Path-integral derivation of black-hole radiance*

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The Feynman path-integral method is applied to the quantum mechanics of a scalar particle moving in the background geometry of a Schwarzschild black hole. The amplitude for the black hole to emit a scalar particle in a particular mode is expressed as a sum over paths connecting the future singularity and infinity. By analytic continuation in the complexified Schwarzschild space this amplitude is related to that for a particle to propagate from the past singularity to infinity and hence by time reversal to the amplitude for the black hole to absorb a particle in the same mode. The form of the connection between the emission and absorption probabilities shows that a Schwarzschild black hole will emit scalar particles with a thermal spectrum characterized by a temperature which is related to its mass, M, by $T = \hbar c^3/8\pi GMk$. Thereby a conceptually simple derivation of black-hole radiance is obtained. The extension of this result to other spin fields and other black-hole geometries is discussed.

I. INTRODUCTION AND SYNOPSIS

The Feynman path-integral method[1] is a natural way to formulate the quantum mechanics of matter fields moving in curved background spacetimes.[2-4] In this method the amplitude $K(x,x')$ for a particle to propagate from one space time point x' to another x is expressed as an integral over all the paths connecting the two points. The integral has the form

$$K(x,x') \sim \sum_{\text{paths}} e^{iS(x,x')/\hbar}, \tag{1.1}$$

where $S(x,x')$ is the classical action for a particular path connecting x' and x. The amplitude K is called the propagator.

This formulation of quantum mechanics has several advantages. Because the sum is over paths in the four-dimensional space time and because S is a four-dimensional scalar the expression for K is manifestly covariant. Expressing the propagator as a sum over paths gives it a direct physical interpretation. Since the propagator is expressed directly as a functional integral a problem of finding an approximate form for it reduces immediately to a problem of approximating the functional integral. For these reasons the Feynman path-integral method is an attractive way to do quantum mechanics in curved space times. In this paper we shall use the path-integral method to derive the thermal radiation emitted by black holes.[5] In the following we shall give a qualitative outline of our methods and results. The details and proofs will be presented in the subsequent sections.

Figure 1 shows the Penrose diagram for the Schwarzschild geometry. The unshaded part of this diagram represents the geometry outside a spherically symmetric collapsing body. The shaded part should be replaced by the geometry inside. Let us now consider the probability that a particle is emitted by the black hole and detected by an observer a constant distance away in a positive-frequency mode peaked about some point A when there are no incoming particles in the distant past. This probability can be related to the amplitude to propagate from some point B on the future singularity to the observation point A. This in turn can be represented as a sum over paths of the form in Eq. (1.1), where the paths summed over are those which begin at the point B on the future singularity and end at the observation point A. A typical such path (BCA) is shown in Fig. 1. These are exactly the paths which correspond to a pair of particles being created (near C), one of which falls into the black hole and the other of which propagates out to the observer. We do not sum over paths which start on $\mathscr{I}^-$ since they would represent the propagation of incoming particles in the distant past. We do not sum over paths which pass through the shaded region since that should properly be replaced by the interior of the collapsing star and will not contribute to the particle production at late times (as we shall show in more detail subsequently). We consider only propagation from the future singularity.

If one attempts to evaluate the production probability by applying the method of stationary phase to the integrals involved, then it is readily seen that there are no real stationary paths which connect the future singularity to a positive-frequency mode for a stationary exterior observer. Such paths would be permissible classical paths with the positive energy connecting the two surfaces and there are none (although these are classical paths with negative energy). However, if the coordinates of the point B on the future singularity are displaced to complex values then stationary-phase paths can be found. These paths are, in fact, in the real manifold and connect the *past* singularity to the external observer. Thus, the amplitude to propagate to the external observer from a point B on the future singularity can be related to the amplitude to propagate from a corresponding point D on the past singularity. In turn, by symmetry under time reversal, the modulus of the latter amplitude is the same as that of the amplitude to propagate from the time-reversed point of A into the future singularity at the time-reversed point of D. This shows that by appropriately distorting the contours of integration into the complex coordinate plane the amplitude for a black hole to emit particles can be related to the amplitude for it to absorb. If we consider the amplitude for the black hole to emit particles in a definite mode with energy E as measured by a distant observer then this connection is

$$(\text{probability to emit a particle with energy } E) = e^{-2\pi E/\kappa} \times (\text{probability to absorb a particle with energy } E)\,, \tag{1.2}$$

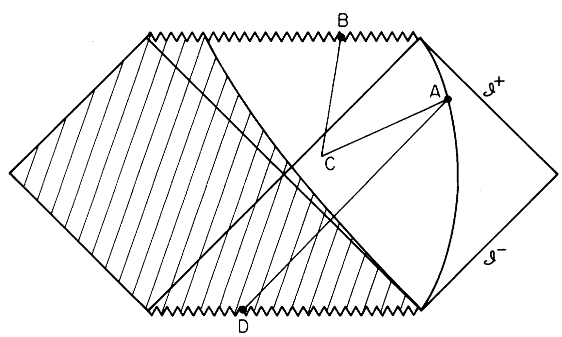


FIG. 1. The unshaded part of the diagram represents the Schwarzschild geometry outside a spherically symmetric collapse. The world line O is that of an observer, who remains outside the black hole at a fixed radius. The nonstationary path BCA corresponds to particle production by the black hole. A pair of particles is created near C. One falls into the future singularity at B while the other propagates out to the observer and is detected at A. The amplitude that a particle is produced by the black hole and detected by the observer in a given mode at late times can be expressed as an integral over the amplitude to propagate between a point B on the future singularity and a point A on the observer's world line. In turn the propagator can be expressed as a sum over the paths which connect these points. By analytically continuing the point B into the complexified Schwarzschild space, the amplitude to propagate to A from a real point B on the future singularity can be related to the amplitude to propagate to A from a reflected point D on the past singularity. This latter process is just the time-reversed proces of absorption of a particle by the black hole. In this way the probability for a black hole to emit a scalar particle can be related to the probability for it to absorb one. The relation implies the thermal radiance.

where κ is the surface gravity of the black hole; in the case of a Schwarzschild black hole of mass M, $\kappa = 1/4M$. (Here as in the following we are using units where $\hbar = c = G = 1$.) This connection between emission and absorption is exactly that necessary to establish the result of Ref. 5 that a Schwarzschild black hole emits particles with a thermal spectrum corresponding to a temperature $T = \kappa/2\pi k$. To see this, imagine surrounding the black hole by a thermal cavity and adjusting the temperature, T, until the whole system is in equilibrium. The temperature of the radiation is then the temperature of the black hole. In equilibrium the rate of emission particles by the black hole must exactly equal the rate of absorption. Since the ratio of the probability of having N photons in a particular mode in the cavity with energy E to the probability of having $N-1$ photons in the same mode is $\exp(-E/kT)$ this equilibrium condition will be true when $T = \kappa/2\pi k$.

In the following we shall give the details of this simple argument. In Sec. II the quantum mechanics of scalar particles moving in a Schwarzschild background geometry is formulated in terms of path integrals. Section III contains a derivation of the necessary analytic properties of the propagator on the complexified manifold, and in Sec. IV the thermal radiation from a Schwarzschild black hole is deduced. Section V discusses the generalization to Reissner-Nordström and Kerr black holes and higher-spin particles.

II. PATH-INTEGRAL QUANTUM MECHANICS OF A SCALAR PARTICLE IN A SCHWARZSCHILD BACKGROUND

In this section we shall formulate the quantum mechanics of a scalar particle moving in a curved background geometry in terms of Feynman path integrals. Our work here is a generalization and interpretation of that of Feynman for a scalar field in a flat-space background,[7] and as far as the general formulation of path integrals in curved backgrounds goes closely parallels the previous work of the DeWitts.[2-3] The considerations of this section are intended to motivate the definition of the propagator in Sec. III as a solution of the inhomogeneous scalar wave equation with certain boundary conditions. As a consequence in the present section we shall not be uniformly mathematically precise, but this is a familiar situation when working with path integrals.

The path of a scalar particle through spacetime may be specified by giving the four coordinates x^α as a function of a parameter time w. Letting x stand for all four coordinates we write this as $x=x(w)$. Suppose we consider the motion of a particle which starts at a spacetime point x' at $w=0$ and arrives at x at $w=W$. An action functional which describes the classical motion of such a particle is

$$S[x(w)]=\frac{1}{4}\int_0^W dw\, g(\dot{x},\dot{x})\,, \tag{2.1}$$

where g is the metric on the curved spacetime and $\dot{x}$ represents the tangent vector whose components are dx^α/dw. The path which extremizes S satisfies the geodesic equation with w as an affine parameter. Thus, for timelike paths w may be taken to be a constant multiple of the proper time while for spacelike paths it could be taken to be the same multiple of the proper distance.

The action functional of Eq. (2.1) is not the usual one in which the integrand is $[-g(\dot{x},\dot{x})]^{1/2}$. However, (2.1) is a perfectly valid classical action which has obvious advantages for a path-integral formulation in that it is quadratic in the four-velocities (see the Appendix). In contrast to the usual form, the action in Eq. (2.1) continues analytically from timelike to spacelike paths. In addition it gives correctly the relativistic quantum mechanics of a scalar particle in flat spacetime and is therefore a natural generalization to curved backgrounds. We shall not discuss other choices of the action further here.

The basic assumption of the Feynman path-integral method is that the amplitude for a particle to travel a particular path in spacetime is proportional to $\exp\{iS[x(w)]\}$. To have a clearer idea of what this means imagine dividing the parameter time w into many small intervals at values w_i. The amplitude for observations of spacetime position at *each* parameter time w_i to yield the set of values $\{x_i=x(w_i)\}$ is proportional to $\exp\{iS[x(w)]\}$ in the limit as the intervals become infinitesimally separated. Amplitudes for more restricted sets of observations may be constructed by summing this amplitude over the unobserved positions. For example, the amplitude that an observation of spacetime position at one parameter time yields the value x' and a second observation a parameter time W later yields the value x is

$$F(W,x,x')=\int \delta x[w]\exp\left[\tfrac{1}{4}i\int_0^W g(\dot{x},\dot{x})dw\right]\,, \tag{2.2}$$

where the integral is over all the unobserved positions at parameter times between 0 and W. In other words, the integral is a functional integral over all paths which have $x(0)=x'$ and $x(W)=x$.

The parameter w has been introduced as an observable which plays a role analogous to ordinary time in nonrelativistic quantum mechanics.[8] However, there is no experiment in which it is directly observed (since particles do not carry clocks). All physical observations can be obtained from the amplitude $K(x,x')$ for a particle to be localized at two spacetime points x' and x. $K(x,x')$ is called the propagator. This amplitude can be constructed in two steps: first by summing over all paths which connect x' to x in a given parameter time W and then by summing[9] over the unobserved value, W. The first sum is just F in Eq. (2.2). In the second sum an appropriate weight must be assigned to each elapsed parameter time W. In flat space if the scalar particle has a rest mass m this weight is[7] $i\exp(-im^2W)$. It is natural to adopt this also for the curved-space case. The expression for the propagator then takes the form[10]

$$K(x,x')=i\int_0^\infty dW\exp(-im^2W)F(W,x,x')\,, \tag{2.3}$$

where F is given by Eq. (2.2). The restriction of the integral to positive W is the requirement that the particles always propagate forward in parameter time.

From this definition it easily follows that $K(x,x')$ is symmetric in x and x'. Letting $w=W-w'$ leaves the action in Eq. (2.1) unchanged but interchanges x and x' in the sense that as functions of w', $x(0)=x$ and $x(W)=x'$. Thus $F(W,x,x')$ is symmetric in x and x' and it follows immediately from Eq. (2.3) that $K(x,x')$ is also.

Equations (2.2) and (2.3) are the basic relations which are needed to define the quantum mechanics of a free scalar particle moving in a curved spacetime. Before this definition is complete the integrals in Eqs. (2.2) and (2.3) need to be given meaning. We now turn to this question but for simplicity and definiteness restrict our attention to scalar particles propagating in the Schwarzschild geometry.

The first problem is the definition of the path integral in Eq. (2.2). To solve this we analytically continue the variables in this formal expression to values where the integral is well defined. In particular w and W are continued to negative imaginary values $-i\omega$ and $-i\Omega$, respectively, and the coordinates are continued to a domain where the metric has signature +4. In the case of the Schwarzschild geometry which in Kruskal coordinates z, y, θ, φ has the form

$$ds^2 = (32M^3e^{-r/2M}/r)(-dz^2+dy^2)+r^2d\Omega^2\,, \tag{2.4}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $r(y,z)$ defined by

$$-z^2+y^2 = (r/2M-1)e^{r/2M}\,, \tag{2.5}$$

this can be accomplished by letting $z = i\zeta$ and keeping ζ real. Then the analytically continued metric γ is given by the line element

$$d\sigma^2 = (32M^3e^{-r/2M}/r)(d\zeta^2+dy^2)+r^2d\Omega^2\,, \tag{2.6}$$

with r now defined by

$$\zeta^2+y^2 = (r/2M-1)e^{r/2M}\,. \tag{2.7}$$

The analytically continued expression for F becomes

$$F(\Omega,x,x') = \int \delta x[\omega]\exp\left[-\tfrac{1}{4}\int_0^\Omega \gamma(\dot{x},\dot{x})d\omega\right]\,, \tag{2.8}$$

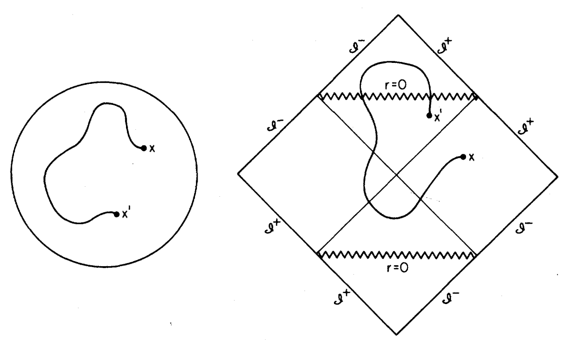
where x is to be understood as $x^\alpha = (\zeta, y, \theta, \varphi)$ and $\dot{x}$ as $dx^\alpha/d\omega$. The space covered by the coordinate ranges $-\infty<\zeta<\infty$, $-\infty<y<\infty$, $0\le\theta\le\pi$, $0\le\varphi<2\pi$ is complete, has the topology $R^2\times S^2$, and, since r ranges only over values greater than $2M$, the metric γ is regular with the exception of the trivial polar singularities at $\theta=0$ and π. The integration now extends over all paths in this space which start with $\omega=0$ at x' and end with $\omega=\Omega$ at x (see Fig. 2). This path integral can be precisely defined.[11] Our basic assumption[12] is that the function F when defined in this way and analytically continued back to real values of the coordinates and parameter time variable gives the correct propagator defined heuristically by Eq. (2.2).

This procedure not only gives definition to the integral in Eq. (2.2) but also identifies the class of paths over which integration is done. Imagine taking a particular path in the space with positive-definite metric and continuing both coordinates and parameter to complex values. The complexified path is now a two-dimensional sheet in the space of complex coordinates given by the four complex analytic functions $x(\omega)$ of the complex parameter ω. The analytic functions are completely fixed by their real values for real ω. What does the class of paths defined above look like when the analytic continuation reaches real values of W and the contours of the path integration are deformed to real coordinate values? It seems clear that the resulting class of paths will not be confined by any finite boundaries in the Kruskal coordinates. In particular, they will cross and recross the singularities at $r=0$. These singularities are poles in the metric considered as functions of the complex coordinates. The path integral across the singularity is defined by giving a prescription for which way the contour of integration goes around the pole. In turn this is determined by the analytic continuation of the path from the positive-definite section and the deformation of the contours to real values everywhere except near $r=0$. If the paths cross $r=0$ then they extend into the Schwarzschild geometries with negative mass. This is illustrated schematically in Fig. 2(b).

The actual computation of $F(\Omega,x,x')$ is greatly facilitated by noticing that it satisfies a parabolic partial differential equation

$$\frac{\partial F}{\partial\Omega} = \tilde{\Box}^2 F\,, \tag{2.9}$$

where $\tilde{\Box}^2 = \gamma^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta$ and $\tilde{\nabla}_\alpha$ indicates covariant differentiation with respect to the metric γ. The derivation of this result is reviewed in the Appendix. The boundary conditions on Eq. (2.9) which yield



(a) (b)

FIG. 2. (a) A compactified representation of a constant θ, constant φ slice of the positive-definite spacetime whose metric is given in Eq. (2.6). The heavy circle represents infinity. There are no singularities. A typical path connecting two points x' and x is shown. (b) A Penrose diagram for the Schwarzschild geometry showing in addition the regions of negative mass (or $r\le 0$) above and below the singularities. A typical member of the class of paths continued analytically to this real section from the positive-definite spacetime represented in (a) is shown. Such paths may cross and recross the singularities at $r=0$. Integrations over paths which cross the singularities are specified by choosing contours of integration which are the analytic continuations of those in the positive-definite section.

F as defined by the path integral are first that

$$F(0,x,x')=\delta(x,x')\,, \tag{2.10}$$

where $\delta(x,x')$ is the four-dimensional δ function in the space with positive-definite metric [equal to $\delta^{(4)}(x-x')\gamma^{-1/2}$]. Second, F must vanish as x approaches the infinity of the space with positive-definite metric. The first boundary condition is simply the requirement that at $\Omega=0$ the particle be localized at x'. The second follows from the exponential damping of the integral as x tends to infinity.

The solution to Eq. (2.8) and associated boundary conditions will be analytic in x where the metric is and analytic in Ω except at the origin.[13] The solution can thus be continued back to physical values of the coordinates and parameter time. There it will satisfy the analytically continued equation

$$i\frac{\partial F}{\partial W}=-\Box^2 F\,, \tag{2.11}$$

where $\Box^2=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ and ∇_α is covariant differentiation with respect to the metric g. Equation (2.11) can be thought of as the Schrödinger equation for propagation in the parameter time W.

The second problem involved in giving meaning to Eq. (2.3) for the propagator is the integral over W. To resolve this we need the asymptotic form of $F(W,x,x')$ for both small and large W. The behavior of F for small W is essentially given by the definition of the path integral itself. For small W and fixed x and x' the action [Eq. (2.1)] for a typical path will become large. The only paths which contribute significantly to the path integral are the stationary paths, i.e., the geodesics connecting x and x'. For these paths the action is

$$S(W,x,x')=\tfrac{1}{4}s(x,x')/W\,, \tag{2.12}$$

where $s(x,x')$ is the square of the geodesic distance between x and x'. If there were a single geodesic connecting x and x' then for small W we could write

$$F(W,x,x')=\exp[is(x,x')/(4W)]N(W,x,x'),$$

where $N(W,x,x')$ is a real normalization factor. Equation (2.11) can then be solved for small-W behavior of N. When note is taken of the identity

$$(\nabla_\alpha s)(\nabla^\alpha s)=4s\,, \tag{2.13}$$

one finds that

$$N(W,x,x')=D(x,x')W^{-2}+\cdots\,, \tag{2.14}$$

where D is independent of W; its exact form will not concern us. The reader will recognize this as essentially the WKB approximation to the solution of Eq. (2.11). Considerably more detail on the derivation can be found in Refs. 2 and 8(d).

In general there will be several geodesics connecting x and x'. In that case the small-W behavior of F will be

$$F(W,x,x')=\sum_c e^{is_c(x,x')/(4W)}\times W^{-2}[D_c(x,x')+O(W^{-1})]\,, \tag{2.15}$$

where the sum is over each class of geodesics which connect x to x'. This small-W behavior will not be uniformly valid over the whole range of values of x. In particular where neighboring geodesics which start at x' intersect (caustics) we expect the approximation to break down. For example, $s(x,x')$ will have a branch point at such an intersection, but we know from general considerations that F has none.

If Eq. (2.15) is integrated over a smooth function of x then as W tends to zero there will be a significant contribution to the integral only for values of x for which $s_c(x,x')$ is nearly stationary keeping x' fixed. This will happen only for x close to x'. In other words, $F(0,x,x')$ is proportional to a δ function. The proportionality factor is just unity because the normalization factors in Eq. (2.15) must also give rise to Eq. (2.10). Thus,

$$\lim_{W\to 0}F(W,x,x')=\delta(x,x')\,. \tag{2.16}$$

For large values of Ω standard estimates[13] for the solution of the parabolic equation (2.9) defined by the boundary condition in Eq. (2.10) show that $F(\Omega,x,x')$ decreases at least as fast as Ω^{-2} for x and x'. Physically this is nothing more than the spreading of an initially localized wave packet with increasing Ω. Thus, at large Ω, F can be expressed as

$$F(\Omega,x,x')=\Omega^{-2}[F_0(x,x')+F_1(x,x')\Omega^{-1}+\cdots]\,. \tag{2.17}$$

We will assume that expansion can be continued back to real values of W and x by term. Thus, in particular we have for large W

$$F(W,x,x')=O(W^{-2})\,. \tag{2.18}$$

The large-W behavior of F shows that the integral in Eq. (2.3) always converges at the upper limit. This is not the case at the lower limit where F diverges as W^{-2} [Eq. (2.15)]. To make the integral finite we shall insert a convergence factor $\exp(-\epsilon/W)$, where ϵ is a small positive constant. Physical quantities are to be computed with ϵ finite and then the limit $\epsilon\to 0$ is to be taken. Thus,

$$K(x,x')=i\int_0^\infty dW\exp(-im^2W-\epsilon/W)F(W,x,x')\,. \tag{2.19}$$

This method of regularization corresponds physically to the requirement that the particle propagate forward in the parameter time W. It correctly gives the usual Feynman propagator for a scalar particle in flat spacetime as we shall show in the next section. With Eq. (2.19) our definition of the propagator is now essentially precise. We shall now examine the consequences.

III. ANALYTICITY PROPERTIES OF THE PROPAGATOR

In the preceding section an integral representation for the Feynman propagator $K(x,x')$ was derived in terms of the propagator for a definite parameter time W, $F(W,x,x')$. It follows immediately from Eq. (2.9), the parameter time Schrödinger equation, Eq. (2.11), and the boundary conditions, Eqs. (2.15) and (2.17), that $K(x,x')$ is a solution of the inhomogeneous wave equation in the Schwarzschild background

$$(\Box^2 - m^2)K(x,x') = -\delta(x,x')\,. \tag{3.1}$$

As an alternative to the path integral, $K(x,x')$ could be defined as a solution to Eq. (3.1) with suitable boundary conditions. This approach is a useful one because some properties of $K(x,x')$ can be deduced directly from the differential equation and because it is more easily generalizable to the propagation of particles with higher spin. In this section we shall derive the boundary conditions for $K(x,x')$ from its path-integral definition.

First, we illustrate the procedure with the example of a massless scalar particle in flat space. The solution of the parameter time Schrödinger equation for F which satisfies the boundary condition of Eq. (2.10) is

$$F(W,x,x') = i(4\pi W)^{-2}e^{is(x,x')/(4W)}\,, \tag{3.2}$$

where $s(x,x')$ denotes the square of the Minkowski interval between x and x'. From Eq. (2.19) the propagator is then

$$K(x,x') = -\frac{i}{4\pi^2}\frac{1}{s(x,x')+i\epsilon}\,, \tag{3.3}$$

which is the correct Feynman propagator. In any coordinates in which the Minkowski metric is analytic, $s(x,x')$ will be an analytic function of the coordinates and $K(x,x')$ will also be analytic except at the poles where $s(x,x') = -i\epsilon$. These poles correspond to the null geodesics connecting x and x'. It is regularity at infinity plus the location of these poles in the complex coordinate plane that uniquely fixes $K(x,x')$ as a solution of the inhomogeneous wave equation. More concretely, in the usual rectangular Minkowski coordinates with $x=(t,\vec{x})$ and $x'=(t',\vec{x}')$, $K(x,x')$ has poles at $t-t' = \pm(|\vec{x}-\vec{x}'|-i\epsilon)$. Thus $K(x,x')$ is that solution of the inhomogeneous wave equation which is regular at infinity and for which the singularities corresponding to propagation along future-directed null geodesics lie below the real t axis, while those corresponding to propagation along the past-directed null geodesics lie above the real t axis. Elsewhere in the complex t plane $K(x,x')$ is analytic. We shall now consider the analogous boundary conditions for Eq. (3.1) in the Schwarzschild geometry.

To begin with let us consider the case in which x' is exterior to the black hole and x lies on the horizon. It is convenient to use null Kruskal coordinates U and V in which the Schwarzschild metric takes the form

$$ds^2 = -(32M^3e^{-r/2M}/r)dUdV + r^2d\Omega^2\,, \tag{3.4}$$

with r defined by

$$UV = (1-r/2M)e^{r/2M}\,. \tag{3.5}$$

On the horizon $r=2M$ and either $U=0$ or $V=0$. We analytically continue the nonzero member of the pair (U,V) to complex values and refer to the surface thus obtained as the complexified horizon. Since the metric is analytic in the Kruskal coordinates on the complexified horizon, the function $F(W,x,x')$ will also be analytic there. Any singularities in $K(x,x')$ will therefore come from the end points of the integration over W. From the asymptotic expression [Eq. (2.17)] for $F(W,x,x')$ at large values of W one easily sees that the integral converges for large W for all complex values of x. Any singularities of $K(x,x')$ must therefore come from the $W=0$ end point. To analyze these, divide the interval $[0,\infty]$ in W into two pieces $[0,W_0]$ and $[W_0,\infty]$, where W_0 is small. The integral from W_0 to infinity gives a contribution $K_0(x,x')$ to $K(x,x')$ which is analytic in x. In the part from 0 to W_0 our expression for the small-W behavior of $F(W,x,x')$ may be used wherever it is valid. The result for $K(x,x')$ when $m=0$ is

$$K(x,x') = K_0(x,x') - i\sum_c \frac{e^{is_c(x,x')/4W_0}}{s_c(x,x')+i\epsilon}D_c(x,x')\,. \tag{3.6}$$

This expression can be used to continue $K(x,x')$ to values of x off the complexified horizon. With $m\neq 0$, $K(x,x')$ also has singularities whenever $s_c(x,x') = -i\epsilon$, i.e., slightly displaced from wherever there is a null geodesic connecting x' to the complexified horizon. We shall now locate these points.

To start with we shall show that all null geodesics which start from real values of x' intersect the complexified horizon on the real section, i.e., at real values of U and V. For definiteness let us consider first the geodesics which connect a real x' exterior to the hole to the future horizon. Instead of the affine parameter V on the horizon it is convenient to use the Killing time v related to it by $V=\exp(\kappa v)$. This is just the familiar advanced time of the Eddington-Finkelstein coordinates which cover the region $V\geq 0$. Complex null geodesics may be represented by giving the four coordinates as functions of a complex affine parameter which we shall call λ. The geodesic is thus really a two-dimensional sheet in the complex coordinate space.

From the relation $V=\exp(\kappa v)$, a given value of v and one displaced from it by $\mathrm{Im}v=2\pi/\kappa$ correspond to the same value of V. A consequence of this is that by studying the null geodesics whose v coordinates are confined to a strip of width $2\pi/\kappa$ in the complex plane one learns about the null geodesics for all other values of v. It is convenient to choose a strip which includes the real v axis. We may then suppose that at $\lambda=0$ the coordinates assume the real starting values $v', r', \theta', \varphi'$. Our question is what complex values of v in this strip with real values of θ and φ and $r=2M$ lie on the two-dimensional sheet which represents a complex null geodesic?

Null geodesics in this stationary spherically symmetric spacetime may without loss in generality be taken to be in the equatorial plane, $\theta=\pi/2$. They are characterized by two constants of the motion e and l which may take complex values. The definitions of these constants are

$$e=\left(1-\frac{2M}{r}\right)\frac{dv}{d\lambda}, \quad l=r^2\frac{d\varphi}{d\lambda}\,. \tag{3.7}$$

Then from the null condition $g(\dot{x},\dot{x})=0$ we derive the familiar expressions

$$v-v'=\int_{r'}^{2M}\frac{dr}{1-2M/r}\left\{1-\frac{1}{[1-b^2r^{-2}(1-2M/r)]^{1/2}}\right\}, \tag{3.8}$$

and

$$\varphi-\varphi'=\int_{r'}^{2M}\frac{b\,dr/r^2}{[1-b^2r^{-2}(1-2M/r)]^{1/2}}\,. \tag{3.9}$$

Here we have written b for the impact parameter l/e. The multiplicative arbitrariness in the affine parameter λ implies that the invariant ratio b is sufficient to completely characterize a particular null geodesic.

The purely real geodesics connecting x' to the future horizon correspond to real values of b between 0 and $3\sqrt{3}M$. For these values, $r=r'$ and $r=2M$ can be connected by a purely real contour. For this reason for real b between 0 and $3\sqrt{3}M$ it is convenient to take the cuts of $f(r)=[1-b^2r^{-2}(1-2M/r)]^{1/2}$ to avoid the positive real r axis and to have $r=r'$ and $r=2M$ on the same sheet of the Riemann surface of f. The complex analytic structure of f for other values of b is then fixed by analytic continuation in that variable.

A given complex value of b and a given contour in the r plane connecting $r=r'$ with $r=2M$ should determine a unique complex null geodesic. For example, for every real b between 0 and $3\sqrt{3}M$ and a purely real contour between $r=r'$ and $r=2M$ there is a unique real null geodesic. For the same value of b a second contour which could not be obtained from the first by a smooth distortion would determine a different complex null geodesic.

However, it is easily verified from Eqs. (3.8) and (3.9) that there are no poles in the integrands of these equations to prevent one contour which connects $r=r'$ and $r=2M$ on the given sheet from being distorted into any other. The most convenient choice for the contour, therefore, is simply to take it to lie along the real axis provided that it does not intersect a cut of $f(r)$.

For a given contour the integral in Eq. (3.9) defines a function φ/b which is a multivalued function of b^2. In order to obtain a unique connection between φ and b it will be necessary to restrict b^2 to a given sheet of the Riemann surface of φ/b. We shall call this the physical sheet. This sheet must include the real axis between $b^2=0$ and $b^2=27M^2$ which corresponds to the physical real null geodesics. There is a branch point of the function φ/b at $b^2=27M^2$. It is therefore convenient to define the physical sheet as the plane cut along the positive real axis from $27M^2$ to infinity and containing the physical real values from $b^2=0$ to $27M^2$. It is then easily verified that the contour in Eq. (3.9) can always be chosen to lie along the real axis. With b^2 on the physical sheet the integral in Eq. (3.9) then defines a unique connection between φ and b.

Of interest in the present instance are null geodesics which have real values of φ at the end point. An elementary analysis of the integral in Eq. (3.9) shows that since the contour can be chosen real for b^2 on the physical sheet, the imaginary part of the integrand is always of one sign and the imaginary part of φ does not vanish unless b^2 is real and between 0 and $27M^2$. However, these values of b mean that v at the end point will also be real. Thus, complex null geodesics starting from x' intersect the complexified future horizon only for real values of V. A similar conclusion clearly holds for the past horizon. There are then singularities of K on the complexified horizon slightly displaced from the real values of U and V at which null geodesics from x' intersect the horizon according to the relation $s(x,x')=-i\epsilon$. We shall now determine the direction of these displacements.

Suppose that x_0 is the end point on the future horizon of a real null geodesic which starts at x'. If V_0 represents the value of V associated with x_0, then for values of V near to V_0, s will behave as

$$s(x,x')=\left(\frac{\partial s}{\partial V}\right)_{x_0}(V-V_0)+\cdots \ . \tag{3.10}$$

A positive value of $(\partial s/\partial V)_{x_0}$ means that the solution of $s(x,x')=-i\epsilon$ is in the lower half plane while a negative value means that it is in the upper half plane. To determine the correct sign let $k\delta V$ be the displacement vector from the null geodesic to a neighboring geodesic which starts at x' but ends on the future horizon a small affine parameter distance δV to the future of V_0. Thus on the horizon $k=\partial/\partial V$. If l is the tangent vector to the null geodesic then on the horizon $l\cdot k<0$. The equation of geodesic deviation implies that

$$d^2(l\cdot k)/d\lambda^2=0\ , \tag{3.11}$$

and this relation can be used to propagate $l\cdot k$ back along the null geodesic to $\lambda=0$, where both geodesics originate. One finds

$$l\cdot k=c\lambda\ , \tag{3.12}$$

where c is a negative constant. Since the tangent vector along the neighboring geodesic is the sum of l and $k\delta V$, Eq. (3.12) implies that the neighboring geodesic is timelike. Thus $s(x,x')$ is negative as x runs along the neighboring curve and $(\partial s/\partial V)_{x_0}<0$. The singularities of the propagator corresponding to $s(x,x')=-i\epsilon$ therefore lie in the upper half plane and the propagator will be analytic in the lower half V plane on the complexified horizon.

In a similar manner the analytic properties of the propagator on the past complexified horizon can be deduced. For x' located at a real point outside the black hole, and x on the complexified past horizon, $K(x,x')$ will be analytic in the upper half U plane.

The analytic properties which we have deduced from the path integral for the propagator on the complexified horizon may now be considered as boundary conditions which *define* the propagator as a particular solution of the inhomogeneous wave equation, Eq. (3.1). For all of our subsequent results we could have started from this definition of the propagator in terms of its analytic properties on the complexified horizon, but such a definition would lack the physical motivation which our definition in terms of the path integral gives.

The inhomogeneous scalar wave equation together with the boundary conditions just deduced may be used to derive the analytic properties of the propagator for regions other than the complexified horizon. To complete the program outlined in the Introduction we shall be concerned in particular with the analytic properties in the Schwarzschild coordinate t. This is connected to the null coordinates U and V through the relations

$$\left.\begin{aligned}U&=(1-r/2M)^{1/2}e^{(r-t)/4M}\\ V&=(1-r/2M)^{1/2}e^{(r+t)/4M}\end{aligned}\right\}\ U>0,\quad V>0\quad\text{(region II)} \tag{3.13a}$$

$$\left.\begin{aligned}U&=-(r/2M-1)^{1/2}e^{(r-t)/4M}\\ V&=(r/2M-1)^{1/2}e^{(r+t)/4M}\end{aligned}\right\}\ U<0,\quad V>0\quad\text{(region I)} \tag{3.13b}$$

and similar relations with the signs of U and V changed in the quadrants reflected in the origin of the U-V plane. These relations are indicated schematically on a Penrose diagram in Fig. 3.

For definiteness let us first consider the case when x' is exterior to the black hole and x is in region II. The portion of the future horizon with

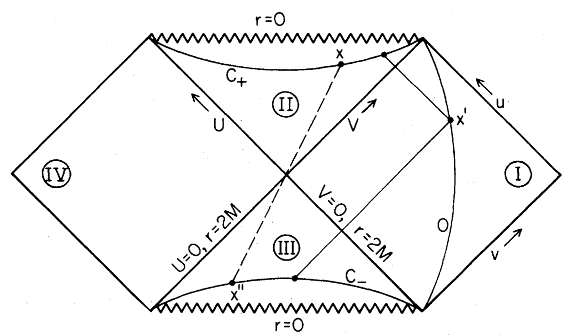


FIG. 3. A Penrose diagram for the Schwarzschild geometry. The amplitude for a black hole to emit particles which are detected in a mode of energy E by an observer in region I may be related [Eq. (4.4)] to the integral of $\exp(-iEt)$ times the propagator to go from a point x on a surface C_+ of constant r in region II to a point x' on the detector's world line in region I. The integral is over the coordinate t on C_+. The propagator is analytic in the coordinate t except for singularities at those values where a null geodesic from x intersects the complexified surface C_+. One of these values is the real value of t corresponding to the intersection with C_+ of the radial real future-directed null geodesic from x. As shown in the text, if x has a time coordinate with an imaginary part $-4\pi M$, it actually corresponds to the point x'' in region III, which is x reflected in the origin. There is thus another singularity in t with imaginary part $-4\pi M$ corresponding to the radial real past-directed null geodesic from x intersecting the surface C_- which is the reflection of C_+. The two singularities just discussed are repeated at intervals of $8\pi M$ in Imt. The location of the singularity is illustrated in Fig. 4(a).

$V \geq 0$ together with the part of the past horizon with $U \geq 0$ are a complete characteristic Cauchy surface for region II. The propagator in the interior region is uniquely determined through the differential equation by the initial data on this Cauchy surface. These are just the values of the propagator on the relevant parts of the horizon.

Complex values of t with r, θ, and φ fixed correspond to certain complex values of U and V according to Eq. (3.13a). If we let $t = \tau + i\sigma$ then in particular

$$U = |U| e^{-i\sigma/4M}, \quad V = |V| e^{i\sigma/4M}. \qquad (3.14)$$

The problem of determining the propagator at a certain complex value of t may be considered as a problem of solving the wave equation, Eq. (3.1), for a fixed value of σ in the real coordinates $|U|$ and $|V|$. Since the metric is independent of t the equation is hyperbolic for any value of σ and the characteristic initial-value problem is well posed. The analyticity of the propagator on the complexified horizon in the upper half U plane and in the lower half V plane implies that the Cauchy data for this real problem are regular provided $-4\pi M < \sigma < 0$. The standard existence and uniqueness theorem for the hyperbolic characteristic initial value problem guarantees that there will be a solution for the propagator for this range of σ. To determine whether the resulting solution is analytic in t we need only verify that the Cauchy-Riemann condition is satisfied. This is (a bar denotes complex conjugation)

$$\left(\frac{\partial K}{\partial \bar{t}}\right)_r = \frac{1}{4M}\left(\bar{V}\frac{\partial K}{\partial \bar{V}} - \bar{U}\frac{\partial K}{\partial \bar{U}}\right) = 0\,, \qquad (3.15)$$

where the derivative with respect to $\bar{t}$ is being taken at constant r. Evidently this condition is satisfied by the data for $-4\pi M < \sigma < 0$ since K is an analytic function of U and V in the appropriate half planes on the complexified horizon. Furthermore $(\partial/\partial\bar{t})_r$ commutes with $\Box^2$ so that determining $(\partial K/\partial \bar{t})_r$ may be regarded as a problem of solving the wave equation with zero data on the characteristic Cauchy surfaces. The unique answer to this problem is $(\partial K/\partial\bar{t})_r = 0$. One concludes, therefore, that for x' in the exterior region and fixed r, θ, φ in the region $U > 0$, $V > 0$, $K(x, x')$ is analytic in t in a strip of width $4\pi M$ below the real axis.

The strip of analyticity cannot be extended above the real axis because immediately above it there are singularities corresponding to the real null geodesics which connect a value of t on the surface of given r to x' (see Fig. 3). The strip of analyticity cannot be pushed below $\sigma = -4\pi M$ either. From Eq. (3.14), this value of σ corresponds to a U and V which are again in the real section but reversed in sign. The propagator $K(x, x')$ with x in region II when continued in t to $t - 4\pi M i$ then equals the propagator from a point x'' in region III to x' in the exterior region. The point x'' is just x reflected in the origin of the U-V plane. This identity will be the basis of our derivation of black-hole radiance in the next section, but for the present we simply note that it implies that immediately below the line $\sigma = -4\pi M$ there are singularities corresponding to the real null geodesic which connect a point on the surface of constant r in region III to x'. The singularities in this case lie below the real axis because x'' lies in region III and from the relations analogous to Eq. (3.14) the small positive imaginary value of U and a negative value of V which locate the pole correspond to a negative imaginary value of t.

In this way the analytic properties of the propagator $K(x, x')$ in the variable t become apparent. For fixed θ, φ, r in region II with x' located in region I they are illustrated in Fig. 4(a). The propagator is periodic in $\sigma = \mathrm{Im}(t)$ with period $8\pi M$. The regions of analyticity corresponding to the upper half U plane and lower half V plane on the complexified horizon are the shaded strips of width $4\pi M$. There are periodic singularities corresponding to the real null geodesics which connect x' to a point on the curve of given θ, φ, r either in the past or in the future.

If x and x' are both located in region I the propagator is still periodic in σ with period $8\pi M$ as a consequence of Eq. (3.13b). Now, however, there are real values of t both in the past and in the future of x' for which there are real null geodesics connecting it to the fixed r, θ, φ curve. For the values to the future of x' the corresponding singularities are displaced slightly above the real t axis. For the values of t to the past, the corresponding singularities are displaced slightly below the real t axis. These singularities are shown in Fig. 4(b). There are no singularities corresponding to real null geodesics near $\sigma = \pm 4\pi M$ since the corresponding real points lie in region IV, every point of which is separated by a spacelike interval from x'.

The propagator $K(x, x')$ is periodic in imaginary t because it follows from the path-integral definition that the propagator is an analytic function of the Kruskal coordinates U and V except at the singularities we have described. However, the Schwarzschild coordinate, t, has a logarithmic singularity as a function of U and V and is multivalued; it is defined only modulo $8\pi i M$. Thus if the propagator has a singularity at some value of U and V, it will have periodic singularities at intervals of $8\pi i M$ when expressed as a function of t. The propagator is similar to that suggested by Unruh.[15] By contrast, the propagator proposed by

Boulware[14] is not periodic in t because it is not analytic on the two horizons.

The existence of periodic singularities in t implies that observers moving on lines of constant r, θ, φ in the extended Schwarzschild solution will detect particles. This is very similar to the fact that, as Unruh has pointed out, observers moving on world lines of uniform acceleration in Minkowski space will also detect particles. On the other hand, an observer moving along either of the two horizons will not see any particles. This is an illustration of the fact that the concept of particles is observer-dependent.[16] In the next section we shall show that the propagator constructed here will give for observers at a constant distance from the black hole the same rate of particle production as was obtained in Ref. 5 through a study of the mixing of positive and negative frequencies in a gravitational collapse.

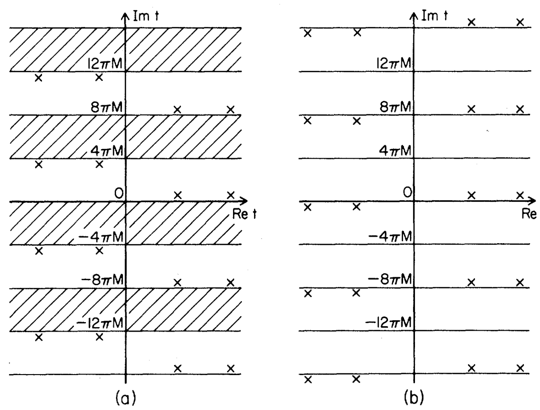


FIG. 4. The analytic structure of the propagator in the complex t plane. (a) shows the analytic structure of $K(x, x')$ for x' fixed outside the black hole (region I) and fixed values of r, θ, φ inside the future horizon (region II). The propagator is periodic in $\mathrm{Im}(\sigma)$ with period $8\pi M$ as a consequence of the relation of t [Eq. (3.14)] to the coordinates U and V in which the metric is analytic everywhere except at the physical singularity. The shaded regions are the regions of analyticity in t which are deduced from the analyticity of the propagator in the upper half U plane and lower half V plane on the complexified horizon. The crosses locate the singularities which correspond to the real null geodesics which connect x' to the curve of constant r, θ, φ. A typical situation is illustrated in Fig. 3. There are singularities immediately above the real axis corresponding to the null geodesics which connect x' to the fixed r, θ, φ curve in region II. There are singularities below the $\mathrm{Im}\,t=-4\pi M$ line corresponding to the real geodesics which connect x' to the fixed r, θ, φ curve in region III. In each case there is an infinite sequence of singularities (only a few of which are shown) which arises because there are null geodesics which spiral an arbitrary number of times near $r=3M$ and thus can connect the fixed r, θ, φ curve to x' at increasingly large values of $|t-t'|$. The singularities at other values of $\mathrm{Im}\,t$ are duplicates of these as a consequence of the periodicity of the propagator in $\mathrm{Im}\,t$ with period $8\pi M$. (b) shows the similar analytic structure when both x and x' are in Region I. The propagator remains periodic in $\mathrm{Im}\,t$ with period $8\pi M$. There are now infinite sequences of null geodesics which connect x' to a curve of fixed r, θ, φ in the future and in the past. Correspondingly there are singularities above and below the real t axis. These are periodically repeated in $\mathrm{Im}\,t$ with period $8\pi M$.

IV. BLACK-HOLE RADIANCE

In the preceding section we demonstrated the analyticity of the propagator $K(x, x')$ in a strip in the t plane when x is in region II and x' is in region I of the Schwarzschild geometry. We shall now use this analyticity to derive the thermal radiation from a Schwarzschild black hole.

Suppose we surround a Schwarzschild black hole by particle detectors at some large constant radius R. These detectors measure particles coming out from inside the surface in modes $f_j(t', r', \theta', \varphi')$ which are purely positive-frequency (with respect to t') solutions of the scalar wave equation. The amplitude that a particle is detected in a mode $f_i(x')$ having started in a mode $h_j(x)$ on some surface which bounds a region interior to R is

$$-\int d\sigma^{\mu}(x')\int d\sigma^{\nu}(x)\bar{f}_i(x')\overleftrightarrow{\partial}_{\mu}K(x', x)\overleftrightarrow{\partial}_{\nu}h_j(x), \tag{4.1}$$

where the integral over x' is taken over the surface $r'=R$ and the integral over x is over the interior bounding surface. The notation $a\overleftrightarrow{\partial}_{\mu}b$ means $ab_{,\mu}-a_{,\mu}b$.

Suppose for the moment that the particle detectors are confined to a time interval $t'\in(-t_1', t_1')$, where t_1' is very large. Eventually the limit $t_1'\to\infty$ will be taken. The interior bounding surface mentioned above can then be taken to be a spacelike surface through the precollapse star at $-t_1'$, a complete spacelike surface inside the future horizon and a timelike surface connecting this to the $r'=R$ surface outside at $t'=t_1'$. The spacelike surface inside the future horizon will be taken to be part of a constant-r surface C_+ outside the matter and a spacelike extension inside it. The complete spacelike surface inside the future horizon could have been chosen to be the future singularity were it not convenient to avoid mathematical complications associated with the singularity in the metric at $r=0$ by keeping it away from those points.

We now calculate the total probability that a particle is measured by a detector in a mode $f_i(x')$ which is peaked in time about some late time t_0'. This probability will be the sum of the square of the amplitude in Eq. (4.1) over all modes $h_j(x)$ which are consistent with our knowledge of the bounding surface. There will be no contribution from modes which are localized on the spacelike part at $t'=-t_1'$ because we are assuming that there are no scalar particles in the initial state. The contribution of the other two surfaces is significantly restricted by the fact that the propagator $K(x,x')$ in the exterior geometry of a collapsing star at late times will be a function only of the difference $t'-t$ as a consequence of the time transition invariance of the Schwarzschild geometry. Thus if $f_i(x')$ is peaked about a late time t_0' only times t comparable to t_0' will contribute in Eq. (4.1). In particular there will be no contribution from the timelike surface which starts at t_1' since by choosing t_1' large enough this only intersects values of t much larger than t_0'. Furthermore, the only part of the spacelike surface inside the horizon which contributes is the part with $t\approx t_0'$ and for sufficiently late times this is well outside the matter. The conclusion of this is that if one is interested in the probability of production of particles at late times the details of the collapse may be ignored. The only part of the amplitude in Eq. (4.1) which contributes to the probability of particle production is that in which the integral over x is taken over the complete spacelike surface inside the horizon, and this can be idealized as a surface C_+ of constant r between 0 and $2M$ in the exact Schwarzschild geometry. We are explicitly assuming here that the propagator between a point interior to the future horizon and a point outside the black hole in the geometry of a collapsing star is well approximated at late times by the propagator we have obtained in the analytically extended Schwarzschild metric.

To compute the total probability that a particle is detected we next note that there is no information on the state of the particle on the future singularity. The total probability is obtained by summing the modulus squared of Eq. (4.1) over a complete set of states on C_+. It is not necessary to carry this sum out in detail to derive the black-hole radiance as we shall now show.

Of chief interest is the amplitude for a black hole to emit a mode with a definite positive energy E. The time dependence of such a mode is $f\sim\exp(-iEt')$. Because of the time translation invariance the modes $h_i(x)$ in the complete set on C_+ may also be classified into modes with the time dependence $\exp(-iEt)$, although since t is a spacelike coordinate inside the horizon, E is not to be interpreted as a local energy. The fact that K is a function only of the difference $t'-t$ means that the integral over t and t' in Eq. (4.1) will lead to an energy-conservation δ function. When probabilities are computed the formal square of the δ function will be replaced by a density-of-states factor in the usual way. Factoring out this energy-conservation δ function there remains of Eq. (4.1).

$$-\int d\sigma(\vec{R}')\int d\sigma(\vec{R})\Big[\bar{f}_i(\vec{R}')\frac{\overleftrightarrow{\partial}}{\partial r}\mathcal{E}_E(\vec{R}',\vec{R})\frac{\overleftrightarrow{\partial}}{\partial r}h_j(\vec{R})\Big], \tag{4.2}$$

where $\vec{R}$ and $\vec{R}'$ denote the coordinates $r,\ \theta,\ \varphi$ and $r',\ \theta',\ \varphi'$ respectively, $f_i(\vec{R})$ and $h_j(\vec{R})$ denote the angular parts of the respective modes, and $d\sigma(\vec{R})$ and $d\sigma(\vec{R}')$ are appropriately weighted angular integrals. The crucial information about the emission is contained in the amplitude $\mathcal{E}_E$ defined by

$$\mathcal{E}_E(\vec{R}',\vec{R})=\int_{-\infty}^{+\infty}dt\,e^{-iEt}K(0,\vec{R}';t,\vec{R})\,. \tag{4.3}$$

Making use of the symmetry of $K(x,x')$ under interchange of x and x' this can also be written in what will be the more convenient form

$$\mathcal{E}_E(\mathrm{R}',\mathrm{R})=\int_{-\infty}^{+\infty}dt\,e^{-iEt}K(t,\vec{R};0,\vec{R}')\,. \tag{4.4}$$

The amplitude $\mathcal{E}_E$ is the component with energy E of the amplitude to propagate from the surface C_+ to a point $(0,\vec{R}')$ outside the black hole.

Following the program outlined in the Introduction we now wish to relate the amplitude for emission as contained in Eqs. (4.2) and (4.4) to an amplitude for the black hole to absorb a particle in the same mode by distorting the contours of integration into the complex plane. To do this it is enough to concentrate on the amplitude $\mathcal{E}_E(\vec{R}',\vec{R})$ and distort the contour of the t integration in Eq. (4.4) downward by an amount $-4\pi Mi$. This distortion is permissible since the main result of the preceding section is that $K(t,\vec{R};0,\vec{R})$ is analytic in a strip of width $4\pi M$ below the real axis. Equation (4.4) becomes

$$\mathcal{E}_E(\vec{R}',\vec{R})=e^{-\pi E/\kappa}\int_{-\infty}^{+\infty}dt\,e^{-iEt}K(t-i\pi/\kappa,\vec{R};0,\vec{R}')\,, \tag{4.5}$$

where we have written the surface gravity of the black hole κ instead of $1/4M$. Equation (3.14) shows that translating t by an amount $-i\pi/\kappa$ is equivalent to reflecting the Kruskal coordinates U and V in the origin. The integral in Eq. (4.5) can thus be interpreted as the component with energy E of the amplitude to propagate to a point $(0,\vec{R}')$ outside the black hole from the surface C_- in region III, which is C_+ reflected in the origin of the U-V plane. If this integral is inserted in

Eq. (4.2) in place of $\mathcal{E}_E$ we obtain minus the amplitude for a particle to be detected in a mode f_i of definite energy E having started on C_- in region III in a mode h_j with the same energy. The minus sign occurs because the appropriate normal to the surface C_- is reversed. By time-reversal invariance the modulus squared of this amplitude is exactly equal to the modulus squared of the amplitude for the black hole to *absorb* a particle which starts at $(0, \vec{R}')$ in a mode f_i with energy E and arrives at C_+ in a mode h_j with the same energy. When the sum over the complete set of states h_j is performed we have the general relation

(probability for a Schwarzschild black hole to emit a particle in a mode with energy E)

$$= e^{-2\pi E/\kappa} \times (\text{probability for a Schwarzschild black hole to absorb a particle in the same mode}). \quad (4.6)$$

This is the fundamental connection between emission and absorption stated in the Introduction. This connection shows that a black hole will emit particles with a thermal spectrum characterized by a temperature $T = \kappa/2\pi k$. Thus we recover[17] the result of Ref. 5.

V. ROTATION, CHARGE, AND SPIN

In this section we shall comment on the generalization of our results to particles with higher spin and to black holes with rotation and charge.

For fields of spin greater than zero it is difficult to express the propagator in terms of an integral over paths. However, we shall assume that the analytic properties of the higher-spin propagators are the same of those we have derived for the scalar field. Namely, we shall assume that if U and V are affine parameters along the past and future horizons, respectively, both increasing toward the future, the propagator from a point outside the black hole to a point on the horizon is analytic in the upper half U plane and in the lower half V plane. If this assumption is taken then the generalization of our results to higher-spin particle is immediate since the derivation of black-hole radiance given in the preceding section for Schwarzschild black holes and that to be given below for more general black holes depend only on this analyticity property and the correct construction of the emission and absorption amplitudes. We now proceed to the generalization of our argument to the Kerr and Reissner-Nordström black holes. For simplicity we treat these two cases separately, leaving it to the reader to join the arguments together for the general rotating charged black hole.

A. The Kerr black hole

Figure 5(a) shows the familiar Penrose diagram for the axis of a Kerr black hole. Following our argument for the Schwarzschild case, the amplitude for the black hole to emit a particle can be related to an integral of the propagator $K(x, x')$ in which x' is fixed outside the black hole and x is integrated over a spacelike surface interior to the horizon which divides the past and future. It is convenient to call this surface C_+ and take for it a surface of constant r in the usual Boyer-Lindquist coordinates such that $r_- < r < r_+$. In order to generalize our result to the Kerr geometry we shall thus need a set of coordinates in which the metric is regular over at least regions I, II, and III shown in Fig. 5(a). Fortunately Carter[18] has given such a set of coordinates. To avoid lengthy redefinition we shall use his notation wherever it does not differ from that used elsewhere in this paper. The reader is referred to Carter's paper for symbols not defined here.

Region I in the Kerr geometry can be covered by Boyer-Lindquist coordinates (t, r, θ, φ) with $r > r_+$ (Carter uses $\hat{t}, \hat{\varphi}$ for our t, φ). Regions II and III can be covered by a similar patch with $r_- < r < r_+$. Regions I and II can be covered by a coordinate patch of the Kerr-Newman form involving an advanced time v. Similarly regions I and III can be covered by a patch involving a retarded time u. (Carter uses u for our v and $-w$ for our u.) In region I these coordinates are connected by rela-

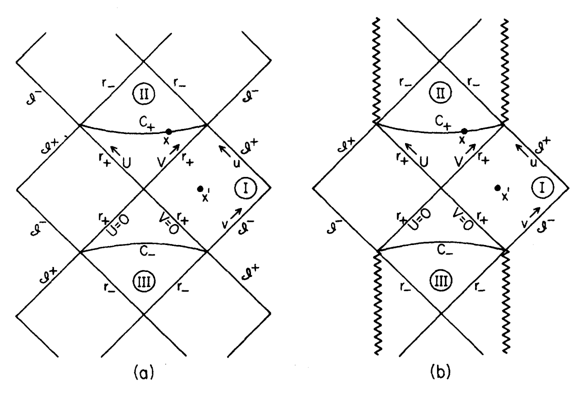


FIG. 5. (a) The Penrose diagram for the axis of a Kerr black hole. (b) The Penrose diagram for a Reissner-Nordström black hole.

tions of the form

$$t=u+f(r), \tag{5.1}$$

$$t=v-f(r), \tag{5.2}$$

where $df/dr=(r^2+a^2)/\Delta$. Carter then introduces a new azimuthal Killing angle φ^+ defined by

$$\varphi^+=\varphi-\omega_+ t, \tag{5.3}$$

where $\omega_+=a/(r_+^2+a^2)$ is the angular frequency of the horizon. He also introduces two new null coordinates x and y which correspond to our U and V. These are defined as

$$U=-e^{-\kappa u},\quad V=e^{\kappa v},\quad \text{region I}, \tag{5.4a}$$

$$U=e^{-\kappa u},\quad V=e^{\kappa v},\quad \text{region II}, \tag{5.4b}$$

$$U=-e^{-\kappa u},\quad V=-e^{\kappa v},\quad \text{region III}, \tag{5.4c}$$

where $\kappa=\frac{1}{2}(r_+-r_-)/(r_+^2+a^2)$ is the surface gravity of the black hole. Thus V is Carter's x while U is the negative of Carter's y. In this new (U, V, r, φ^+) patch the metric is analytic in regions I, II, and III [see Carter's Eq. (26)]. The future horizon is located at $U=0$ and V is an affine coordinate along it. The past horizon is at $V=0$ and U is an affine coordinate along it. The coordinate r may be considered to be defined in terms of U and V by Eqs. (5.1), (5.2), and (5.4). Figure 5(a) gives a schematic representation of the various definitions.

In the Schwarzschild case the amplitude for a black hole to emit a particle with energy E was ultimately related to an integral $\mathcal{E}_E(\vec{R}',\vec{R})$ over a surface of constant r inside the future horizon of $\exp(-iEt)$ times the propagator to go from a point x on that surface to a point x' outside the hole. In the Kerr case we will be interested in the amplitude for the emission of a particle of energy E and an angular momentum along the axis of rotation m (m is not to be confused with the rest mass). In a similar fashion this amplitude can be related to an integral of the form

$$\mathcal{E}_{Em}(\vec{R}',\vec{R})=\int_{-\infty}^{+\infty}dt\int_0^{2\pi}d\varphi\, e^{-i(Et-m\varphi)}K(t,\varphi,\vec{R};0,0,\vec{R}'). \tag{5.5}$$

Here K is the propagator in the Kerr geometry which, because of the time-translation invariance and axial symmetry, depends only on the difference in the t and φ coordinates of x and x'. The quantities $\vec{R}$ and $\vec{R}'$ stand for the r, θ coordinates of x and x'. The integral is over the surface C_+. Arguments similar to those given in the Schwarzschild case will show that, for fixed r, θ, φ^+, K is analytic with U in the upper half plane and V in the lower half plane. We can therefore distort the contour of the t integration downward by an amount $-i\pi/\kappa$ keeping r, θ, φ^+ fixed since that amounts to rotating U by an angle π and V by an angle $-\pi$. Keeping φ^+ fixed means from Eq. (5.3) that $\varphi\to\varphi-i\pi\omega_+/\kappa$ in the process. Thus,

$$\mathcal{E}_{Em}(\vec{R}',\vec{R})=e^{-\pi(E-m\omega_+)/\kappa}\int_{-\infty}^{+\infty}dt\int_0^{2\pi}d\varphi K(t-i\pi/\kappa,\varphi-i\pi\omega_+/\kappa,\vec{R};0,0,\vec{R}'). \tag{5.6}$$

Since this displacement of t is equivalent to [see Eq. (5.4)] $U\to-U$ and $V\to-V$ the integration is now over the reflected surface of constant r which is in region III and which we have shown as C_- in Fig. 5(a). The remaining integral can be related to the amplitude for the black hole to absorb a particle of energy E and angular momentum m. Thus we have

(probability for a Kerr black hole to emit a mode with energy E and angular momentum m)

$$=e^{-2\pi(E-m\omega_+)/\kappa}\times\text{(probability for a Kerr black hole to absorb a mode with } E \text{ and } m). \tag{5.7}$$

This is exactly the relation necessary to establish that a rotating black hole will emit particles with an expected number per mode proportional to $\{\exp[(E-m\omega_+)2\pi/\kappa]-1\}$ so that $\kappa/2\pi k$ may be interpreted as the black hole's temperature.[5,6]

B. Reissner-Nordström black hole

The situation with the Reissner-Nordström black hole is similar. Here, however, we shall investigate the amplitude for the black hole to emit particles of charge q. Figure 5(b) shows the Penrose diagram for the Reissner-Nordström geometry. The propagator for a particle of charge q is a solution of the wave equation

$$g^{\alpha\beta}(\nabla_\alpha-iqA_\alpha)(\nabla_\beta-iqA_\beta)K(x,x')=-\delta(x,x'), \tag{5.8}$$

in the Reissner-Nordström background geometry. In the usual gauge the only nonvanishing component of A_μ is

$$A_t(x)=e/r, \tag{5.9}$$

where e is the charge on the black hole. However, such a gauge K cannot be expected to be analytic in the upper half U plane or lower half V plane as is required by our boundary conditions because the components of $A_\mu(x)$ will not be analytic on the horizon in coordinates which are analytic there. For example, if we use the (u, r, θ, φ) coordinates which are analytic on the future horizon, $A_r(x) = (e/r)(1 - 2M/r + e^2/r^2)^{-1}$, which diverges at $r = r_+$. A gauge in which A_μ is stationary and regular on both horizons can be found by making the transformation

$$A_\mu \to A_\mu + \Lambda_{,\mu}, \tag{5.10}$$

where

$$\Lambda = \Phi t, \tag{5.11}$$

and Φ is the potential on the horizon equal to e/r_+. In the new gauge A_μ will be analytic in the same domain as the metric but it will not vanish at infinity. This means that the time dependence of a mode of energy E at large distance from the hole will not be $\exp(-iEt)$ but rather $\exp[-i(E - q\Phi)t]$.

As in the Schwarzschild and Kerr cases the amplitude for the black hole to emit a particle with energy E and charge q can be expressed in terms of an integral of the propagator over a constant-r surface C_+ which lies between the future horizon at r_+ and the inner horizon at r_-. By the time invariance of the propagator the amplitude for emission depends on the integral

$$\mathcal{E}_E(\vec{R}', \vec{R}) = \int_{-\infty}^{+\infty} dt\, e^{-i(E-q\Phi)t}\, K(t, \vec{R}; 0, \vec{R}'), \tag{5.12}$$

where $\vec{R}$ stands for the coordinates r, θ, φ.

In the new gauge K will be analytic in the upper half U plane and the lower half V plane. The coordinates U and V here are related to those of Carter as described in Eq. (5.4) and above. As in the Kerr case this analyticity implies that the contour of the t integration in Eq. (5.12) may be distorted downward in the complex t plane by an amount $-i\pi/\kappa$ keeping r fixed since this amounts to rotating U by π and V by π. Thus,

$$\mathcal{E}_E(\vec{R}', \vec{R}) = e^{\pi(E-q\Phi)/\kappa} \int_{-\infty}^{+\infty} dt\, e^{-i(E-q\Phi)t} \times K(t - i\pi/\kappa, \vec{R}; 0, \vec{R}'). \tag{5.13}$$

Since displacing t by $-i\pi/\kappa$ is equivalent to the reflection $U \to -U$ and $V \to -V$ this integral may be written

$$\mathcal{E}_E(\vec{R}', \vec{R}) = e^{\pi(E-q\Phi)/\kappa} \int_{-\infty}^{+\infty} dt\, e^{-iEt} K(t, \vec{R}; 0, \vec{R}'). \tag{5.14}$$

where the integral is now over the reflected surface C_- illustrated in Fig. 5(b). The integral in Eq. (5.14) can be related, as before, to the amplitude for the black hole to absorb a particle of charge q. Following through the arguments which led to Eq. (4.6) we have

(probability for a Reissner-Nordström black hole to emit a particle of charge q and energy E
$= e^{-2\pi(E-q\Phi)/\kappa} \times$ (probability for a Reissner-Nordström black hole
to absorb a particle of charge q and energy E). (5.15)

This is exactly the relation necessary to establish that a rotating black hole will emit scalar charged particles with an expected number per mode proportional to $\{\exp[(E - q\Phi)/kT] - 1\}^{-1}$, where $kT = \kappa/2\pi$. (See Refs. 5 and 19.)

APPENDIX: DERIVATION OF THE DIFFUSION EQUATION FOR $F(\Omega, x, x')$

The amplitude $F(\Omega, x, x')$ is defined by the path integral

$$F(\Omega, x, x') = \int \delta x[\omega] \exp\left[-\tfrac{1}{4}\int_0^\Omega \gamma(\dot{x}, \dot{x}) d\omega\right]. \tag{A1}$$

where γ is the positive-definite metric of Eq. (2.6), $\dot{x} = dx/d\omega$, and the sum ranges over all paths with $x(0) = x'$ and $x(\Omega) = x$. We shall now derive the diffusion equation [Eq. (2.8)] for F and in the process discuss the interpretation of the differential $\delta x[\omega]$.

Divide the interval $[0, \Omega]$ up into $N+1$ intervals each ϵ long. With a natural assignment of the weight to the integrals over spacetime the path integral in Eq. (2.1) may be interpreted as

$$F(\Omega, x, x') = \lim_{N\to\infty} \int \frac{d^4x_N}{A} [\gamma(x_N)]^{1/2} \int \frac{d^4x_{N-1}}{A} \cdots \int \frac{d^4x_1}{A} [\gamma(x_1)]^{1/2} \exp\left[\sum_{i=0}^{N} S(\epsilon, x_{i+1}, x_i)\right], \tag{A2}$$

where $x_0 = x'$, $x_{N+1} = x$, and S is given by integral

$$S(\epsilon, x_{i+1}, x_i) = \tfrac{1}{4}\int_0^\epsilon d\omega\, \gamma(\dot{x}, \dot{x}), \tag{A3}$$

evaluated along the geodesic path which connects x_i at $\omega = 0$ with x_{i+1} at $\omega = \epsilon$. The constant A is a normalization fixed by the requirement that the amplitude for a particle to propagate from one point to *any* other point in the spacetime is unity. This is equivalent to

$$A = \int d^4x [\gamma(x)]^{1/2} \exp[-S(\epsilon, x, x')]. \tag{A4}$$

The reason that Eq. (A2) is correct is that as ϵ becomes smaller and smaller the action for the paths which connect fixed x_i to x_{i+1} will become larger and larger and hence their contribution to the integral will be exponentially damped. The dominant contribution will come from the stationary path for which S is a minimum. This is a geodesic between the points.

The diffusion equation can be derived by considering the relation between $F(\Omega + \epsilon, x, x')$ and $F(\Omega, x, x')$. From Eq. (A2) this is

$$F(\Omega + \epsilon, x, x') = \int d^4y [\gamma(y)]^{1/2} \exp[-S(\epsilon, x, y)] \times F(\Omega, y, x')/A. \tag{A5}$$

Write $y = x + z$ and let the integration be over z. On the right expand S and F in powers of z. On the left expand F in powers of ϵ. Analysis of the integral shows that only the first few terms of the expansion on the right contribute to the part of F linear in ϵ giving an expression for $\partial F/\partial\Omega$. This analysis has been carried out by Cheng in a general coordinate system.[4] However, the calculations are considerably simplified if a Riemann normal coordinate system is introduced at x. In such a coordinate system

$$\gamma_{\alpha\beta}(y) = \delta_{\alpha\beta} - \tfrac{1}{3}R_{\alpha\gamma\beta\delta}z^\gamma z^\delta + O(z^3). \tag{A6}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker δ. From the definition of normal coordinates it follows that the geodesics from the origin are straight lines so that $\dot{x}^\alpha = z^\alpha/\epsilon$ and

$$S(\epsilon, x, x + z) = \tfrac{1}{4}\delta_{\alpha\beta}z^\alpha z^\beta/\epsilon. \tag{A7}$$

From the form of S we see that the only significant part of the integral as $\epsilon \to 0$ comes from the region where $z^\alpha \sim \epsilon^{1/2}$. Thus only terms $O(z^2)$ in the expansion of $(\gamma)^{1/2}$ and F need to kept under the integral and the limits may be extended to infinity. Then, the zeroth-order term in an expansion of Eq. (A5) gives the normalization condition

$$A = \int d^4z \exp(-\tfrac{1}{4}\delta_{\alpha\beta}z^\alpha z^\beta/\epsilon) = (4\pi\epsilon)^2. \tag{A8}$$

Because the odd integrations in z vanish, the first-order term in ϵ is

$$\epsilon\frac{\partial F}{\partial\Omega} = B^{\alpha\beta}(\tfrac{1}{2}F_{,\alpha\beta} - \tfrac{1}{6}R_{\alpha\beta}F). \tag{A9}$$

where the curvature term comes from the expansion

$$\gamma^{1/2} = 1 - \tfrac{1}{6}R_{\alpha\beta}z^\alpha z^\beta + O(z^3), \tag{A10}$$

and $B^{\alpha\beta}$ is the integral

$$B^{\alpha\beta} = \int d^4z (z^\alpha z^\beta/A) \exp(-\tfrac{1}{4}\delta_{\mu\nu}z^\mu z^\nu/\epsilon) = 2\epsilon\delta^{\alpha\beta}. \tag{A11}$$

Using this expression we finally arrive at the diffusion equation for F by replacing the partial derivatives in the normal coordinates by covariant derivatives $\tilde{\nabla}_\alpha$ with respect to the metric γ. The equation is

$$\frac{\partial F}{\partial\Omega} = (\gamma^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta - \tfrac{1}{3}R)F. \tag{A12}$$

This is exactly Eq. (2.8) taking into account the fact that R vanishes for Schwarzschild geometry.

The factor $-R/3$ in Eq. (A12) is a consequence of our particular choice of weight in the coordinate integrals in Eq. (A2). If we had replaced $\exp[-S(\epsilon, x_{i+1}, x_i)]$ in the integrals with

$$[\gamma(x_{i+1})/\gamma(x_i)]^{s/2} \exp[-S(\epsilon, x_{i+1}, x_i)],$$

then the resulting equation would have been

$$\frac{\partial F}{\partial\Omega} = \left[\gamma^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + \tfrac{1}{3}(s-1)R\right]F. \tag{A13}$$

Thus any amount of scalar curvature can be had both here and in the equation for K by the appropriate choice of the action. For the vacuum black-hole solutions we are considering R vanishes and these equations are all identical and we will not consider this issue further. However, see the remarks in Ref. 3.

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