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Bound states, tachyons, and restoration of symmetry in the $1/N$ expansion*

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An extensive analysis of the $1/N$ expansion of O(N)-symmetric $\lambda\phi^4$ theory in four dimensions shows it to be a consistent approximation method. It is confirmed that the ground state of the theory is O(N)-symmetric, and that spontaneous symmetry breaking is not possible in the large-N limit. The Green's functions are free of tachyons if constructed relative to this ground state. A natural upper bound is derived for the parameters of the theory to ensure the existence of a ground state. In the strong-coupling domain there exist a bound state and a resonance [in the identity representation of the O(N) group], which disappear in the weak-coupling regime. It is shown that, to leading order in N, a zero-mass interacting "charged" boson cannot be sustained in this theory. If the boson mass goes to zero, the model becomes a free-field theory.

I. INTRODUCTION

As part of the continuing effort to understand the physical content of quantum field theories, techniques have been developed which avoid a conventional perturbation expansion in the coupling constants of the theory. One particular method, which has attracted considerable attention as a tool for the study of spontaneous symmetry breakdown, is the $1/N$ expansion[1,2] of the Green's functions of O(N)-symmetric field theories. Initial studies of this method,[1,2] when applied to $\lambda\phi^4$ theory in four (space-time) dimensions, unfairly cast doubt on the validity of the method. The heretofore unresolved difficulty was the presence of tachyons in the Green's functions of the theory constructed relative to the natural candidate for the ground state of the model. The consequences of this tachyon propagate through the theory, since it was shown by Root[3] that the $1/N$ corrections to the effective potential make this particular vacuum state unstable as a direct result of the tachyons in the Green's functions.

More recent studies[4] have exploited the fact that the effective potential $V(\phi^2)$ is a double-valued function of ϕ^2 for small ϕ^2. It turns out that the ground state of the theory is to be found from the branch of $V(\phi^2)$ not previously considered. If the Green's functions are constructed relative to the true vacuum, they become free of tachyons. Further, the ground state of the theory in the large-N limit *always* has the O(N) symmetry of the original Lagrangian. Spontaneous symmetry breakdown and Goldstone phenomena are not possible in this limit.

Although previous workers[4] have shown that it was possible to avoid tachyons in the theory, their results are incomplete. In particular, only special parameter choices were treated, while Root's criticism[3] and other important issues were not considered at all. Therefore, the work of Ref. 4 does not resolve the consistency of the $1/N$ expansion of O(N)-symmetric $\lambda\phi^4$ theory in four dimensions. In order to avoid ambiguities and confusion related to particular choices of renormalization mass, we carry out our analyses in a renormalization-invariant manner throughout. As a result we are able to consider all possible (renormalization invariant) parameter choices in a complete and coherent manner, and not just special cases. The issues we face are all addressed to the consistency of the $1/N$ expansion in four dimensions.

Let us summarize the organization of our paper in view of the large number of topics considered. In Sec. II we review the salient features of the theory, and define renormalization-invariant variables for the description of the model. This gives us the necessary tools for our subsequent analysis. We study the ground state of the model in Sec. III, and confirm that spontaneous symmetry breaking is not possible in the large-N limit. We find a natural upper bound for the free parameters of the theory if a stable ground state is to exist. The Green's functions of the theory, computed to leading order in N, are studied in Sec. IV. It is shown that tachyons are always absent if the Green's functions are constructed relative to the vacuum defined by the ground-state branch of the gap equation. In Sec. V it is established that both a bound state *and* a resonance occur if the natural interaction strength of the model is sufficiently strong, while neither occurs for weak coupling. [These bound-state structures appear in the identity representation of the O(N) group.] We then prove that, to leading order in N, a zero-mass interacting "charged" boson cannot be sustained in this theory, a result which is "*natural*" in the technical sense of being valid for *arbitrary* parameter sets in the model. Finally, we compute bound-state and resonance masses, and show that the parameter upper bound found in Sec. III corresponds to the limit in which the bound state *unavoidably* becomes a tachyon. The $1/N$ corrections to the effective potential are considered in Sec. VI. We conclude, up to next-to-leading order in $1/N$, that there exists a finite, real region of $V(\phi^2)$ in the neighborhood of the ground state at $\phi^2=0$, *if* the $1/N$ corrections are evaluated relative to the ground-state branch of the gap equation. In Sec. VII we establish criteria for the domain of validity of the $1/N$ expansion. We show in Sec. VIII that the large-ϕ^2 behavior of $V(\phi^2)$ cannot be computed reliably by the $1/N$ expansion. Even the qualitative behavior of the $V(\phi^2)$ is not stable to $1/N$ corrections in the limit of large ϕ^2. In Sec. IX we attempt to draw some lessons from these exercises.

The over-all conclusion of the extensive analysis is that there are no remaining objections to the $1/N$ expansion for four-dimensional scalar field theories. Therefore, it must be considered a consistent approximation procedure. It is not known if the $1/N$ expansion correctly characterizes the properties of the complete theory, or if the results can be extrapolated to small N. These are unanswered questions.

II. THE MODEL

The O(N)-symmetric $\lambda\phi^4$ theory is described by the (unrenormalized) Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}\mu_0{}^2\Phi^2-\frac{\lambda_0}{4!N}(\Phi^2)^2, \qquad (2.1)$$

where Φ_a is an N-component quantum field and $\Phi^2=\sum_{a=1}^N\Phi_a\Phi_a$. To leading order in $1/N$, the effective potential satisfies[1,2]

$$\frac{dV(\phi^2)}{d\phi^2}=\tfrac{1}{2}\chi, \qquad (2.2)$$

with χ related to the (constant) classical field ϕ by the (unrenormalized) gap equation

$$\chi=\mu_0{}^2+\frac{\lambda_0}{6}\left(\frac{\phi^2}{N}\right)+\frac{\lambda_0}{6}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\chi}, \qquad (2.3)$$

where the integral is over Euclidean momenta. Renormalized parameters μ^2, λ, and g are defined by the equations

$$\frac{\mu^2}{g}=\frac{\mu_0{}^2}{\lambda_0}+\tfrac{1}{6}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}, \qquad (2.4)$$

$$\frac{1}{g}=\frac{1}{\lambda_0}+\tfrac{1}{6}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}, \qquad (2.5)$$

and

$$\frac{1}{g}=\frac{1}{\lambda}+\frac{1}{96\pi^2}, \qquad (2.6)$$

where M^2 is an arbitrary renormalization mass, and the integrals are over Euclidean momenta. (The parameter g is most convenient for studying the gap equation, while λ appears naturally in the Green's functions.) After renormalization

$$\chi=\mu^2+\frac{g}{6}\left(\frac{\phi^2}{N}\right)+\frac{g}{96\pi^2}\chi\ln(\chi/M^2). \qquad (2.7)$$

Noting that the Euclidean integral in (2.3) is real when χ is real and positive, one defines the logarithm to be real for this range of χ.

From Eqs. (2.4)–(2.6) we see that (μ^2/g) is a renormalization-invariant parameter, although μ^2, g, and λ individually are M^2 dependent. In fact even the absolute signs of g and μ^2 are renormalization mass dependent since the effective potential with parameters g, μ^2, and M^2 can be mapped to one with parameters $-g$, $-\mu^2$, and M'^2, where $M'^2=M^2\exp(192\pi^2/g)$. Thus μ^2 only serves as an intermediate renormalized mass parameter which renders the model finite. Since μ^2 depends on M^2, one should not expect it to be the physical meson mass. In general it is not. The dimensionless coupling parameter g depends on M^2 in such a way that

$$\frac{1}{g(M_1)}=\frac{1}{g(M_2)}+\frac{1}{96\pi^2}\ln(M_2{}^2/M_1{}^2). \qquad (2.8)$$

As a result

$$M_1{}^2\exp[96\pi^2/g(M_1)]=M_2{}^2\exp[96\pi^2/g(M_2)], \qquad (2.9)$$

and similarly for $\lambda(M)$. In order to avoid ambiguities of interpretation, we shall present our analysis in terms of renormalization-invariant quantities throughout. We define the renormalization-invariant quantities χ_0 and $\rho(\phi^2)$, where

$$\chi_0=M^2\exp(96\pi^2/g) \qquad (2.10)$$

and

$$\chi(\phi^2)=\rho(\phi^2)\chi_0. \qquad (2.11)$$

Note that $\rho(\phi^2)$ is dimensionless, while χ_0 has dimensions of mass squared. The gap equation becomes

$$\rho\ln\rho=-\frac{96\pi^2}{\chi_0}\left(\frac{\mu^2}{g}\right)-\frac{16\pi^2}{\chi_0}\left(\frac{\phi^2}{N}\right), \qquad (2.12)$$

which demonstrates that $\chi(\phi^2)$ is determined once the the two renormalization-invariant parameters χ_0 and (μ^2/g) are specified. It is to be observed that a natural scale for ϕ^2/N is set by $(\chi_0/16\pi^2)$, a remark which will be helpful in the discussion in Sec. VII of the domain of validity of the $1/N$ expansion. Further, at $\phi^2=0$, the gap equation only depends on the single parameter $(96\pi^2/\chi_0)(\mu^2/g)$.

Certain qualitative features of the effective potential can be deduced without a numerical solution. First, we observe that the *leading-order* effective potential has no lower bound as a consequence of Eqs. (2.2) and (2.7), since

$$\chi = \frac{\mu^2 + (g/6)(\phi^2/N)}{1-(g/96\pi^2)\ln(\chi/M^2)}, \tag{2.13a}$$

$$\chi \underset{\phi^2\to\infty}{\sim} -\frac{16\pi^2(\phi^2/N)}{\ln(\chi/M^2)}. \tag{2.13b}$$

Thus

$$\mathrm{Re}\chi \underset{\phi^2\to\infty}{\sim} \frac{-16\pi^2(\phi^2/N)}{\ln(\phi^2/M^2)} \tag{2.14}$$

and

$$\mathrm{Re}\frac{dV(\phi^2)}{d\phi^2} \underset{\phi^2\to\infty}{\sim} \frac{-8\pi^2(\phi^2/N)}{\ln(\phi^2/M^2)}, \tag{2.15}$$

a result which is independent of the parameters of the theory. In Sec. VIII we show that this result does not persist in higher orders of the expansion. When the $1/N$ correction is included, $V(\phi^2)$ increases with ϕ^2 for arbitrarily large ϕ^2. Although these values of ϕ^2 are outside the domain of validity of the expansion, it indicates that (2.15) is not a definitive prediction of the theory.

The leading-order effective potential is not everywhere real,[2] becoming complex for ϕ^2 sufficiently large. From (2.2) and (2.7) one has

$$\mathrm{Im}V(\phi^2)\neq 0 \text{ for all } \phi^2>\phi_b^2, \tag{2.16}$$

where the branch point appears at

$$\phi_b^2/N = -6\mu^2/g + \frac{1}{16\pi^2}M^2\exp\left(\frac{96\pi^2}{\lambda}\right) \tag{2.17a}$$

$$= -6\mu^2/g + \frac{e^{-1}}{16\pi^2}\chi_0, \tag{2.17b}$$

which corresponds to

$$\chi_b \equiv \chi(\phi_b^2) = M^2 e^{96\pi^2/\lambda} = e^{-1}\chi_0, \tag{2.18}$$

which is a renormalization invariant. We shall subsequently show that the existence of this branch point does not signal a fundamental failure of the $1/N$ expansion, but rather indicates that $V(\phi^2)$ is a double-valued function of ϕ^2 for $\phi^2<\phi_b^2$. The real part of $V(\phi^2)$ in leading order reaches a maximum at ϕ_c^2 before beginning its plunge toward negative infinity, where

$$\phi_c^2/N = -6\mu^2/g + \frac{1}{32\pi^2}M^2\exp(96\pi^2/g)$$
$$= -6\mu^2/g + \frac{1}{32\pi^2}\chi_0$$
$$> \phi_b^2/N. \tag{2.19}$$

For $\phi^2<\phi_b^2$ one requires a solution of (2.12) to determine $\rho(\phi^2)$ and then $V(\phi^2)$. In Fig. 1 we plot the function $\rho\ln\rho$ as a function of ρ for real values of ρ. From the figure, we see that ρ is a double-valued function of $\rho\ln\rho$ for $-e^{-1}\le\rho\ln\rho\le 0$. Define these two branches as

$$\text{branch I: } \rho_{\mathrm{I}}<e^{-1}$$
$$\text{and} \tag{2.20}$$
$$\text{branch II: } \rho_{\mathrm{II}}>e^{-1},$$

with branches I and II of $V(\phi^2)$ that portion of the effective potential determined by branches I or II of $\rho(\phi^2)$, and with a similar definition given to branches I and II of $\chi(\phi^2)$. From Eqs. (2.2), (2.11), and (2.12) and Fig. 1 we can generate a solution for $\mathrm{Re}V(\phi^2)$. These are shown in Figs. 2, 3, and 4 for $\mu^2/g>0$, $\mu^2/g<0$, and $\mu^2/g=0$, respectively. The solid line indicates the domain for which the potential is real, while the dashed line indicates that $\mathrm{Im}V(\phi^2)\neq 0$. Note that $V(\phi^2)$ on branch II is *always less* than $V(\phi^2)$ on branch I for the same value of ϕ^2. The dividing point between the branches, defined by Eq. (2.20), corresponds to the branch point at ϕ_b^2, as can be observed from (2.11), (2.18), and (2.20).

Another relevant feature of the effective potential is the renormalization invariant

$$\chi(0)=\rho(0)\chi_0, \tag{2.21}$$

where

$$\rho(0)\ln\rho(0) = -\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right). \tag{2.22}$$

Comparing (2.22) with (2.12), we see that

$$\rho(\phi^2)\ln\rho(\phi^2)<\rho(0)\ln\rho(0) \tag{2.23}$$

for all $0\le\phi^2\le\phi_b^2$, thus on branch II of the gap equation

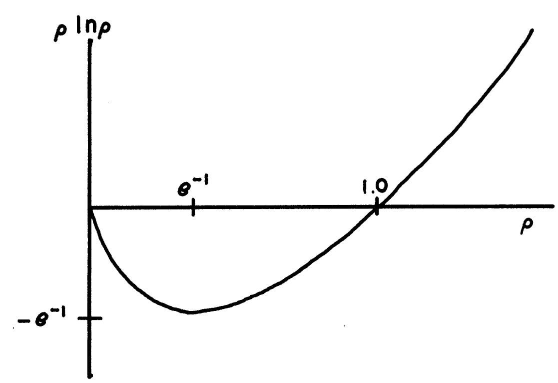

FIG. 1. A graph of $\rho\ln\rho$ versus ρ for real values of ρ.

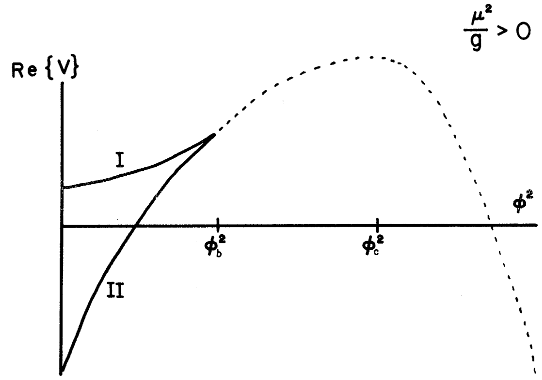

FIG. 2. A graph of Re $V(\phi^2)$ versus ϕ^2 for $\mu^2/g>0$, where the solid line indicates the domain for which the effective potential is real, while the dashed line indicates that Im $V(\phi^2)\neq 0$.

$$\chi_{II}(0)\geq\chi_{II}(\phi^2)\geq\chi_b=\chi(\phi_b{}^2)>0 \text{ for } 0\leq\phi^2\leq\phi_b{}^2. \tag{2.24}$$

We shall show in the next two sections that the evaluation of the Green's functions relative to the ground state defined by branch II is a necessary condition for the removal of tachyons from the theory. Further, we show that depending on the parameter values (χ_0, μ^2/g) the theory exhibits either a bound state *and* a resonance, or no bound state or resonance. If the spectrum of the renormalized theory contains a bound state which is not contained in the spectrum of the bare Lagrangian, one should *not* expect the effective potential and Green's functions to go over to the weak-coupling loop expansion as a boundary condition. The inequivalence of the physical branch of the gap equation to the vacuum defined by loop expansion, the presence of the branch point at $\phi_b{}^2$, and the inequivalent particle spectrum of bare and renormalizaed theories appear to be different aspects of the same phenomenon. This

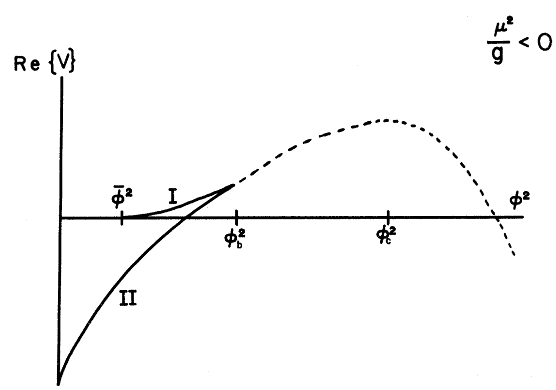

FIG. 3. Same as Fig. 2, except $\mu^2/g<0$.

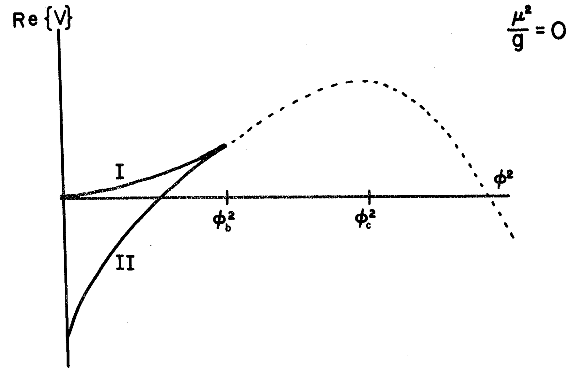

FIG. 4. Same as Fig. 2, except $\mu^2/g=0$.

situation, reminiscent of superconductivity, will be developed in subsequent sections.

III. GROUND STATE

The effective potential to leading order is[1,2]

$$V(\phi^2,\chi)=-\frac{3N}{2g}\chi^2+\tfrac{1}{2}\chi\phi^2+\frac{3N\mu^2}{g}\chi+\frac{N\chi^2}{128\pi^2}\left[2\ln\left(\frac{\chi}{M^2}\right)-1\right] \tag{3.1}$$

up to an additive constant, with χ satisfying the gap equation. This can be expressed in the explicitly renormalization-invariant form by using (2.10) and (2.11) to obtain

$$N^{-1}V(\phi^2,\chi)=3\chi_0\rho\left[\left(\frac{\mu^2}{g}\right)+\frac{1}{6}\left(\frac{\phi^2}{N}\right)+\frac{\chi_0}{384\pi^2}(2\rho\ln\rho-\rho)\right]. \tag{3.2}$$

From (2.2) observe that the local minimum of $V(\phi^2,\chi)$ is attained only if either $\chi=0$ or $\phi^2=0$. From (2.7)

$$\chi=0 \tag{3.3a}$$

requires

$$(\mu^2/g)<0 \tag{3.3b}$$

and implies

$$\overline{\phi}^2/N=-6\mu^2/g \tag{3.3c}$$

at this local minimum, in which case

$$V(\overline{\phi}^2,0)=0 \tag{3.4}$$

for the particular choice of additive constant in (3.1). The minimum described by Eqs. (3.3) and (3.4) lies on branch I of the effective potential. For all other cases the local minima are found at $\phi^2=0$.

Substituting from the gap equation (2.12), we find

$$V(\phi^2,\chi(\phi^2))=\frac{N\chi(\phi^2)}{2}\left[\frac{1}{2}\left(\frac{\phi^2}{N}\right)+3\left(\frac{\mu^2}{g}\right)-\frac{\chi(\phi^2)}{64\pi^2}\right]. \quad (3.5)$$

Thus

$$V(0,\chi(0))=\frac{3N\chi(0)}{2}\left[\left(\frac{\mu^2}{g}\right)-\frac{\chi(0)}{192\pi^2}\right]. \quad (3.6)$$

If $(\mu^2/g)<0$, there exists one *real* solution $\chi_{II}(0)>0$, so that

$$V(0,\chi_{II}(0))<0, \quad (3.7)$$

which means the local minimum of branch II *always* lies below the minimum on branch I, described by Eqs. (3.3) and (3.4). Thus, spontaneous symmetry breaking is *not* possible in this model.[4] The O(N)-symmetric ground state always lies *below* the vacuum state which could give Goldstone phenomena. If $(\mu^2/g)=0$, with $\chi_0>0$, then Eq. (2.22) has two solutions

$$\rho_I(0)=0 \quad (3.8a)$$

and

$$\rho_{II}(0)=1 \quad (3.8b)$$

belonging to branches I and II, respectively. It is obvious that (3.7) again holds, and is the lowest minimum in this case also.

Finally, for $(\mu^2/g)>0$ the analysis is slightly more complicated because the quantity in brackets in (3.6) is not of a single sign. The most elegant argument for finding the ground state is indirect. First, we have

$$\frac{dV(\phi^2)}{d\phi^2}=\tfrac{1}{2}\chi(\phi^2), \quad (2.2)$$

which is renormalization invariant from (2.11) and (2.12). Then note that the two branches coincide at $\phi^2=\phi_b{}^2$, and further

$$\left.\frac{dV(\phi^2)}{d\phi^2}\right|_{\phi_b{}^2}=\tfrac{1}{2}\chi_I(\phi_b{}^2)=\tfrac{1}{2}\chi_{II}(\phi_b{}^2)=\tfrac{1}{2}\chi_b. \quad (3.9)$$

From (2.17) we have

$$\phi_b{}^2=0 \quad (3.10a)$$

if

$$\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)=e^{-1}, \quad (3.10b)$$

which is the same as

$$(96\pi^2)\left(\frac{\mu^2}{g}\right)=\chi_b. \quad (3.10c)$$

Therefore, if

$$\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)\geq e^{-1}, \quad (3.11)$$

the effective potential is *everywhere complex*, that is, (3.11) is a *forbidden* domain for a consistent theory. Therefore, we must restrict ourselves to the parameter range delineated by

$$\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)<e^{-1}:\ \text{allowed domain.} \quad (3.12)$$

From (2.22) and Fig. 1, we see that if (3.12) is satisfied and $(\mu^2/g)>0$ there are *two real* solutions for $\rho(0)$ and thus for $\chi(0)$. Combining (2.11), (2.20), and (2.24) with this observation we infer that

$$\chi_{II}(0)\geq\chi_{II}(\phi^2)\geq\chi_b\geq\chi_I(\phi^2)\geq\chi_I(0) \quad (3.13)$$

in the allowed parameter domain. Now using (3.13) with (3.9) to integrate back to $\phi^2=0$, we see that the effective potential on branch II *always* lies lower than branch I. This proves the desired result.

The conclusions of this section are as follows:

(i) If

$$\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)\geq e^{-1}, \quad (3.14)$$

the effective potential is everywhere complex, and no consistent theory is possible.

(ii) If

$$\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)<e^{-1}, \quad (3.15)$$

the ground state of the model occurs at $\phi^2=0$, and branch II of the effective potential *always* lies below branch I.

IV. GREEN'S FUNCTIONS

The (unrenormalized) effective action of the model, to leading order in N, is[1,2]

$$\Gamma=\int d^4x\left(\tfrac{1}{2}\phi_a\Box\phi_a+\frac{3}{2}\frac{N}{\lambda_0}\chi^2-\tfrac{1}{2}\chi\phi^2-\frac{3N\mu_0{}^2}{\lambda_0}\chi\right)-\frac{N}{2}\operatorname{tr}\ln(-\Box+\chi), \quad (4.1)$$

where tr denotes the trace operator considered as an integral operator in Euclidean four-space. All Green's functions of the model can be constructed from (4.1). In particular, the (unrenormalized) ϕ-χ matrix inverse propagator, in the presence of external fields, is[2]

$$\underline{D}^{-1}(-k^2,\phi,\chi)=\begin{bmatrix}(k^2+\chi)\delta_{ab} & \phi_a\\ \phi_b & -3N\left[\frac{1}{\lambda_0}-\bar{B}(\chi,k^2,\Lambda^2)\right]\end{bmatrix} \quad (4.2)$$

for Euclidean momenta k^2. The quantity $\overline{B}(\chi, k^2, \Lambda^2)$ is given by a divergent integral over Euclidean momenta,

$$\overline{B}(\chi, k^2, \Lambda^2) = -\tfrac{1}{6}\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+\chi)[(k+p)^2+\chi]}$$
$$= -\frac{1}{96\pi^2}\left\{\ln\left(\frac{\Lambda^2}{\chi}\right) + 1 - 2\left(\frac{k^2+4\chi}{k^2}\right)^{1/2}\ln\left[\frac{(k^2+4\chi)^{1/2}+\sqrt{k^2}}{2\sqrt{\chi}}\right]\right\}. \quad (4.3)$$

The renormalization is accomplished by substituting Eqs. (2.4) and (2.5) into (4.2), providing us with the renormalized propagator

$$\underline{D}^{-1}(-k^2, \phi, \chi) = \begin{bmatrix} (k^2+\chi)\delta_{ab} & \phi_a \\ \phi_b & -3N\left[\frac{1}{\lambda} - B(\chi, k^2, M^2)\right] \end{bmatrix}. \quad (4.4)$$

It turns out that the finite functions in (4.4) can be represented as

$$B(\chi, k^2, M^2) = \overline{B}(\chi, k^2, \Lambda^2) - \overline{B}(M^2, 0, \Lambda^2). \quad (4.5)$$

For convenience we define a new function $f(s, \chi)$ by the relation

$$B(\chi, k^2, M^2) = B(\chi, 0, M^2) + [f(-k^2, \chi) - f(0, \chi)], \quad (4.6)$$

where

$$B(\chi, 0, M^2) = \frac{1}{96\pi^2}\ln\left(\frac{\chi}{M^2}\right) \quad (4.7)$$

and

$$f(-k^2, \chi) = \frac{1}{48\pi^2}\left(\frac{k^2+4\chi}{k^2}\right)^{1/2} \times \ln\left[\frac{(k^2+4\chi)^{1/2}+\sqrt{k^2}}{2\sqrt{\chi}}\right] \quad (4.8)$$

for $k^2 > 0$. From Eqs. (2.6), (2.8), (4.4)–(4.8), we write $D^{-1}_{\chi\chi}(-k^2, \phi, \chi)$ as a renormalization invariant,

$$-\frac{1}{3N}D^{-1}_{\chi\chi}(-k^2, \chi) = -\frac{1}{96\pi^2}[1 + \ln\rho(\phi^2)] + [f(0, \chi) - f(-k^2, \chi)]. \quad (4.9)$$

The analytic function $f(s, \chi)$, characteristic of two-particle unitarity for mesons of mass $m = \sqrt{\chi}$, has the following properties for $k^2 > 0, \chi \neq 0$:

(a) $f(-k^2, \chi) > 0$,

(b) $f(0, \chi) = \frac{1}{48\pi^2}$, (4.10)

(c) $f(-k^2, \chi)$ increases monotonically for increasing $k^2 > 0$.

A. Absence of tachyons

It is known[2] that if the Green's functions are evaluated relative to the vacuum defined by branch I of the gap equation, tachyons will always be present, indicating that the ground state of the theory cannot be found on branch I. We now demonstrate[5] that the Green's functions are free of tachyons if they are computed relative to branch II of $V(\phi^2)$, so that the ground state of the theory is attained on this branch. Since

$$0 \geq [f(0, \chi) - f(-k^2, \chi)] > -\infty \text{ for } k^2 \geq 0, \quad (4.11)$$

tachyons will be absent from $D^{-1}_{\chi\chi}$ if

$$\frac{1}{\lambda} - B(\chi, 0, M^2) < 0, \quad (4.12)$$

i.e., if

$$[1 + \ln\rho(\phi^2)] > 0, \quad (4.13)$$

since $D^{-1}_{\chi\chi}(k^2)$ will be free of zeros for Euclidean $k^2 \geq 0$. Since $\rho_{\rm I}(\phi^2) < e^{-1}$, we find tachyons throughout branch I. Recall that the ground state occurs on branch II at $\phi_a = 0$, so that (4.4) is diagonal when computed relative to this vacuum. Combining the definition of branch II, Eq. (2.20),

$$\rho_{\rm II}(\phi^2) > e^{-1}, \quad (4.14)$$

with (4.13), we see that $D^{-1}_{\chi\chi}(-k^2, \chi)$ is free of tachyons for the *entire* range of branch II.

To summarize the results of this section, we have shown that tachyons are absent from the Green's functions *if* the Green's functions are constructed relative to the ground state of the theory. The previously discovered tachyons were symptomatic of constructing Green's functions relative to the wrong vacuum state, and are not a signal of a fundamental failure of the theory. Note that tachyons have been removed by finding the ground state of the theory in leading order, and not by appealing to higher-order corrections in $1/N$.

V. PARTICLES SPECTRUM

The ground state of the theory occurs at $\phi^2 = 0$, on branch II of $V(\phi^2)$. From (4.4), we find the elementary meson mass

$$m^2 = \chi_{\rm II}(0) > e^{-1}\chi_0, \quad (5.1)$$

which is renormalization invariant, as it must be. Since

$$\frac{\delta^3\Gamma}{\delta\phi_a(x)\delta\phi_b(y)\delta\chi(z)} = -\delta_{ab}\delta^4(x-y)\delta^4(y-z), \quad (5.2)$$

the meson-meson scattering amplitude is

$$T_{ab,cd}(s,t,u) = \delta_{ab}\delta_{cd}D_{\chi\chi}(s,m^2) + \text{crossed terms}, \quad (5.3)$$

where s,t,u are the usual Mandelstam variables. Thus, the singularities of $D_{\chi\chi}(s,m^2)$ are physically observable, renormalization invariant, and free of tachyons. Whatever bound states and resonances are present in the theory emerge from the (timelike) zeros of $D_{\chi\chi}^{-1}$ These dynamically generated states will all belong to the identity representation of the O(N) group. The χ propagator in the timelike (Minkowski) region, obtained from analytic continuation of (4.9), is

$$-\frac{1}{3N}D_{\chi\chi}^{-1}(s,m^2) = -\frac{1}{96\pi^2}\left[1+\ln\left(\frac{m^2}{\chi_0}\right)\right] + [f(0,m^2)-f(s,m^2)] \quad (5.4a)$$

$$= -\frac{1}{96\pi^2}[1+\ln\rho_{II}(0)] + [f(0,m^2)-f(s,m^2)]. \quad (5.4b)$$

By analytic continuation of (4.8) one obtains the representation

$$f(s,m^2) = \frac{1}{96\pi^2}\left[-2\left(\frac{4m^2-s}{s}\right)^{1/2}\arctan\left(\frac{4m^2-s}{s}\right)^{1/2} + \pi\left(\frac{4m^2-s}{s}\right)^{1/2}\right] \text{ for } 0\le s\le 4m^2 \quad (5.5)$$

and

$$f(s,m^2) = \frac{1}{96\pi^2}\left\{2\left(\frac{s-4m^2}{s}\right)^{1/2}\ln\left[\frac{\sqrt{s}+(s-4m^2)^{1/2}}{2m}\right] - i\pi\left(\frac{s-4m^2}{s}\right)^{1/2}\right\} \text{ for } s\ge 4m^2 \quad (5.6)$$

with the limiting cases

$$f(s,m^2) \underset{s\to 4m_-^2}{\sim} \frac{1}{96\pi^2}\left(\frac{1}{2m}\right)[\pi(4m^2-s)^{1/2} - (4m^2-s) + \cdots], \quad (5.7a)$$

$$\underset{s\to 4m_+^2}{\sim} \frac{1}{96\pi^2}\left(\frac{1}{2m}\right)[-i\pi(s-4m^2)^{1/2} + (s-4m^2) + \cdots], \quad (5.7b)$$

and

$$f(0,m^2) = \frac{1}{48\pi^2}. \quad (5.7c)$$

Two trivial, but useful, limits are

$$[f(0,m^2) - f(4m^2,m^2)] = \frac{1}{48\pi^2} \quad (5.8a)$$

and

$$[f(0,m^2)-f(s,m^2)] \underset{s\to+\infty}{\sim} -\frac{1}{96\pi^2}(\ln s - i\pi). \quad (5.8b)$$

We plot $\text{Re}[f(0,m^2)-f(s,m^2)]$ versus s in Fig. 5, which helps us visualize the behavior of $D_{\chi\chi}^{-1}(s,m^2)$.

A. Criterion for bound states

From (4.13) we have

$$-[1+\ln\rho_{II}(0)] < 0, \quad (5.9)$$

while

$$-\infty < \text{Re}[f(0,m^2)-f(s,m^2)] \le \frac{1}{48\pi^2} \quad (5.10)$$

for $0\le s\le\infty$, with the upper limit attained for $s=4m^2$. Comparing (5.9) with (5.10), and con-

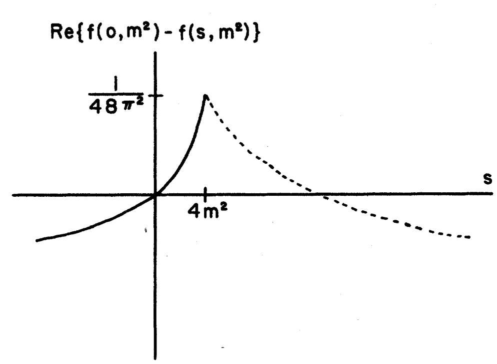

FIG. 5. A graph of Re[$f(0,m^2)-f(s,m^2)$] versus s, where $f(s,m^2)$ is given in Eqs. (4.8), and (5.5)–(5.8). A dashed line indicates where the function is complex.

sulting Eq. (5.4) and Fig. 5, we see that there will be

(1) a bound state *and* a resonance *if*

$$[f(0,m^2)-f(4m^2,m^2)]=\frac{1}{48\pi^2}\geq\frac{1}{96\pi^2}[1+\ln\rho_{II}(0)]>0 \qquad (5.11)$$

and

(2) neither a bound state nor a resonance *if*

$$\frac{1}{96\pi^2}[1+\ln\rho_{II}(0)]>\frac{1}{48\pi^2}. \qquad (5.12)$$

Combining (5.13) with (2.10), (2.21), and (5.1), there will be a bound state if

$$\ln\rho_{II}(0)<1, \qquad (5.13a)$$

i.e.,

$$\rho_{II}(0)<e. \qquad (5.13b)$$

Thus

$$\rho_{II}(0)\ln\rho_{II}(0)<e \qquad (5.14)$$

is also required. However, recall that branch II is characterized by

$$\rho_{II}(0)>e^{-1}. \qquad (5.15)$$

Considering (2.22) with (5.14)

$$e^{-1}>\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right)>-e \qquad (5.16)$$

describes the range of parameters which gives rise to a bound state and a resonance, where the upper limit in (5.16) keeps us within the allowed domain defined by (3.12). If the upper limit in (5.16) is exceeded, the binding energy of the bound state exceeds $2m$, producing a tachyon, and an effective potential which is everywhere complex. We plot in Fig. 6 the domains defined by Eqs. (3.14) and (5.16).

The following results are immediate:

(1) For $\mu^2/g=0$, χ_0 finite, there is always a bound state and a resonance, and a *finite* meson mass m.

(2) For $\mu^2/g>0$, with χ_0 large enough to satisfy (5.16), there is always a bound state and a resonance.

(3) The case $(\mu^2/g)<0$ gives either

(i) a bound state and resonance if

$$\frac{\chi_0}{96\pi^2}\geq-e^{-1}(\mu^2/g) \qquad (5.17)$$

or

(ii) neither a bound state nor a resonance if

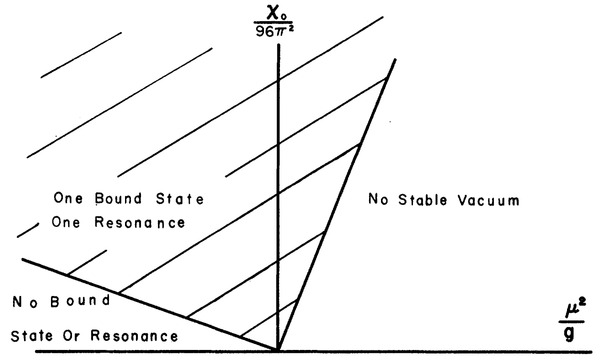

FIG. 6. A graph of $z=\chi_0/96\pi^2$ versus $y=\mu^2/g$. If (a) $y>e^{-1}z$ there is no stable ground state, (b) $-ez<y<e^{-1}z$ there is a bound state and a resonance, (c) $y<-ez$ there are no bound states. Only $z>0$ is relevant throughout.

$$\frac{\chi_0}{96\pi^2}<-e^{-1}\frac{\mu^2}{g}\ . \qquad (5.18)$$

The S matrix of the model is obtained from the renormalized version of (4.1) by relating one-particle irreducible (1PI) Green's functions to the connected Green's functions. Treelike structures result, but with clothed propagators and vertices. Notice that when bound states are present, one should consider *external* χ lines in the *on-shell* S-matrix elements because of the bound-state structure present in the χ propagator. There are nontrivial 1PI χ n-point functions due to the last term in Eq. (4.1), which can be interpreted as n-sided polygonal Feynman diagrams with n internal ϕ lines for the elementary mesons of mass m, together with n external χ lines. Note that to leading order in N, no additional bound states or resonances are produced by this construction. Thus, the spectrum of states is quite sparse compared to what one desires for a realistic theory.

B. Zero-mass theory

Let us consider the zero-mass theory $m^2=0$, which requires $\chi_0=0$, $(\mu^2/g)<0$ and fixed. This limit entails $\rho_{II}(0)\to+\infty$, but from (2.22)

$$m^2=\chi_0\rho_{II}(0)=-96\pi^2\left(\frac{\mu^2}{g}\right)\frac{1}{\ln\rho_{II}(0)}\xrightarrow[\chi_0\to 0]{}0, \qquad (5.19)$$

so that $m^2=0$ can be achieved. Then

$$\lim_{m^2\to 0}D_{\chi\chi}^{-1}(s,m^2)=\frac{-N}{32\pi^2}\lim_{m^2\to 0}[1-\ln\rho_{II}(0)-96\pi^2f(s,m^2)], \qquad (5.20a)$$

so that

$$D_{\chi\chi}(s,m^2) \underset{m^2\to 0}{\sim} -\left(\frac{1}{3N}\right)\left(\frac{\mu^2}{g}\right)^{-1} m^2, \tag{5.20b}$$

which means the four-point meson-meson scattering amplitude vanishes like m^2. [From (5.18) we see that we have no bound states present.] Now consider the $2n$-point *connected* meson Green's functions obtained from the renormalized version of (4.1). These involve tree diagrams with dressed ϕ and χ propagators, as well as diagrams containing a single closed ϕ loop diagram. From (5.20b) we may write the zero-mass limit of the connected $2n$-point meson Green's function, $G^{(2n)}_{\text{conn}}(m^2)$, in the schematic fashion

$$G^{(2n)}_{\text{conn}}(m^2) \underset{m^2\to 0}{\cong} (m^2)^{n-1} G^{(2n)}_{\text{tree}} + (m^2)^n G^{(2n)}_{\text{1-loop}}, \tag{5.21}$$

where $G^{(2n)}_{\text{tree}}$ ($G^{(2n)}_{\text{1-loop}}$) is the $2n$-point connected Green's function evaluated in *ordinary* perturbation theory in the tree (one-loop) approximation to leading order in N.

Because of the over-all factors of m^2, the zero-mass limit of our O(N) model is more infrared convergent than massless ϕ^4 theory, evaluated in perturbation theory. It is known that the Green's functions of massless ϕ^4 theory are infrared finite in any finite order of perturbation theory if exceptional momenta are avoided. Thus here

$$G^{(2n)}_{\text{conn}}(m^2) \underset{m^2\to 0}{\cong} 0 \text{ for } n \geq 2.$$

Therefore, the zero-mass theory in the large-N limit is a free-field theory. [If $\mu^2/g = 0$ first *and* then $\chi_0 = 0$, $D_{\chi\chi}^{-1} \underset{m^2\to 0}{\cong} \ln(s/m^2)$, which replaces m^2 by $(1/\ln m^2)$ in Eq. (5.21). Nevertheless, the argument still goes through.]

This result is anticipated from the Federbush-Johnson theorem,[6] since the meson propagator behaves as a free field, and the four-point function vanishes. Nevertheless, it is interesting to see how the theorem is implemented in detail. Thus the limit $m^2 \to 0$ reduces the model to a free-field theory in leading order. The other possible massless limit, $\mu^2/g \to 0$, considered in Eq. (5.16) ff., leads to a *finite* meson mass m by dimensional transmutation, since $\chi_0 \neq 0$. Thus, a zero-mass, interacting "charged" boson cannot be sustained in this theory, in accord with conventional wisdom. This result is "natural" in the technical sense of being valid for arbitrary allowed parameter sets $(\chi_0, \mu^2/g)$.

C. Mass of the bound state

We now compute the mass of the bound state and resonance masses, as obtained from the zeros of $D_{\chi\chi}^{-1}(s,m^2)$ for timelike s. It will be shown that the mass of the bound state varies continuously from $m_B^2 = 4m^2$ to $m_B^2 = 0$, as one sweeps over the bound-state region of Fig. 6 from left to right. From (5.4b)

$$D_{\chi\chi}^{-1}(m_B^2, m^2) = 0 \tag{5.22}$$

implies

$$[1 + \ln\rho_{II}(0)] = 96\pi^2[f(0,m^2) - f(m_B^2, m^2)]. \tag{5.23}$$

1. Strong binding

Recall from (2.22) that

$$\rho_{II}(0)\ln\rho_{II}(0) = -\left(\frac{96\pi^2}{\chi_0}\right)\left(\frac{\mu^2}{g}\right), \tag{5.24}$$

so that

$$\rho_{II}(0) = e^{-1} \tag{5.25a}$$

corresponds to

$$m_B^2 = 0 \tag{5.25b}$$

and

$$\left(\frac{\chi_0}{96\pi^2}\right) = e\left(\frac{\mu^2}{g}\right). \tag{5.25c}$$

Thus a zero-mass bound state occurs for parameters corresponding to the extreme-right edge of the bound-state region of Fig. 6. The resonance mass for this limit is obtained from

$$\text{Re} D_{\chi\chi}^{-1}(s_R, m^2) = 0, \tag{5.26}$$

which implies

$$\text{Re} f(s_R, m^2) \simeq 1/48\pi^2 \tag{5.27a}$$

and

$$\text{Re}\, s_R \simeq 13.5 m^2 \simeq 3.4(4m^2). \tag{5.27b}$$

Similarly

$$\text{Im} D_{\chi\chi}^{-1}(s_R, m^2) \simeq -0.06N. \tag{5.28}$$

From Eqs. (2.17b) and (5.25c), we have

$$(\phi_b^2/N) = 0 \tag{5.29}$$

in this limit, which means that the effective potential becomes everywhere complex just when the bound-state mass $m_B^2 = 0$. If

$$\frac{\chi_0}{96\pi^2} < e(\mu^2/g),$$

$m_B^2 < 0$, which shows that when the branch point reaches $\phi_b^2 = 0$, one obtains the limiting case in which *unavoidable* tachyons first appear in the theory, spoiling the stability of the vacuum.

2. *Weak binding*

If

$$\rho_{II}(0)=e, \tag{5.30a}$$

then

$$\frac{\chi_0}{96\pi^2}=-e^{-1}\left(\frac{\mu^2}{g}\right), \tag{5.30b}$$

which implies

$$m_R{}^2=m_B{}^2=4m^2. \tag{5.30c}$$

This limit corresponds to the left-hand edge of the bound-state region of Fig. 6. Then

$$(\phi_b{}^2/N)=\frac{1}{16\pi^2}(e+e^{-1})\chi_0. \tag{5.31}$$

Let us also consider the case

$$(\mu^2/g)=0, \tag{5.32a}$$

which gives, according to (5.24),

$$\rho_{II}(0)=1$$

and

$$m^2=\chi_0. \tag{5.32b}$$

Then from (5.22)

$$1-2\left(\frac{4m^2-m_B{}^2}{m_B{}^2}\right)^{1/2}\tan^{-1}\left(\frac{m_B{}^2}{4m^2-m_B{}^2}\right)^{1/2}=0, \tag{5.33}$$

with the result

$$m_B{}^2\simeq 0.845(4m^2). \tag{5.34}$$

Similarly

$$\mathrm{Res}_R\simeq 1.9(4m^2). \tag{5.35}$$

Therefore, even in the middle of the bound-state region of Fig. 6,

$$(4m^2-m_B{}^2)/4m^2\ll 1. \tag{5.36}$$

Equation (5.32a) combined with (2.17b) sets the branch point at

$$(\phi_b{}^2/N)=\frac{e^{-1}}{16\pi^2}\chi_0=\frac{e^{-1}}{16\pi^2}m^2. \tag{5.37}$$

D. Dimensionless coupling parameters

Since both $g(M)$ and $\lambda(M)$ depend on the renormalization mass M, neither is well suited for a description of the *intrinsic* interaction strength. We propose an (*ad hoc*) definition, which interpolates between the various limits discussed in this paper. Let us define the renormalization-invariant, dimensionless parameter

$$\hat{g}=\frac{-(\chi_0/96\pi^2)(g/\mu^2)}{e-(\chi_0/96\pi^2)(g/\mu^2)} \tag{5.38a}$$

$$=\frac{g/96\pi^2}{(g/96\pi^2)-(\mu^2/\chi_b)}, \tag{5.38b}$$

where χ_b is defined by (2.18). Then,

$$\hat{g}=0:\ \text{free field}, \tag{5.39a}$$

$$0<\hat{g}\le\frac{1}{1+e^2}<1:\ \text{weak-coupling with no-bound states}, \tag{5.39b}$$

$$\frac{1}{1+e^2}\le\hat{g}\le\infty:\ \text{strong-coupling with bound state}, \tag{5.39c}$$

$$\hat{g}<0:\ \text{no ground state} \tag{5.39d}$$

relate the various domains appropriate to this parameter. Thus all $\hat{g}>0$ is allowed, with increasing $\hat{g}$ corresponding to one's naive expectations for a coupling parameter. Unfortunately we have found no operational definition of $\hat{g}$ aside from (5.39).

VI. EFFECTIVE POTENTIAL TO ORDER $1/N$

A formal expression for the $1/N$ correction to the effective potential has been given by Root.[3] Restricting his analysis to branch I of the gap equation (in the sense of this paper), he showed that branch I becomes everywhere complex in next-to-leading order, a result directly traceable to the presence of tachyons in the Green's functions computed to leading order. Since we know from Sec. IV that tachyons are absent if the Green's functions are computed relative to the branch II it becomes essential to reexamine Root's results[3] in the light of our finding.

An expression for $\partial V/\partial\phi^2$ accurate to next-to-leading order in $1/N$ is given by Root in Eq. (3.37) of Ref. 3. (His B corresponds to our λB.) The actual expression, which involves a sum of a number of complicated terms, will not be required here. It suffices to note that *each* of these terms is given by a *Euclidean* integral of the form

$$I(\chi,\phi^2;a,b,c,d)=\int\frac{d^4k}{(2\pi)^4}[\det\underline{D}(-k^2,\phi,\chi)]^a\frac{1}{(k^2+\chi)^b}\frac{1}{[1/\lambda-B(\chi,k^2,M^2)]^c}\left[\frac{\partial B(x,k^2)}{\partial\chi}\right]^d, \tag{6.1}$$

where a, b, c, and d are zero or positive integers, χ satisfies the gap equation, and from (4.4)

$$-3N\det \underline{D}(-k^2,\phi,\chi)=\left\{(k^2+\chi)\left[\frac{1}{\lambda}-B(\chi,k^2,M^2)\right]+\frac{\phi^2}{3N}\right\}^{-1}. \tag{6.2}$$

In order for $\partial V/\partial\phi^2$ and $V(\phi^2)$ to be real, each of the terms of the form (6.1) should be separately real. [We conservatively assume that there are no favorable cancellations among the sum of terms which contribute to (6.1).] Since the integration in (6.1) is over Euclidean momenta, $\partial V/\partial\phi^2$ will become complex only when a singularity appears in the integrand of (6.1). We may forget about the convergence of (6.1), which is taken care of by renormalization, while infrared problems are avoided exactly as in leading order, by introduction of the renormalization mass M^2.

Since

$$\frac{\partial B(\chi,k^2,M^2)}{\partial\chi}=\frac{\partial B(\chi,k^2)}{\partial\chi}$$

$$=\frac{1}{6}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+\chi)[(k+p)^2+\chi]}\left\{\frac{1}{(p^2+\chi)}+\frac{1}{[(k+p)^2+\chi]}\right\}, \tag{6.3}$$

the term $\partial B(\chi,k^2)/\partial\chi$ is never singular for $0\le k^2\le\infty$. Similarly, $1/(k^2+\chi)^b$ is regular throughout the integration domain for $\chi(\phi^2)$ real and positive, which is true throughout branch II, with $0\le\phi^2\le\phi_b^2$.

The function

$$\left[\frac{1}{\lambda}-B(\chi,k^2,M^2)\right]=-\frac{1}{3N}D_{\chi\chi}^{-1}(k^2,\chi(\phi^2))\neq 0 \tag{6.4}$$

for $\rho(\phi^2)$ on branch II, and $0\le\phi^2<\phi_b^2$, as is evident from (2.20) and (4.4). That is, (6.4) has no zeros on branch II since tachyons are absent for all ϕ^2 on this branch.

Now from (6.4) and (4.4)

$$\det\underline{D}(-k^2,\phi,\chi)=-[D_{\chi\chi}^{-1}(k^2,\chi)(k^2+\chi)+\phi^2]^{-1}. \tag{6.5}$$

Since $(k^2+\chi)D_{\chi\chi}^{-1}(k^2,\chi)$ monotonically *decreases* for increasing (Euclidean) k^2, the smallest value of ϕ^2 for which (6.5) can vanish occurs at $k^2=0$. Thus

$$\det\underline{D}(0,\phi_p,\chi_p(\phi_p^2))=0 \tag{6.6}$$

defines the value ϕ_p^2 for which $\det\underline{D}$ has a pole at $k^2=0$. For convenience define

$$\Phi_p^2=\frac{16\pi^2}{\chi_0}\left(\frac{\phi_p^2}{N}\right) \tag{6.7}$$

so that Φ_p^2 satisfies the pair of transcendental equations

$$2\Phi_p^2=\rho_p(1+\ln\rho_p) \tag{6.8}$$

and

$$\Phi_p^2=-\frac{96\pi^2}{\chi_0}\left(\frac{\mu^2}{g}\right)-\rho_p\ln\rho_p. \tag{2.12}$$

We can combine (6.8) with (2.12) to obtain a single transcendental equation for Φ_p^2, i.e.,

$$\xi_p(1+3\ln\xi_p)=-2a, \tag{6.9}$$

where

$$\xi_p\equiv 3\Phi_p^2+a \tag{6.10}$$

and

$$a=\frac{96\pi^2}{\chi_0}(\mu^2/g). \tag{6.11}$$

The solution of (6.9) requires a numerical evaluation, so that we only present solutions of (6.9) for the special cases corresponding to the boundaries of the various domains of Fig. 6. Our discussion is restricted to branch II of the gap equation, since this is the only case of interest.

1. Case $a=e^{-1}$. This corresponds to the boundary between the bound-state and tachyon regions. The solution is

$$\xi_p=e^{-1}, \tag{6.12a}$$

or

$$\Phi_p^2=\phi_p^2=0. \tag{6.12b}$$

Thus the $1/N$ correction to the effective potential is also everywhere complex in this case.

2. Case $a=0$. This case, discussed in Eqs. (5.32)–(5.37), is the same as $(\mu^2/g)=0$, which is within the bound-state domain. The solution of (6.9) gives

$$\Phi_p^2=0.239, \tag{6.13a}$$

with

$$\phi_p^2/\phi_b^2=0.65 \tag{6.13b}$$

and

$$\phi_b^2/N=\frac{e^{-1}}{16\pi^2}\chi_0$$

$$=\frac{e^{-1}}{16\pi^2}m^2. \tag{5.37}$$

3. Case $a=-e$. This corresponds to the bound-

ary between the bound-state and weak-coupling regions. Here

$$\Phi_p{}^2 = 1.5, \tag{6.13a}$$

with

$$\phi_p{}^2/\phi_b{}^2 = 0.5 \tag{6.13b}$$

and

$$\phi_b{}^2/N = \frac{\chi_0}{16\pi^2}(e+e^{-1}). \tag{5.31}$$

4. Case $a \to -\infty$. This corresponds to the zero-mass or free-field limit. Here both

$$\phi_p{}^2 \to \infty \tag{6.14a}$$

and

$$\phi_b{}^2 \to \infty, \tag{6.14b}$$

but with

$$\phi_p{}^2/\phi_b{}^2 < 1. \tag{6.14c}$$

It is obvious from the above discussion, and from examination of (6.9)–(6.11), that

$$0 < \phi_p{}^2 < \phi_b{}^2 \text{ for } a < e^{-1}, \tag{6.15}$$

with the effective potential restricted to branch II. Thus, we conclude that there *always* exists a *finite*, *real* region in the neighborhood of $\phi^2 = 0$ for the effective potential evaluated to the first two orders in $1/N$, and restricted to branch II. This real region extends over the range

$$0 \le \phi^2 \le \phi_p{}^2. \tag{6.16}$$

VII. DOMAIN OF VALIDITY

The $1/N$ expansion is initially defined in terms of (unrenormalized) vacuum graphs,[1] organized according to the over-all power of N multiplying each graph, with an infinite number of graphs present in each order of N. Once the theory has been renormalized the connection with Feynman graphs is less immediate, particularly because of the presence of bound states in the renormalized theory. Since the vacuum is not expected to go smoothly into the vacuum of the loop expansion,[8] one cannot use the conventional criteria for establishing the domain of validity of the $1/N$ expansion when bound states are present. Further, any criterion we do set should be independent of the renormalization mass, as this is only an artifact of a particular renormalization convention.

It seems to us that a minimal requirement is that the $1/N$ correction to the effective potential be small compared to the leading-order contribution. Recall from (3.2) that the effective potential can be written in renormalization-invariant form as

$$N^{-1}V(\phi^2,\chi) = 3\chi_0\rho\left(\frac{\mu^2}{g}\right)\left[1+\frac{1}{6}\left(\frac{g}{\mu^2}\right)\left(\frac{\phi^2}{N}\right) + \left(\frac{g}{\mu^2}\right)\left(\frac{\chi_0}{384\pi^2}\right)\rho(2\ln\rho-1)\right], \tag{7.1}$$

where the quantity inside the square brackets is dimensionless, and the quantity outside sets the mass scale of the effective potential.

Since we do not have available an explicit expression for the $1/N$ corrections, we substitute a simpler criterion, i.e., that the quantity inside the curly bracket in (7.1) not be of order N, term by term. One then obtains the conditions

$$\frac{1}{N} \ll 1, \tag{7.2}$$

$$\frac{1}{6}\left|\frac{g}{\mu^2}\right|\left(\frac{\phi^2}{N}\right) \ll N, \tag{7.3}$$

and

$$\frac{1}{N}\left|\frac{g}{\mu^2}\right|\left(\frac{\chi_0}{384\pi^2}\right)\rho(\phi^2)[2\ln\rho(\phi^2)-1] \ll 1. \tag{7.4}$$

Equations (7.2)–(7.4) fulfill our naive expectations for limits on the allowed domain of the $1/N$ expansion. If one uses the gap equation, (7.4) reduces to (7.2) and (7.3). Note that

$$-\left(\frac{g}{\mu^2}\right)\frac{\chi_0}{96\pi^2}\rho(0)\ln\rho(0) = 1, \tag{7.5}$$

so that these inequalities are always satisfied at $\phi^2 = 0$.

Let us ask if the branch point at $\phi_b{}^2$ is inside the domain of validity of the expansion.

From (2.17)

$$\frac{1}{6}\left(\frac{g}{\mu^2}\right)\left(\frac{\phi_b{}^2}{N}\right) = -1 + e^{-1}\left(\frac{\chi_0}{96\pi^2}\right)\left(\frac{g}{\mu^2}\right), \tag{7.6}$$

which satisfied (7.3) *except* for $(\mu^2/g) \to 0$. However, from (5.32b) and (5.19)

$$\lim_{\mu^2/g\to 0} N^{-1}V(\phi^2,\chi) = \frac{m^4\rho(\phi^2)}{128\pi^2}\left\{\frac{64\pi^2}{m^2}(\phi^2/N) + \rho(\phi^2)[2\ln\rho(\phi^2)-1]\right\}, \tag{7.7}$$

which indicates that the criteria (7.2)–(7.4) ought to be modified, since $\chi_0(\mu^2/g)$ no longer sets the mass scale of $V(\phi^2)$ in this case. We speculate that (7.2)–(7.4) should be replaced by

$$\frac{64\pi^2}{m^2}\left(\frac{\phi^2}{N}\right) \ll N \tag{7.8}$$

and

$$\rho(2\ln\rho - 1) \ll N \tag{7.9}$$

when $(\mu^2/g)\simeq 0$. Note that $(\mu^2/g)=0$ means

$$\frac{64\pi^2}{m^2}\left(\frac{\phi_b{}^2}{N}\right)=4e^{-1}. \tag{7.10}$$

Therefore, from (7.6), or (7.10) when applicable, we find that $\phi_b{}^2/N$ always satisfies our criteria. We interpret this to mean that the branch point at $\phi_b{}^2$ always lies within the domain of the $1/N$ expansion, and is a true feature of the O(N) model when N is large [cf. (7.2)]. Although we are not as certain of these estimates as the results given in earlier sections, since they are based on some speculation, the conclusions reached seem entirely reasonable.

Finally we consider what happens when $\mu^2/g\to-\infty$, which is the weak-coupling limit. For $\mu^2/g\to-\infty$, and $\Phi_b{}^2\ll 1$, with $\Phi_b{}^2$ defined as in (6.7), it is straightforward to show that

$$V(\phi^2)\simeq\tfrac{1}{2}\mu^2\phi^2+\frac{1}{4!}\frac{g}{N}\phi^4 \tag{7.11}$$

if one is on branch II. Thus, as $\mu^2/g\to-\infty$, branch II goes to the perturbation limit for *small* ϕ^2. However, $g<0$ cannot be excluded, as the infinite number of Feynman diagrams of leading order in $1/N$ provide stability of $V(\phi^2)$ for $\phi^2<\phi_b$, as seen in Fig. 7.

VIII. LARGE-ϕ^2 BEHAVIOR

In Sec. II we showed that the effective potential computed to leading order in N^{-1} behaves as

$$\text{Re}\frac{dV(\phi^2)}{d\phi^2}\underset{\phi^2\to\infty}{\sim}\frac{-8\pi^2(\phi^2/N)}{\ln(\phi^2/M^2)} \tag{8.1}$$

independent of the parameters of the theory, and the particular branch of the gap equation chosen

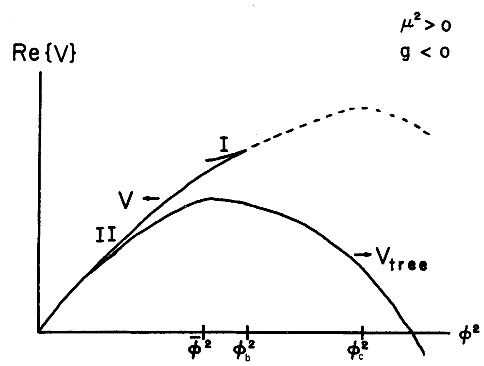

FIG. 7. A graph of Re$V(\phi^2)$ versus ϕ^2 in the weak-coupling limit $\mu^2/g\to-\infty$ to leading order in N. This is compared with $V(\phi^2)$ of the tree approximation for $\mu^2>0$. and $g<0$. Note how the infinite number of Feynman graphs appropriate to the $1/N$ expansion "stabilizes" the effective potential even though $g<0$. A dashed line indicates where the function is complex.

to describe the small-ϕ^2 behavior. Here we argue that the $1/N$ correction to (8.1) increases as $\phi^2\to\infty$, and in fact dominates (8.1) for sufficiently large ϕ^2. Thus, the large-ϕ^2 behavior of the effective potential cannot be reliably calculated by the $1/N$ expansion, since the correction terms to (8.1) are as large or larger than this leading-order result. Thus (8.1) is not a definitive prediction of the model, since even the qualitative conclusions of (8.1) are not to be trusted.

In principle the large-ϕ^2 behavior of the $1/N$ correction to $V(\phi^2)$ can be extracted from Root's expression[3] for $dV(\phi^2)/d\phi^2$ [his Eq. (3.37)], however, this is a very tedious task. However, for sufficiently large ϕ^2 we may use the homogeneous Callan-Symanzik equation to calculate $V(\phi^2)$ in this limit if we know $\beta(\lambda)$ and $\gamma(\lambda)$ to sufficient accuracy. It is obvious that $\beta(\lambda)$, computed in the usual one-loop approximation, coincides with $\beta(\lambda)$ as computed from (2.5) and (2.6). Since there is no wave-function renormalization in leading order, $\gamma(\lambda)=0$ in this approximation. On the basis of this result, we conjecture that $\beta(\lambda)$ and $\gamma(\lambda)$, computed from the first two terms of the loop expansion, and then expanded to the first two orders in $1/N$, are the appropriate functions to be used with the homogeneous Callan-Symanzik for the solution of our problem.

The homogeneous Callan-Symanzik equation for the effective potential is[7]

$$\left[M\frac{\partial}{\partial M}+N\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma(\lambda)\phi^2\frac{\partial}{\partial\phi^2}\right]V(\phi^2,M^2,\lambda)=0 \tag{8.2}$$

for $\phi^2\gg m^2$, the meson mass. A calculation of $\beta(\lambda)$ and $\gamma(\lambda)$ in the two-loop approximation gives[9]

$$(48\pi^2)\beta(\lambda)=\frac{\lambda^2}{3N^2}\left[(N+8)-\frac{\lambda}{48\pi^2}\left(\frac{9N+42}{N}\right)\right] \tag{8.3}$$

and

$$(48\pi^2)^2\gamma(\lambda)=-\tfrac{1}{2}\lambda^2\frac{(N+2)}{N^2}, \tag{8.4}$$

where of course λ depends implicitly on the renormalization mass M^2. Notice that $\beta(\lambda)$ has a nontrivial ultraviolet zero, since

$$\beta(\lambda^*)=0$$

for

$$\frac{\lambda^*}{48\pi^2}=\frac{N(N+8)}{9N+42}. \tag{8.5}$$

Define the dimensionless quantity $v(\phi^2/M^2,\lambda)$ by

$$V(\phi^2,M^2,\lambda)=\phi^4 v(\phi^2/M^2,\lambda). \tag{8.6}$$

Then the solution to (8.2) is[7]

$$v(t,\lambda)=\frac{\lambda'(t,\lambda)}{4!N}\exp\left[2\int_0^t dt'\bar{\gamma}(\lambda'(t',\lambda))\right], \quad (8.7)$$

where

$$t=\ln(\phi^2/M^2), \quad (8.8)$$

$$N^{-1}\frac{d\lambda'(t,\lambda)}{dt}=\bar{\beta}(\lambda'(t,\lambda)), \quad (8.9)$$

$$\lambda'(0,\lambda)=\lambda, \quad (8.10)$$

$$v(0,\lambda)=\frac{\lambda}{4!N}, \quad (8.11)$$

$$[2-\gamma(\lambda)]\bar{\beta}(\lambda)=\beta(\lambda), \quad (8.12)$$

and

$$[2-\gamma(\lambda)]\bar{\gamma}(\lambda)=\gamma(\lambda). \quad (8.13)$$

It is convenient to write

$$\beta(\lambda)=a\lambda^2(1-b\lambda) \quad (8.14)$$

and

$$\gamma(\lambda)=-2c\lambda^2 \quad (8.15)$$

with the (positive) coefficients a, b, and c given by (8.3) and (8.4). Then (8.7)–(8.15) imply

$$v(t,\lambda)=\frac{\lambda'(t)}{4!N}\left[\frac{1-b\lambda'(t)}{1-b\lambda}\right]^{(4/N)(c/ab)}, \quad (8.16)$$

where

$$\frac{Nat}{2}=\left[\frac{1}{\lambda}-\frac{1}{\lambda'(t)}\right]+b\ln\left[\frac{\lambda'(t)}{\lambda}\right] -\left(\frac{b^2+c}{b}\right)\ln\left[\frac{1-b\lambda'(t)}{1-b\lambda}\right]. \quad (8.17)$$

Notice that when b and $c\to 0$, we recover the one-loop result as well as the prediction (8.1) for $V(\phi^2)$. However, (8.16) is qualitatively different from the lowest-order result (8.1), primarily as a result of the nontrivial ultraviolet zero of $\beta(\lambda)$.

In order to exhibit the asymptotic behavior of $V(\phi^2)$, we consider $t\to\infty$ in (8.16) and (8.17) which drives $\lambda'(t)$ toward ultraviolet zero $\lambda=1/b$. [We only consider $\lambda>0$. See the paragraph that follows Eq. (2.7).] The result is

$$\frac{Nat}{2}\simeq-\left(\frac{b^2+c}{b}\right)\ln[1-b\lambda'(t,\lambda)]. \quad (8.18)$$

Thus, combining (8.18) with (8.6) and (8.16), we obtain for $\lambda<1/b$.

$$V(\phi^2)\underset{\phi^2\to\infty}{\simeq}\frac{1}{4!Nb}\left(\frac{1}{1-b\lambda}\right)^{(4/N)(c/ab)}\left(\frac{\phi^2}{M^2}\right)^{-2c/(b^2+c)}\phi^4 \quad (8.19)$$

$$\underset{\phi^2\to\infty}{\simeq}(\text{positive constant})(\phi^2)^{2\{36(3N+14)^2/[36(3N+14)^2+(N+2)(N+8)^2]\}}. \quad (8.20)$$

Since b^2 and c are the same order in the $1/N$ expansion, it is not appropriate to expand the exponent in (8.19) or (8.20). However, note that

$$V(\phi^2)\underset{\phi^2\to+\infty}{\simeq}(\phi^2)^2 \text{ for } N\ll 300,$$
$$V(\phi^2)\underset{\phi^2\to+\infty}{\simeq}(\phi^2)^{648/N} \text{ for } N\gg 300. \quad (8.21)$$

We conclude that (8.1) is not a stable prediction of the $1/N$ expansion. When the next to leading corrections are included, one obtains the qualitatively different result (8.19)–(8.21). This is not surprising, since we are dealing with values of ϕ^2 well outside the domain of validity of the $1/N$ expansion.

IX. CONCLUSIONS

Our detailed conclusions related to various technical issues have already been presented in the individual sections of this paper. Here we present a brief overview of our results so as to underline any lessons that can be extracted from our work.

The most important global conclusion to be drawn from our calculations is that the $1/N$ expansion appears to be a consistent approximation scheme. The tachyons characteristic of bubble sums can be removed by finding the correct vacuum state for the construction of Green's functions. However, the manner in which this comes about is rather surprising in that spontaneous symmetry breakdown cannot occur in the large-N limit, Goldstone phenomena are not possible, and the ground state of the theory is O(N)-symmetric. Since we believe in Goldstone phenomena for small N, there must exist for *critical* value of N above which spontaneous symmetry breakdown is not possible. Unfortunately, since we do not have an intuitive picture of this phenomenon, we cannot give an estimate of the critical value N_c. Certainly clarification of this issue would be valuable.

Another result of particular interest concerns the zero-mass limit. We found, to leading order in N, that a zero-mass, interacting "charged" boson cannot be sustained. If the physical boson mass goes to zero, the theory becomes a free-field theory. On the other hand, if the intermediate renormalized mass $\mu^2/g\to 0$, the elementary boson acquires a mass. These results are natural in the technical sense of being valid for arbitrary parameter sets in the theory. This is in accord with conventional wisdom.

Since the $1/N$ expansion appears to be consistent, it should prove to be a valuable tool for the understanding of the physical content of quantum field theories, particularly in view of the rich and unexpected phenomena already exposed.

Note added. It is interesting to note that the effective coupling constant becomes complex asymptotically since the effective potential $V(\phi^2)$ becomes complex for $\phi^2 > \phi_b{}^2$ (see Figs. 2, 3, and 4). Therefore the asymptotic coupling constant requires the study of the $\beta(\lambda)$ in the Callan-Symanzik equation for complex λ, as is done by Khuri for the simple $\lambda\phi^4$ theory.[10] However, it is simpler to calculate the effective coupling constant directly from Eq. (3.1), and we find that it becomes zero asymptotically. Nevertheless, this does not necessarily mean that the theory is asymptotically free as emphasized by Khuri.[10] The features we obtain are qualitatively similar although not identical to the case studied by Khuri. Were our expansion parameter the self-coupling constant λ_0, the vanishing effective coupling would imply that the expansion was a good one in the asymptotic region and the theory would be asymptotically free. However, here $1/N$ is our expansion parameter, and the small effective coupling constant does not mean that the leading term of the $1/N$ expansion predicts the correct large-ϕ^2 behavior. It simply means that the next-to-leading terms of the $1/N$ expansion are dominant in the asymptotic limit $\phi^2 \to \infty$.

Note added in proof. F. Cooper, G. S. Guralnik, and S. Kasdan [Phys. Rev. D (to be published)] study $\lambda\phi^4$ theory in the random-phase approximation. This is essentially the extrapolation of the O(N) model to $N = 1$. It is unlikely that this is valid, as the terms omitted would then be as large as those retained. See Sec. VII above.

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