exp\left\{iu\left[\omega^2 - \left(\frac{\epsilon_{n_\mu}}{\hbar}\right)^2\right] - i\omega(t-t')\right\} \underline{\Phi}_{n_\mu}(\vec{x}) \underline{\Phi}^{*T}_{n_\mu}(\vec{x}')$$

$$= \frac{i}{2\pi} \int_{-\infty}^{+\infty} du \exp\left[-\frac{i}{\hbar^2}\left(\hbar^2 \frac{\partial^2}{\partial t^2} + \underline{H}^2\right)u\right] \delta(\vec{x}-\vec{x}')\delta(t-t'). \tag{3.9b}$$

It is quite clear that the integrand of (3.9b) is the propagator (with respect to the parameter u) of the equation

$$\left[i\hbar^2 \frac{\partial}{\partial u} - \left(\hbar^2 \frac{\partial^2}{\partial t^2} + \underline{H}^2\right)\right]\underline{Q} = \underline{0}. \tag{3.10}$$

It should be noted here that although the term propagator ordinarily conveys time (propagation in time) it is used here with reference to the parameter u, which essentially plays the role of time in the sense that time parametrizes paths $\vec{x}(t)$ in 3-space, whereas u parametrizes world lines in the 4-dimensional space-time $(\vec{x}(u), t(u))$. In the "wave" equation (3.10) the parameter u has dimensions of time squared. The propagator of (3.10) can be made available through a space-time path (world-line) integral and this will be given later on.

Alternatively one could consider the u propagator of the equation

$$\left(i\hbar^2 \frac{\partial}{\partial u} - \underline{H}^2\right)\underline{G} = \underline{0}, \tag{3.11}$$

from which the special solution $\underline{J}$ can be obtained as

$$\underline{J}(\vec{x}t|\vec{x}'t') = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \int du\, d\omega\, \underline{G}(\vec{x}u|\vec{x}'0) \exp[iu\omega^2 - i\omega(t-t')]. \tag{3.12}$$

We wish now to find the special solution $\underline{J}$ of the wave equation (3.2) for the case of a Dirac electron under the influence of a magnetic field in the z direction. For reasons of convenience, which will become apparent when we reach the point of extracting the spectrum, we shall make use of the Landau gauge for the vector potential

$$\vec{A}^T = (-Bx_2, 0, 0),$$

and our Dirac Hamiltonian will be

$$\underline{H} = \underline{\alpha}_1 c\left(p_1 + \frac{e}{c}Bx_2\right) + \underline{\alpha}_2 cp_2 + \underline{\alpha}_3 cp_3 + mc^2\underline{\beta}, \tag{3.13}$$

where in (3.13) we have taken $e>0$, and hence the electron charge is $-e$. The square of the Hamiltonian (3.13) will be

$$\underline{H}^2 = c^2\left[\left(p_1 + \frac{e}{c}Bx_2\right)^2 + p_2{}^2 + p_3{}^2\right]\underline{I} + c\hbar eB\underline{\alpha}_0 + (mc^2)^2\underline{I}, \tag{3.14}$$

where $\underline{\alpha}_0$ is a (4×4) diagonal matrix with its first two diagonal elements equal to $+1$ and its last two equal to -1. It has to do with the spin-magnetic field interaction energy. Fortunately, (3.14) is in this case diagonal and essentially we have to deal with four decoupled world-line integrals.

For the purpose of obtaining a path (world-line) integral for the propagator of the wave equation (3.10) we can either make appropriate changes of variables and thus reduce our problem to a known algorithm, or proceed via first principles as prescribed in the preceding section. Since the latter procedure, in spite of its capability of yielding the correct path differential, has not received adequate attention in the literature, we wish to employ it here for our subsequent discussion.

It is quite clear that the parameter u in the wave equation (3.10) will label the 4-dimensional points $(\vec{x}(u), t(u))$ of the paths (world lines) for making our u propagator.

An approximate propagator associated with a small change in u and correct to first order in the change is given by

$$\underline{Q}'(\vec{x}t, u'+\Delta u'|\vec{x}'t', u') = \left[\underline{I} - \frac{i}{\hbar^2}\left(\hbar^2 \frac{\partial^2}{\partial t^2} + \underline{H}^2\right)\Delta u'\right] \times \underline{I}\delta(\vec{x}-\vec{x}')\delta(t-t'). \tag{3.15}$$

As before we shall make use of a prime to indicate approximate propagators correct up to first order in Δu, irrespective of whether they deviate from each other for higher orders in Δu.

Next, if we decompose our space-time δ function on the right-hand side of (3.15) into plane waves and let the operator in the square brackets act, we have

$$\underline{Q}'(\vec{x}t, u'+\Delta u'|\vec{x}'t', u') = \frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\left\{1 - i\Delta u'\left[-\omega^2 + \left(ck_1 + \frac{e}{\hbar}Bx_2\right)^2 + (ck_2)^2 + (ck_3)^2 + \left(\frac{mc^2}{\hbar}\right)^2 + \frac{e}{\hbar}Bc\underline{\alpha}_0\right]\right\} \times \underline{I}\exp[i\vec{k}\cdot(\vec{x}-\vec{x}') - i\omega(t-t')]. \tag{3.16}$$

An approximate propagator, correct to first order in $\Delta u'$, can be obtained by replacing the quantity in the curly brackets in the integrand by an exponential with argument

$-i\Delta u' \times$[expression in square brackets next to $\Delta u'$]

for upon expansion to order $\Delta u'$ this coincides with (3.16). With this exponential replacement in (3.16) and after performing the integrations over $\vec{k}$ and ω we find the following short Δu propagator

$$\underline{Q}'(\vec{x}t, u'+\Delta u'|\vec{x}'t', u') = \exp\left\{\frac{i}{4c^2}\left[\left(\frac{\vec{x}-\vec{x}'}{\Delta u'}\right)^2 - c^2\left(\frac{t-t'}{\Delta u'}\right)^2\right]\Delta u' - i\left[\frac{eB}{c\hbar}\left(\frac{x_1-x_1'}{\Delta u'}\right)x_2 + \left(\frac{mc^2}{\hbar}\right)^2 + \frac{eB}{\hbar}c\underline{\alpha}_0\right]\Delta u'\right\} \times\left(\frac{i}{4\pi\Delta u'}\right)^{1/2}\left(\frac{1}{4\pi i c^2\Delta u'}\right)^{3/2}. \tag{3.17}$$

It should be noted here that as far as the vector potential term is concerned the arithmetic-mean expression, $\frac{1}{2}[\vec{A}(\vec{x})+\vec{A}(\vec{x}')]\cdot(\vec{x}-\vec{x}')/\Delta u'$, in the approximate propagator is not required, since $\operatorname{div}\vec{A}=0$.

Expression (3.17) is very good for constructing the exact finite u propagator, for with this we are able to apply the well-known machinery of the path-integral calculus of nonrelativistic quantum mechanics. Thus, upon application of the composition law we can write (in the limit of infinite refinement of the u interval) the space-time path-integral expression for our finite u propagator as

$$\underline{Q}(\vec{x}t, u|\vec{x}'t', 0) = \int \exp\left\{\frac{i}{4c^2}\int_0^u\left[\left(\frac{d\vec{x}}{du'}\right)^2 - c^2\left(\frac{dt}{du'}\right)^2 - 4\frac{e}{\hbar}cBx_2\frac{dx_1}{du'}\right]du'\right\} \times\exp\left[-i\left(\frac{mc^2}{\hbar}\right)^2 u - i\frac{e}{\hbar}cB\underline{\alpha}_0 u\right]\mathfrak{D}[\vec{x}(u'), t(u')], \tag{3.18a}$$

where the space-time path (or world-line) differential is given by

$$\mathfrak{D}[\vec{x}(u'), t(u')] = \left[\prod_{0\le u'<u}(-i)(4\pi c\Delta u')^2\right]\prod_{0<u'<u}d\vec{x}(u')dt(u'). \tag{3.18b}$$

It is worthwhile noticing that in (3.18a) Planck's constant, $\hbar$, has nothing to do with the kinematic portion of the propagator phase

$$\left(\frac{d\vec{x}}{du}\right)^2 - c^2\left(\frac{dt}{du}\right)^2 = \left(\frac{ds}{du}\right)^2$$

(where $-ds/c$, as is well known, gives the proper time $d\tau$), and is not hidden in any way in the parameter u. In contrast the action constant enters the rest mass and interaction energy.

Clearly in (3.18a) the time phase is not coupled to the rest of the phase and so the path integration over time can be readily done by use of the formula for the 1-dimensional free-particle propagator, with appropriate replacements. Furthermore, with similar reinterpretation of the quantities contained in the rest of the phase (in the first exponential) the path integration can be achieved through that for the propagator of an electron in a magnetic field for the nonrelativistic case. We then have

$$\underline{Q}(\vec{x}t, u|\vec{x}'t', 0) = -ic^{-3}(4\pi u)^{-2}\frac{(mc^2\omega_c u/\hbar)}{\sin(mc^2\omega_c u/\hbar)}\exp\left[-\frac{i}{4}\frac{(t-t')^2}{u}\right] \times\exp\left\{\frac{i}{\hbar}\left[\frac{m\omega_c}{4}\cot\left(mc^2\omega_c\frac{u}{\hbar}\right)(\vec{x}_\perp-\vec{x}_\perp')^2 + \frac{m}{2}\omega_c(x_1+x_1')(x_2-x_2')\right] + i\frac{(x_3-x_3')^2}{4c^2u}\right\} \times\exp\left[-i\left(\frac{mc^2}{\hbar}\right)u\right]\exp\left(-i\frac{mc^2\omega_c}{\hbar}\underline{\alpha}_0 u\right), \tag{3.19}$$

where ω_c is the usual cyclotron frequency, $\omega_c = eB/mc$. The last exponential in (3.19) is the only one that involves a matrix, but this can be brought down by noticing that the powers of $i\underline{\alpha}_0$ behave like those of the imaginary unit matrix, i.e., $(i\underline{\alpha}_0)^2 = -\underline{I}$, and so

$$\exp(-i\underline{\alpha}_0 mc^2\omega_c u/\hbar) = \underline{I}\cos(mc^2\omega_c u/\hbar) - i\underline{\alpha}_0 \sin(mc^2\omega_c u/\hbar)$$
$$= \tfrac{1}{2}(\underline{I}-\underline{\alpha}_0)\exp\left(imc^2\hbar\omega_c\frac{u}{\hbar^2}\right) + \tfrac{1}{2}(\underline{I}+\underline{\alpha}_0)\exp\left(-imc^2\hbar\omega_c\frac{u}{\hbar^2}\right). \tag{3.20}$$

Our special propagator $\underline{J}(\vec{x}t|\vec{x}'t')$ from which the Dirac propagator can be obtained through (3.4) is, according to (3.9b), given by

$$\underline{J}(\vec{x}t|\vec{x}'t') = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\underline{Q}(\vec{x}t, u|\vec{x}'t', 0)du. \tag{3.21}$$

If we introduce in (3.21) the variable $u=-1/\alpha$, our result coincides with one given by Källén,[9] which was obtained by eigenfunction summation.

It is now a matter of routine exercise to combine (3.9b) with (3.4) and (3.21) for obtaining an expression for the required propagator of our Dirac equation as

$$\underline{K}(\vec{x}t|\vec{x}'t') = \left(i\hbar\frac{\partial}{\partial t} + \underline{H}\right)\frac{i}{2\pi}\int_{-\infty}^{+\infty}\underline{Q}(\vec{x}t, u|\vec{x}'t', 0)du. \tag{3.22}$$

The form (3.22) for the propagator of the Dirac equation is quite convenient for propagating a given spinor and as far as this purpose is concerned we shall pursue the evaluation no further.

Next, we wish to extract the energy spectrum $\{\epsilon_{n_\mu}\}$ of a Dirac electron in a magnetic field. For this purpose we have to cast our Dirac propagator in the form (3.8) or, more simply, just find the eigenfunction expansion of the propagator $\underline{G}(\vec{x}, u|\vec{x}', 0)$. Clearly, the propagator $\underline{G}$ can be obtained from the propagator $\underline{Q}$ by removing from it the factor coming from the path integral over the purely time paths. We have

$$\underline{G}(\vec{x}, u|\vec{x}', 0) = (4\pi ic^2u)^{-3/2}\frac{(mc^2\hbar\omega_c u/\hbar^2)}{\sin(mc^2\hbar\omega_c u/\hbar^2)}$$
$$\times\exp\left\{\frac{i}{\hbar}\left[\frac{m}{4}\omega_c\cot\left(mc^2\hbar\omega_c\frac{u}{\hbar^2}\right)(\vec{x}_\perp-\vec{x}_\perp')^2 + \frac{m}{2}\omega_c(x_1+x_1')(x_2-x_2')\right] + i\frac{(x_3-x_3')^2}{4c^2u}\right\}$$
$$\times\left\{(\underline{I}-\underline{\alpha}_0)\exp\left[-imc^2(mc^2-\hbar\omega_c)\frac{u}{\hbar^2}\right] + \tfrac{1}{2}(\underline{I}+\underline{\alpha}_0)\exp\left[-imc^2(mc^2+\hbar\omega_0)\frac{u}{\hbar^2}\right]\right\}. \tag{3.23}$$

The part of the propagator in front of the last curly brackets in (3.23), which involve the matrices $(\underline{I}+\underline{\alpha}_0)$, can be written in eigenfunction expansion form using (with appropriate replacements) the eigenfunctions of the nonrelativistic electron in a constant magnetic field applied in the z direction, when the Landau gauge is employed (see Landau and Lifshitz, Ref. 10, p. 425). In fact by writing (3.11) in the form

$$\left(i\hbar\frac{\partial}{\partial u} - \frac{1}{\hbar}\underline{H}\right)\underline{G} = \underline{0},$$

we just find that u plays the role of t and that $\hbar/2c^2$ plays the role of the mass in the Landau formulas. Therefore, we have for $\underline{G}$

$$\underline{G}(\vec{x}, u|\vec{x}', 0) = \sum_{n,\vec{k}}\Phi_{n,\vec{k}}(\vec{x})\Phi^*_{n,k}(\vec{x}')\exp\left[-\frac{i}{\hbar^2}(\epsilon^{(0)}_{n,\vec{k}})^2u\right]\sum_{\mu=1}^{4}\underline{\eta}_\mu\underline{\eta}_\mu^T\exp\left[-imc^2(mc^2-s_\mu\hbar\omega_c)\frac{u}{\hbar^2}\right], \tag{3.24a}$$

where $\Phi_{n,\vec{k}}$ are the usual nonrelativistic eigenfunctions composed of Hermite polynomials and plane waves:

$$\Phi_{n,\vec{k}}(\vec{x}) = H_n\left(\left(\frac{m\omega_c}{\hbar}\right)^{1/2}\left(x_2-\frac{\hbar}{m\omega_c}k_1\right)\right)\exp\left\{-\frac{m\omega_c}{\hbar}\left[x_2-\left(\frac{\hbar}{m\omega_c}\right)^{1/2}\right]^2\right\}\exp(i\vec{k}_\perp\cdot\vec{x}_\perp) \tag{3.24b}$$

and where the eigenvalue $(\epsilon^{(0)}_{n,\vec{k}})^2$ can be obtained from the corresponding nonrelativistic eigenvalue formula

$$\left(n+\frac{1}{2}\right)\frac{e\hbar B}{Mc} + \frac{\hbar^2k_3^2}{2M}\quad(n=0,1,2,\ldots)\quad(-\infty<k_3<+\infty),$$

with the appropriate replacement for the mass. (Note that we have used capital M for the mass in the nonrelativistic formula, just to avoid confusion.) We have

$$(\epsilon^{(0)}_{n,\vec{k}})^2 = (2n+1)mc^2\hbar\omega_c + (\hbar ck_3)^2. \tag{3.24c}$$

Finally the $\underline{\eta}_\mu$ stand for the 4-dimensional unit vectors

$$\underline{\eta}_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad \underline{\eta}_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad \underline{\eta}_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad \underline{\eta}_4=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \tag{3.24d}$$

and s_μ is the spin index

$$s_\mu=\begin{cases}+1 & \text{for } \mu=1,2\\ -1 & \text{for } \mu=3,4.\end{cases} \tag{3.24e}$$

The quantity given by $\sum_{\mu=1}^4$ in (3.24a) relates to the rest mass and spin energy. The introduction of the unit 4-vectors $\underline{\eta}_\mu$ has enabled a suitable decomposition of the matrices $(\underline{I}+\underline{\alpha}_0)$, and so by combining the two exponentials associated with them in (3.23) we find immediately that the energy levels of the Dirac electron in a magnetic field are given by

$$\epsilon_{n,k_3;s}=[(2n+1)mc^2\hbar\omega_c+(c\hbar k_3)^2 + mc^2(mc^2+s\hbar\omega_c)]^{1/2}, \tag{3.25}$$

depending on the quantum numbers n, k_3 $(n=0,1,2,\ldots)$, $-\infty<k_3<+\infty$, and the spin quantum number s, $s=\pm1$.

Expanding (3.25) about the rest-mass energy $[mc^2 \gg$ magnetic energy and $(c\hbar k_3)^2]$ we obtain the nonrelativistic energy after removing the mc^2.

Finally for $n=0$, $k_3=0$, and $s=-1$ one obtains no zero-point energy from the magnetic field as a result of the spin interaction.

The eigenvectors of (3.24a) are given by the column vector functions

$$\underline{\Phi}_{(n\vec{k})_\mu}(\vec{x})=\Phi_{n\vec{k}}(\vec{x})\underline{\eta}_\mu,$$

and these are the ones needed to be fed into (3.8) for obtaining the corresponding eigenfunction expansion of the Dirac propagator.

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