Charge effects in a static, spherically symmetric, gravitating fluid

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A previously developed method for making the Einstein-Maxwell equations determinate for a static, multicomponent fluid also permits the calculation of the macroscopic electric field in such a body. As a practical application of these equations, a simple, two-component model for a white dwarf star has been constructed and its properties examined for arbitrary central densities. It is found that, for typical white dwarf densities, the internal electric field is quite small and other characteristics are almost identical to those calculated by the Chandrasekhar classical white dwarf model. For very large central densities, however, internal fields can be quite large, with a general correlation between large mass densities and large charge densities.

I. INTRODUCTION

In a previous paper[1] we developed a method for making the Einstein-Maxwell equations determinate for a spherically symmetric, multicomponent fluid in static equilibrium. In an attempt to determine the effects of these new equations on an actual physical system, we have constructed a relatively simple model for a two-component white dwarf star. Traditional models for the white dwarf are nonrelativistic and electrically neutral.[2] One argues that the quantity $GM(R)/Rc^2$, where G is the gravitational constant and $m(R)$ is the mass integral over the volume of the star, will be small compared to unity and that the relativistic effects will be negligible.[3] Also, although a small electric field is needed to support the pressureless nuclei against gravitational collapse,[4] the star is treated essentially in terms of only one charge component, where charge neutrality is assumed. With the complete system of equations, it is possible to calculate the individual particle distributions and, since the particles are charged, the electric field within the star. If the assumptions of the standard theory are valid, we cannot expect the charge and gravitational effects of this model to be large. Nevertheless, the exact radial dependence of the electric field in the white dwarf is not without interest.

In our search for large charge and gravitational effects, we have carried out calculations for central mass densities larger than those reasonable for a white dwarf. These are to be viewed as possible reference calculations, exact for a mixture of noninteracting and nonreacting charged Fermi and Bose gases at zero temperature and arbitrary density.

In Sec. II we develop the model for the charged, relativistic white dwarf star, and in Sec. III we show the assumptions needed to reduce this to the Chandrasekhar classical model. Section IV describes the calculations to be performed. In Sec. V we obtain numerical results for central mass densities typical of the standard white dwarf. We find that, with a few exceptions, these agree closely with the Chandrasekhar results. In this section we also calculate the electric field and charge density for a typical white dwarf as functions of the radial distance from the center of the star. Finally, in Sec. VI we explore the consequences of large central mass densities. Here, the deviations from the Chandrasekhar model are large, and some unexpected effects occur. Section VII summarizes our results.

II. DEVELOPMENT OF THE THEORY

We take the white dwarf star to be a static, spherically symmetric body of electrons and Fe^{56} nuclei in equilibrium at zero temperature. At pressures typical for this star, mean atomic distances are less than the Bohr orbit for the K-shell electrons, and bound electrons cannot exist.[5] The system is approximated by a noninteracting mixture of a perfect Bose gas of Fe^{56} nuclei and a perfect Fermi gas of electrons. More realistic expressions for the pressure and mass-energy density which incorporate particle interactions in some approximation have been used in previous white dwarf calculations.[6] Also, nuclei other than Fe^{56} are expected to be present at high central densitites.[6] We shall not attempt to take any of these corrections into account here. Although the formalism developed in Ref. 1 is capable of including these effects, the computational effort is formidable. We adopt the simplest possible equation of state (perfect gases) and look for charge imbalance and relativistic effects in the results.

Using quantum statistics, Chandrasekhar[7] has shown that, in the rest frame of a volume element at zero temperature, the pressure P_e and mass-energy density ρ_{m_e} of the electrons can be written in terms of their number density n_e as

$$P_e = \frac{\pi m_e^4 c^5}{h^3}\{(\tfrac{2}{3}\bar{n}_e - \bar{n}_e^{1/3})(\bar{n}_e^{2/3}+1)^{1/2} + \ln[\bar{n}_e^{1/3} + (\bar{n}_e^{2/3}+1)^{1/2}]\}, \quad (1)$$

$$\rho_{m_e} = \frac{\pi m_e^4 c^3}{h^3}\{(2\bar{n}_e + \bar{n}_e^{1/3})(\bar{n}_e^{2/3}+1)^{1/2} - \ln[\bar{n}_e^{1/3} + (\bar{n}_e^{2/3}+1)^{1/2}]\}, \quad (2)$$

where

$$\bar{n}_e = \frac{3h^3 n_e}{8\pi m_e^3 c^3}, \quad (3)$$

and m_e is the rest mass of the electron. The pressure P_N and mass-energy density ρ_{m_N} of the nuclei are given by

$$P_N = 0, \quad (4)$$

$$\rho_{m_N} = m_N n_N, \quad (5)$$

where m_N is the rest mass of a nucleus, and n_N is the nuclei number density.

Inherent in these equations is the assumption that any radial electric field present in the star varies slowly enough that the electric potential in a small volume element can be considered constant. It is simple to show that a constant electric potential in no way alters Eqs. (1)–(5).[4]

The gravitational metric tensor for a static, spherically symmetric body is given by[8]

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (6)$$

where e^{ν} and e^{λ} are functions of r only. The Einstein-Maxwell equations are

$$\frac{8\pi G}{c^4}T_0{}^0 = -\frac{G\mathcal{E}^2}{c^4 r^4} - \frac{8\pi G}{c^4}\rho_m c^2 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}, \quad (7)$$

$$\frac{8\pi G}{c^4}T_1{}^1 = -\frac{G\mathcal{E}^2}{c^4 r^4} + \frac{8\pi G}{c^4}P = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \quad (8)$$

$$\frac{8\pi G}{c^4}T_2{}^2 = \frac{G\mathcal{E}^2}{c^4 r^4} + \frac{8\pi G}{c^4}P = e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4} + \frac{(\nu'+\lambda')}{2r}\right], \quad (9)$$

where

$$\mathcal{E} = 4\pi\int_0^r (q_e n_e + q_N n_N)e^{+\lambda/2}s^2 ds, \quad (10)$$

$$P = P_e + P_N, \quad (11)$$

and

$$\rho_m = \rho_{m_e} + \rho_{m_N}, \quad (12)$$

where q_e and q_N are the unit charge per electron and nucleus, respectively.

Equations (7)–(12) do not constitute a determinate system of equations. By minimizing the energy of a static, zero-temperature multicomponent star, the authors[1] have determined that the following additional independent equation is required for the two-component case:

$$\frac{\mathcal{E}e^{\lambda/2}n_N q_N - P_N' r^2}{P_N + \rho_{m_N}c^2} = \frac{\mathcal{E}e^{\lambda/2}n_e q_e - P_e' r^2}{P_e + \rho_{m_e}c^2}. \quad (13)$$

This new equation is not without a classical analog. Auluck and Kothari[9] have obtained a similar equation by demanding that the Gibbs free energy of the nuclei be constant throughout the star. With definitions (1), (2), (4), and (5), Eqs. (7), (8), (9), and (13) constitute four equations in four unknowns λ, ν, n_e, and n_N. After some manipulation, these four equations can be shown to be equivalent to the four equations

$$\nu' = -\frac{1}{r} + \frac{r[1/r^2 + (8\pi G/c^2)T_1{}^1]}{1 + (1/r)\int_0^r (8\pi G/c^4)T_0{}^0 s^2 ds}, \quad (14)$$

$$-r^2 P_e' = e^{\lambda}(\rho_{m_e}c^2 + P_e)\left(m + \frac{4\pi G r^3 P}{c^4} - \frac{\mathcal{E}^2 G}{2c^4 r}\right) - \mathcal{E}q_e n_e e^{\lambda/2}, \quad (15)$$

$$0 = e^{\lambda}(\rho_{m_N}c^2)\left(m + \frac{4\pi G r^3 P}{c^4} - \frac{\mathcal{E}^2 G}{2c^4 r}\right) - \mathcal{E}q_N n_N e^{\lambda/2}, \quad (16)$$

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}, \quad (17)$$

where

$$m(r) = \frac{4\pi G}{c^2}\int_0^r \left[s^2(\rho_{m_e} + \rho_{m_N}) + \frac{\mathcal{E}^2}{8\pi c^2 s^2}\right]ds. \quad (18)$$

It should be noted that the sum of Eqs. (15) and (16) is simply the "one-component" equation of hydrostatic equilibrium, the Tolman-Oppenheimer-Volkov equation, which can be obtained from Eqs. (7)–(9) without the use of Eq. (13).

It is possible to reduce Eqs. (14)–(17) to a single second-order differential equation in one unknown. It is convenient to rewrite (10) as

$$n_N = \frac{-q_e n_e}{q_N} + \frac{\mathcal{E}' e^{-\lambda/2}}{4\pi r^2 q_N}. \quad (19)$$

Making the change of variable[10] $y \equiv (\bar{n}_e^{2/3}+1)^{1/2}$ and defining

$$K \equiv m_N^2 G/q_N^2, \quad (20)$$

$$\sigma(y) \equiv y(\tfrac{2}{3}y^2 - \tfrac{5}{3})(y^2-1)^{1/2} + \ln[y + (y^2-1)^{1/2}], \tag{21}$$

$$\delta(y) \equiv y(2y^2+1)(y^2-1)^{1/2} - \ln[y+(y^2-1)^{1/2}], \tag{22}$$

we have

$$P = \frac{\pi m_e^4 c^5}{h^3}\sigma, \tag{23}$$

$$\rho_{m_e} = \frac{\pi m_e^4 c^3}{h^3}\delta, \tag{24}$$

$$\rho_{m_N} = \frac{-m_N q_e n_e}{q_N} + \frac{m_N \mathcal{E}' e^{-\lambda/2}}{4\pi r^2 q_N}. \tag{25}$$

Then, Eqs. (15)–(19) become

$$-r^2 m_e c^2 (y^2-1)^{3/2} y' = e^{\lambda} m_e c^2 y (y^2-1)^{3/2}\left(m + r^3\nu\sigma - \frac{\mathcal{E}^2 G}{2c^4 r}\right) - \mathcal{E} q_e e^{\lambda/2}(y^2-1)^{3/2}, \tag{26}$$

$$0 = \left[e^{\lambda} m_N c^2\left(m + r^3\nu\sigma - \frac{\mathcal{E}^2 G}{2c^4 r}\right) - \mathcal{E} q_N e^{\lambda/2}\right] n_N, \tag{27}$$

$$e^{-\lambda} = 1 - 2m/r, \tag{28}$$

$$m' = \nu r^2 \delta + \frac{\mathcal{E}^2 G}{2c^4 r^2} - \tfrac{8}{3}\nu r^2 I (y^2-1)^{3/2} + \frac{G m_N}{c^2 q_N}\mathcal{E}' e^{-\lambda/2}, \tag{29}$$

$$n_N = -\frac{q_e 8\pi m_e^3 c^3 (y^2-1)^{3/2}}{3 q_N h^3} + \frac{\mathcal{E}' e^{-\lambda/2}}{4\pi r^2 q_N}, \tag{30}$$

where

$$\nu = \frac{4\pi^2 G m_e^4 c}{h^3} \tag{31}$$

and

$$I = \frac{m_N q_e}{m_e q_N}. \tag{32}$$

(The reader will note that the constant I is negative in this case.)

Equating the quantity in large parentheses in Eqs. (26) and (27) and solving for $\mathcal{E}$, we find that

$$\mathcal{E} = \frac{m_N c^2 r^2 y' e^{-\lambda/2}}{q_N(I-y)}. \tag{33}$$

Using (33) to write (27) in terms of y and m, one may then solve for m. Differentiating the resultant expression for m with respect to r and setting this equal to Eq. (29) produces the desired equation for y,

$$y'' = \frac{-2y'}{r} - \frac{3(y')^2}{I-y} - \frac{Kr(y')^3}{(I-y)^2} + \frac{(I-y)D\nu}{\eta(1+2r^2\nu\sigma)}\left[D[\sigma+\delta-\tfrac{8}{3}I(y^2-1)^{3/2}] + \left(1+\frac{Kry'}{I-y}\right)[2\sigma + \tfrac{8}{3}r(y^2-1)^{3/2}y']\right], \tag{34}$$

where

$$D = 1 + \frac{2ry'}{I-y} + \frac{Kr^2(y')^2}{(I-y)^2}, \qquad \eta = K+1 \tag{35}$$

III. REDUCTION TO THE CLASSICAL MODEL

The classical model can be obtained from the relativistic model by making the following approximations:

$$\frac{2m}{r} \ll 1, \tag{36}$$

$$\rho_{m_e} c^2 + P_e \ll \rho_{m_N} c^2, \tag{37}$$

$$\frac{4\pi G r^3 P}{c^4} \ll m(r), \tag{38}$$

$$\frac{\mathcal{E}^2 G}{2c^4 r} \ll m(r), \tag{39}$$

$$\frac{\mathcal{E}'}{4\pi r^2 q_N} \ll n_N. \tag{40}$$

With these assumptions Eqs. (15)–(19) reduce to

$$-r^2 P_e' = -\mathcal{E} q_e n_e, \tag{41}$$

$$0 = \rho_{m_N} c^2 m(r) - \mathcal{E} q_N n_N, \tag{42}$$

$$m = \frac{4\pi G}{c^2}\int_0^r s^2 \rho_{m_N} ds, \tag{43}$$

$$n_N = \frac{-q_e n_e}{q_N}. \tag{44}$$

Adding (41) to (42) and using (44), we find that

$$P' = \frac{-mc^2}{r^2}\rho_{m_N}, \tag{45}$$

which is Chandrasekhar's Eq. VI of the standard model.[11] With Eqs. (1)–(5) and the change of variable $y = (\bar{n}_e^{2/3}+1)^{1/2}$, Eq. (45) can be written

$$y'' = \frac{-2y'}{r} - \frac{8\nu I^2}{3}(y^2-1)^{3/2}. \tag{46}$$

This is the classical "one-component" analog of Eq. (34).

IV. CALCULATIONS

We have solved Eqs. (34) and (46) numerically using extrapolation by rational functions.[12] Boundary conditions are specified at the origin, $r=0$, for y and y'. $y(0)$ can be chosen arbitrarily, but from Eq. (34) it can be seen that $y'(0)$ must be zero if $y''(0)$ is to be finite near $r=0$. The minimum value for y compatible with pressure ionization is ~1.0001 as computed from the volume associated with the Bohr radius. Models with an initial y value of 1.006 are found to be completely ionized over 94% of their volume. Those with larger central densities are more nearly totally ionized. For values of y larger than ~8.00, inverse beta decay will occur, introducing more than one kind of nucleus, and above $y \sim 54$, free neutrons are produced.[4]

Integration of Eqs. (34) and (46) progresses from the origin for increasing r until either $y=1$ (the electron number density is zero) or until the nuclei number density is zero, whichever happens first. When either one of the particle number densities is identically zero, the corresponding equation, (26) or (27), is also identically zero. Further integration of the remaining density involves either (26) or (27) plus Eqs. (28)–(30).

Substituting into Eq. (30) expressions for $\mathcal{E}'$ and $e^{-\lambda/2}$ in terms of y and y' one obtains the following expresion for n_N:

$$n_N = -\frac{8\pi m_e c^3 q_e (y^2-1)^{3/2}}{3h^3 q_N} - \left(\frac{y'}{I-y}\right)^2 \left(\frac{m_N c^2}{4\pi q_N{}^2}\right)\left(\frac{1+2r^2\nu\sigma}{D}\right) + \left(\frac{y'}{I-y}\right)\left(\frac{m_N c^2}{4\pi q_N{}^2}\right)\left[\frac{2r\nu\sigma + \frac{8}{3}r^2\nu(y^2-1)^{3/2}y'}{D}\right]$$
$$+\left(\frac{m_N c^2 \nu}{4\pi q_N{}^2}\right)\left[\frac{1}{\eta D}\left(1+\frac{ry'}{I-y}\right)\right]\left[D[\sigma+\delta-\tfrac{8}{3}I(y^2-1)^{3/2}] + \left(1+\frac{Kry'}{I-y}\right)[2\sigma+\tfrac{8}{3}r(y^2-1)^{3/2}y']\right]. \quad (47)$$

If the value $y=1$ is substituted into Eq. (47), one obtains

$$n_N = -\left(\frac{y'}{I-y}\right)^2\left(\frac{m_N c^2}{4\pi q_N{}^2}\right)\left[\frac{1}{1+2ry'/(I-y)+Kr^2(y')^2/(I-y)^2}\right], \quad (48)$$

which for $y=1$ is equivalent to

$$n_N = -\left(\frac{y'}{I-y}\right)^2\left(\frac{m_N c^2}{4\pi q_N{}^2}\right)e^{-\lambda}. \quad (49)$$

Thus, at $y=1$, the nuclear number density is negative. [If $y'=0$, then the mass of the star will also be zero. See Eq. (55).] This suggests that the nuclei number density has previously gone to zero at some r_N and that for $r > r_N$, Eqs. (26), (28), (29), and (30) should have been used to continue the integration for the electron number density. Thus, at least a thin shell of electrons must surround the star for this model.

In addition to the density-radius relationships, the total mass and maximum radius are considered important characteristics of stars with specified central densities. We are also able to compute the total charge for our models. The expression for the classical limiting mass comes from Eq. (45),

$$m_{cl}(R) = \frac{R^2 y'(R)}{I}, \quad (50)$$

where R is the maximum radius of the star.

For the relativistic model, the star's asymptotic mass-energy, $\Lambda(R)$, is the relevant quantity. We can write Eq. (17) as

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}$$
$$= 1 - \frac{2\Lambda(r)}{r} + \frac{G\mathcal{E}^2(r)}{r^2c^4}, \quad (51)$$

where

$$\Lambda(r) = \frac{4\pi G}{c^2}\int_0^r \left(\rho_m + \frac{\mathcal{E}\mathcal{E}'}{4\pi c^2 s^3}\right)s^2 ds. \quad (52)$$

Thus, at large distances from the star, $\Lambda(R)$ is the observed mass-energy.

From Eqs. (27), (28), and (33), we find that

$$e^{-\lambda}(R) = \frac{1+2R^2\nu\sigma(R)}{D(R)}, \quad (53)$$

$$\mathcal{E}(R) = \frac{m_N c^2}{q_N}\left[\frac{R^2 y'(R)}{I-y(R)}\right]\left[\frac{1+2R^2\nu\sigma(R)}{D(R)}\right]^{1/2}, \quad (54)$$

$$\Lambda(R) = \frac{R}{2}\left[\frac{1+2R^2\nu\sigma(R)}{D(R)}\right]\left\{\frac{KR^2[y'(R)]^2}{[I-y(R)]^2} - 1\right\} + \frac{R}{2}. \quad (55)$$

V. NUMERICAL RESULTS FOR MODERATE CENTRAL DENSITIES

In Table I we have summarized the results of the solution of Eq. (34) for several initial electron number densities within the range expected for a white dwarf. Results of the solution of Eq. (46) are included for comparison. $y(0)$ is the value of y at the center of the star [see Eq. (46)].

As expected, the relativistic model agrees closely with the Chandrasekhar model for small densities. However, at the upper limit the relativistic mass-energy is seen to reach a peak for $y(0) \sim 20$ and then decline while the Chandrasekhar mass increases to the well-known limit. Even without a detailed stability analysis, it is possible to deduce, after the manner of Zel'dovich and

TABLE I. Values of the radius and total mass of white dwarf stars with central density $y(0) \leq 50$ for both the two-component model and the Chandrasekhar model. Note that the total mass peaks at $y(0)=20$ for the two-component model, but continues to increase for the Chandrasekhar model. See Figs. 1 and 4 for the radius and total mass of both models at larger densities.

$y(0)$	Charged white dwarf R (cm)	Charged white dwarf $\Lambda(R)$ (g)	Chandrasekhar white dwarf R (cm)	Chandrasekhar white dwarf $m(R)$ (g)
1.006	2.796×10^{9}	4.250×10^{31}	2.798×10^{9}	4.254×10^{31}
1.02	2.064×10^{9}	1.043×10^{32}	2.065×10^{9}	1.043×10^{32}
1.06	1.556×10^{9}	2.336×10^{32}	1.556×10^{9}	2.337×10^{32}
2.0	6.527×10^{8}	1.385×10^{33}	6.531×10^{8}	1.389×10^{33}
4.0	3.883×10^{8}	2.024×10^{33}	3.886×10^{8}	2.030×10^{33}
6.0	2.868×10^{8}	2.221×10^{33}	2.871×10^{8}	2.231×10^{33}
8.0	2.295×10^{8}	2.304×10^{33}	2.298×10^{8}	2.321×10^{33}
10.0	1.920×10^{8}	2.351×10^{33}	1.922×10^{8}	2.369×10^{33}
20.0	1.069×10^{8}	2.407×10^{33}	1.071×10^{8}	2.444×10^{33}
30.0	7.436×10^{7}	2.406×10^{33}	7.455×10^{7}	2.460×10^{33}
40.0	5.707×10^{7}	2.394×10^{33}	5.725×10^{7}	2.466×10^{33}
50.0	4.631×10^{7}	2.379×10^{33}	4.645×10^{7}	2.469×10^{33}

Novikov,[13] that the white dwarf will not be stable in the regions where the derivative of the total mass-energy with respect to $y(0)$ is negative (see Fig. 1). Thus, $y(0)=20$ is the practical limit on the central density for this white-dwarf-star model.

By way of illustration, we also include the radial dependence of the electric field and charge density for a typical white dwarf. Figure 2 shows a plot of the electric field inside a star of central density $y(0)=6$ as a function of r. The field reaches a maximum at 9.8×10^{7} cm and decreases monotonically thereafter. The maximum value of the field, 4.56×10^{-5} esu/cm^2 (~0.01 V/cm), is comparatively small, as expected. In Fig. 3 the charge density

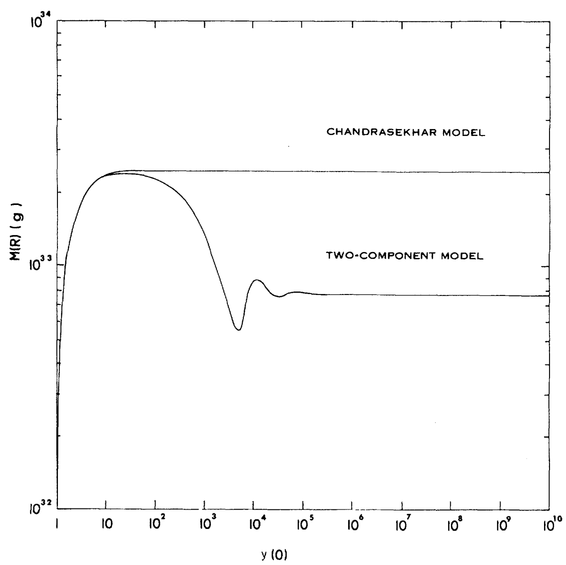

FIG. 1. The total mass-energy, $M(R)$, of the two-component star as a function of the central density $y(0)$. Here, the mass-energy is given in grams to facilitate comparison with the Chandrasekhar mass.

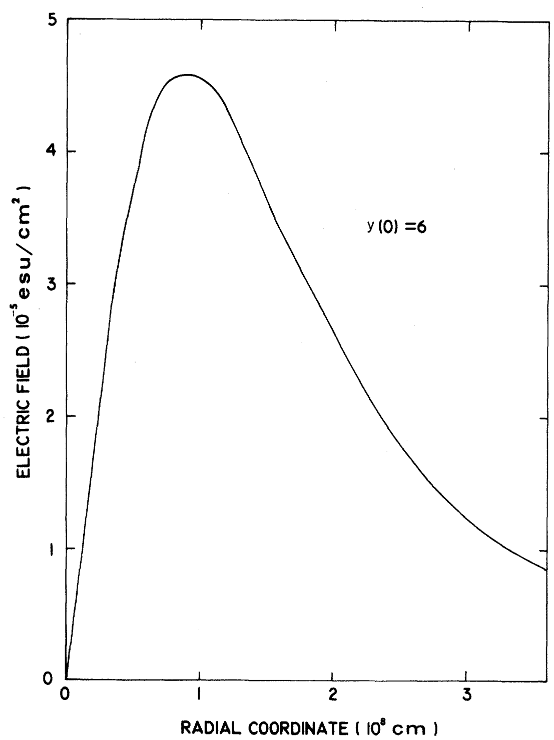

FIG. 2. The electric field as a function of the radial coordinate inside a two-component star with central density $y(0)=6$. It should be noted that the particle density drops to zero beyond $r=2.87\times10^{8}$ cm.

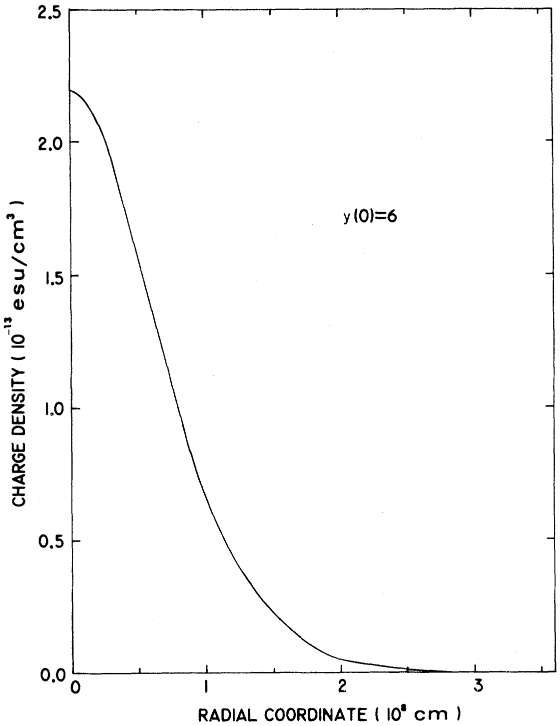

FIG. 3. The charge density as a function of the radial coordinate inside a two-component star with central density $y(0)=6$.

is given as a function of r, also for $y(0)=6$. It is positive over the bulk of the star, although, as mentioned in Sec. IV, there should be a very small negative region near the boundary. [In practice, both number densities fall to zero in an indistinguishably small interval, less than 10^{-8} cm for a central density of $y(0)=6$.] Plots of the electric field and charge density for other moderate central densities are very similar, with the maximum internal field increasing generally as the central mass density increases.

VI. NUMERICAL RESULTS FOR LARGE CENTRAL DENSITIES

For the reasons mentioned previously, this two-particle model is not expected to be realisitic for a white dwarf above $y(0)=20$. Nonetheless, we have carried out the calculation for larger densities with the intent of determining the limiting mass and completing our analysis of the model.

Figure 1 plots the total mass-energy as a function of the central density both for our model and for the Chandrasekhar model. Our asymptotic mass limit is about one third of the Chandrasekhar limit, but our maximum mass is only ~3% smaller. As mentioned above, the star is not expected to be stable for the range of central densities

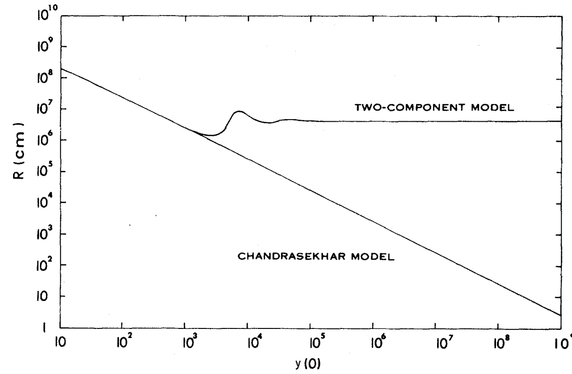

FIG. 4. The maximum radius, R, for the two-component star as a function of the central density $y(0)$.

where this curve has a negative slope. It is also interesting to note that for our model, increasing the central density beyond a certain point does not decrease the maximum radius of the star (Fig. 4). (The Chandrasekhar radius decreases to zero as the central density approaches infinity.) Thus, we do not find a star of finite total mass and zero radius.

Graphs of several additional quantities calculated for the charged, relativistic, two-component model follow. Figure 5 plots the mass density as a function of r for several initial central densities. In general shape these curves resemble those for Cameron's[14] neutron-star model, although our slopes in the straight-line region are steeper and our corresponding maximum masses smaller.

Figure 6 is a plot of the total charge, $\mathcal{E}(R)$, of the star as a function of the central density, $y(0)$.

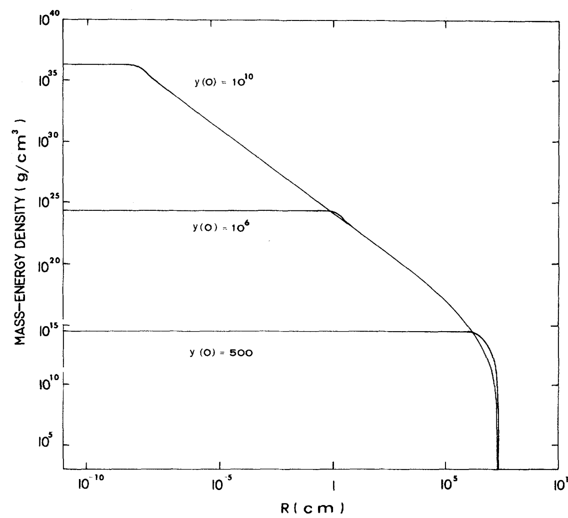

FIG. 5. The mass-energy density, ρ_m, as a function of the radial coordinate for the specified central densities.

It is similar in appearance to the plot for $m(R)$ as a function of $y(0)$ (Fig. 3), with a maximum total charge of 1.2×10^{12} esu and an asymptotic limit of 3.9×10^{11} esu. However, as Fig. 7 shows, the ratio of charge to mass is not constant. There is a direct correlation between the charge-to-mass ratio and the average mass density of the star. The peak of the $\mathcal{E}(R)/m(R)$ curve occurs for a $y(0)$ value of 1.48×10^{3}, for which the average mass density is 3.8×10^{13} g/cm^3. The first minimum occurs for a $y(0)$ value of 6.6×10^{3}, for which the average density is only 2.5×10^{11} g/cm^3. This trend is verified for other points on the curve with the charge-to-mass ratio and the average mass density approaching finite asymptotic values for $y(0)>4\times10^{5}$.

Use of Eq. (48) allows one to calculate the charge density as a function of r and y. Figure 8 shows the charge density at $r=0$ as a function of $y(0)$. Notice that beyond $y(0)\sim2\times10^{4}$, the central charge density increases linearly with $y(0)$. For $y(0)>10^{10}$, this charge imbalance induces internal electric fields stronger than the 10^{16} V/cm needed for electron-positron pair creation[15] (cf. Fig. 2 for the general shape of the internal electric field). Thus, for central densities larger than this, the field will break down and positrons will be introduced. It must be emphasized that the external electric field does not change at all for $y(0)>4\times10^{5}$. Thus, while the internal electric field increases dramatically to balance the effects of increased mass density, the external field attains an asymptotic value of only about 0.02 esu/cm^2 (6 V/cm) at the boundary of the star ($\sim4.46\times10^{6}$ cm). Again we note that this is not a region in which our two-componet model can be considered realistic. However, this instance illustrates a situation in which effects due to charge separation can be quite large.

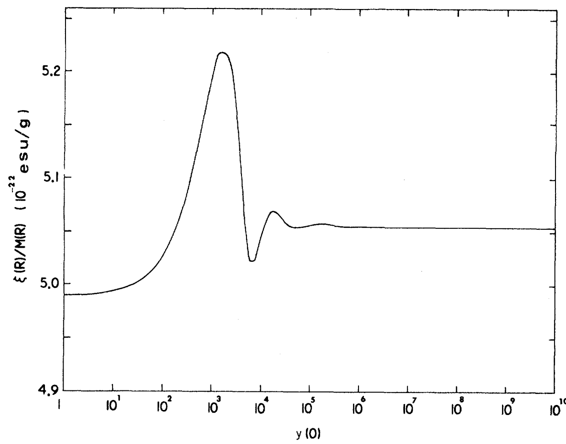

FIG. 7. The ratio of the total charge integral to the total mass-energy integral as a function of the central density $y(0)$.

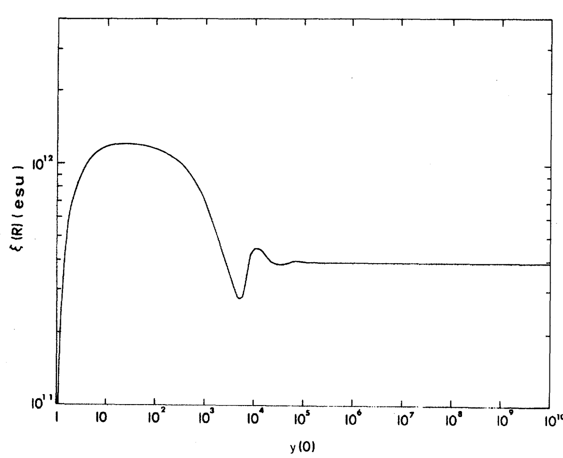

FIG. 6. The total charge integral, $\mathcal{E}(R)$, for the two-component star as a function of the central density, $y(0)$.

VII. SUMMARY

For the low-density white dwarf star, most of the standard results were verified by this study. In addition, we have calculated the electric field

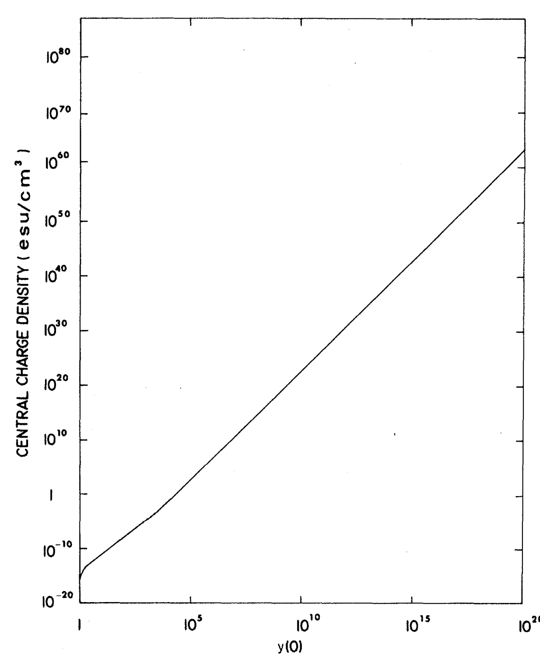

FIG. 8. The central charge density as a function of the central density $y(0)$.

and found it to be weak, as had been predicted.

In the extrapolation to large densities, the following results were obtained:

(1) The total mass-energy, $M(R, y(0))$, does not simply saturate, but peaks at nearly the Chandrasekhar value, drops, and then saturates at about one third the Chandrasekhar limit.

(2) The star radius, $R(y(0))$, does not drop to zero for large $y(0)$ as in the standard model,[2] but saturates at a finite value, $R = 4.46 \times 10^6$ cm.

(3) The charge-to-mass ratio of the star is directly proportional to the average mass density.

(4) For large central mass densities, the central charge density increases as $y(0)$ and eventually produces large internal electric fields. However, the external electric field approaches a low asymptotic value for $y(0) > 4 \times 10^5$.

The first two results are similar to those found for relativistic neutron-star models,[14] but the third and fourth are new.

Even though the two-component, fermion-boson structure is not strictly applicable for large densities, we have carried out an extensive investigation of the model in search of possible charge effects. We are at the present time looking into the more realistic model of a neutron star composed of a compressible neutron-proton liquid and a free electron gas.

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$$y' = \bar{n}_e' / [3\bar{n}_e^{1/3} - (\bar{n}_e^{2/3} + 1)^{1/2}]$$

when both $\bar{n}_e$ and $\bar{n}_e'$ go to zero at the boundary of the star.

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