Hawking radiation and thin shells*

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The properties of quantized scalar and Dirac fields around a collapsing thin shell are discussed and the Hawking radiation exhibited. The radiation is seen to be a result of the collapse process which involves the emission of a pair of particles traveling on either side of the event horizon from the shell. The reaction back on the shell is discussed and it is shown that the radiation only reacts through the metric. The contributions to the stress-energy tensor associated with the radiation are exhibited.

I. INTRODUCTION

In a recent paper Hawking[1] has shown that if the reaction back on the star and metric are neglected, a star collapsing to form a black hole produces thermal radiation characterized by a temperature $T=\hbar c^3/8\pi GMk=0.6\times 10^{-7}\ {}^\circ\mathrm{K}(M_\odot/M)$ and appearing continuously at late times, thus (when taken to an impossible limit) carrying an infinite amount of energy. The radiation is totally independent of the nature of the collapse process; Hawking's argument depends only on the assumed transparency of the star and the space surrounding the event horizon to extremely energetic quanta. During the past year, this radiation has been worked on by many authors[2-10] from a variety of points of view.

In two previous publications, I studied the quantum fields around primordial, Kruskal black holes and showed that there was no radiation; the system both for a scalar[11] and a Dirac[12] field possessed a stable vacuum, and there was no radiation associated with the existence of the event horizon. Here the simplest "physical" system embodying the collapse is studied: a thin shell.

The physical picture which emerges as a result of the considerations of this paper and the other work[1-10] is that there is a flux of particles emerging from the surface of the collapsing object. Part of the particles are reflected back through the event horizon while the remainder escape to infinity. Those which escape exhibit the characteristic thermal distribution modified by the energy-dependent probability of escape. These particles are created in pairs; in the absence of a reaction back on the metric, the other member of the pair emerges from the surface of the collapsing object *after* it has passed the event horizon and proceeds into the singularity at $r=0$. The total energy of the pair of particles as measured by an observer at infinity is zero; if the exterior particle has energy ω, the particle inside the event horizon has energy $-\omega$, although it does have positive energy as measured by a freely falling local observer.

This picture emerges both from the detailed calculations presented below[13] and from the generalization of Hawking's considerations to the radiation which appears inside the event horizon. (Davies, Fulling, and Unruh[10] reach a contrary conclusion in a study which neglects the angular coordinates; for this two-dimensional spacetime, the Green's functions used here are not valid.)

This result and its interpretation depend critically upon the simplicity of the thin-shell model, while Hawking's result depends only on the assumed transparency of the stellar interior to the locally extremely energetic quanta being radiated. These results may be combined: For a collapsing star, the waves of the particular traveling on either side of the event horizon will have the same form as for the shell. Inside the star, the transition to flat space must take place across the entire volume of the star; but it is essentially this transition going backwards along the geodesic of the outgoing radiation which is responsible for the radiation, and Hawking's calculation (extended to include the interior radiation) assures us that the radiation exterior to the star on either side of the event horizon must be the same regardless of the construction of the star. Thus, although the region involved will be different, there must be, even in that case, a flux of virtual particles inside the star which is associated with the radiation.

Hawking's argument depends upon expressing the expectation values which are a sum over a complete set of final states as, instead, a sum over the (better understood) complete set of initial states, plus the assumption that absorption of the wave as it passes through the star may be neglected.

However, present experimental evidence and theoretical speculation suggest that elementary-particle cross sections (including those of photons) remain constant or, perhaps, rise slowly with increasing energy, thus absorption effects might be expected to be important. The absorption is a reflection of the coupling of the radiated fields to the various excitation modes of the star (including those associated with ordinary particle production); thus, when expressing the sum over final-state modes in terms of initial-state modes, *all* modes of the collapsing star must be included. Although it is technically impossibly complicated, one may conjecture that the sum over all modes of the star will, again, produce the Hawking radiation with its thermal characteristics.

The radiation may be expressed in terms of the stress tensor associated with the field whose quanta are being radiated. The complete stress tensor involves the renormalization problem which, although it can be discussed and solved in terms of Riemann normal coordinates around any point, presents technical complications when one attempts to infer the finite part. Nevertheless, because the quantum field is continuous across the shell (the field is taken to have no interaction with the shell except through the metric), the stress tensor will also, even after renormalization, be continuous and there is *no* direct reaction back on the shell from the emitted radiation. Also, the stress-energy tensor may be used to calculate the change in the metric due to the radiation. This will be a large effect because the particles escaping will carry a large amount of energy, reducing the mass of the star or shell. On the inside of the event horizon, the particles of negative Killing energy will produce a corresponding effect on the metric inside the event horizon. This means that the incipient event horizon which the shell is approaching has, by virtue of the emitted radiation, moved to a smaller radius. (Most workers[1-7,9,10] have taken the position that the radiation results from an existing event horizon rather than the collapse of an object *toward* the event horizon; I believe, for reasons adduced here and in Sec. IV, that this is an artifact of the neglect of the reaction of the metric to the radiation.) As the shell approaches this new event horizon, the radiation will increase as the effective temperature rises, thereby moving the incipient event horizon even further in. This process will continue with the shell chasing its event horizon but never quite reaching it until either all the energy is radiated or the intervention of quantum-gravity effects halts the radiation process.

At first only photons and neutrinos can be radiated. The temperature becomes comparable to an electron mass when the Schwarzschild radius of the collapsing object is comparable to an electron Compton wavelength (that is a mass of 8.3×10^{16} g); then electron positron pairs will be radiated, neutralizing any charge (the electric potential will cause preferential emission tending to neutralize the body).[2] This process will continue until the mass of the collapsing object drops to 4×10^{13} g, at which point the Schwarzschild radius is comparable to the proton Compton wavelength and emission of nucleons will begin. By this time, however, a collapsing star of one solar mass only has 2×10^{-20} of its original energy and cannot radiate away all of its baryons. (There does not seem to be any way to induce earlier emission, e.g., by a chemical potential,[4] because there is no long-range force which can change the relative energy of baryons and antibaryons, and the exclusion principle does not affect the emission process because the emission locally is into extremely energetic modes which are not otherwise occupied.)

This picture seems to lead to the conclusion that a collapsing star will chase its event horizon radiating away all its energy, charge, and angular momentum, but retaining its baryons until it reaches zero radius at which point an event horizon does form, encapsulating the remaining baryons and forming a disjoint closed universe. As seen from infinity, the star has collapsed, yielding all its energy, charge, and angular momentum but taking most of its baryons with it. For a one-solar-mass black hole, the time scale is, of course, extremely large. A precise calculation depends on the probability of an outgoing particle at the shell near the incipient event horizon reaching infinity, but a reasonable estimate is 10^{64} years for a one-solar-mass object. During the present epoch, the collapsing object is much colder than its surroundings; thus, it will gain rather than lose energy and will thereby increase its mass until the temperature of the surroundings drops to that of the collapsing object. As a result, the considerations given here are of limited observational interest: Only the very small primordial black holes of mass 10^{15} g or less discussed by Hawking[5,7] can have radiated away within the past lifetime of the universe.

Although the physical picture which emerges is clear and probably correct within the context of any model with a classical metric there are physical grounds for being cautious. The radiation which emerges at very late times arises from distances from the event horizon which are correspondingly small. Further, when the emission of heavier particles begins, the Schwarzschild radius is comparable to the particles' Compton wavelength; at this point modifications of general relativity must certainly arise owing to quantum gravity because the structure of the graviton matter vertices will become important. It may be that these effects, which are enforced by unitarity and the fact that elementary particles are being produced, do not entail any modification of our familiar geometrical interpretation, but until we have a better understanding of quantum gravity we have no such assurance.[14] If measurements are made on the scale at which the phenomena are taking place, one expects that quantum-gravity effects will be important. In Sec. IV, it is shown [Eq. (4.14)] that the energy density of the radiation measured by a freely falling observer within δ of the event horizon and shell is

$$\langle T\rangle \sim \hbar/GM\delta^3 , \tag{1.1}$$

where units with $\hbar$ and G not equal to 1 are used to exhibit the dependence on them and M is the mass of the collapsing object. If a measurement of the metric within a region of volume δ^3 is made, the expected Heisenberg uncertainty in the metric is

$$\delta_H g \sim (\hbar G)^{1/2}/\delta \tag{1.2}$$

and, since the scale of variation is δ, the uncertainty in the Einstein tensor is

$$\delta_H G \sim (\hbar G)^{1/2}/\delta^3 . \tag{1.3}$$

On the other hand, the contribution of $\langle T\rangle$ to the Einstein tensor is

$$\delta_T G \sim \hbar/M\delta^3 \tag{1.4}$$

thus, the ratio of the two contributions,

$$\frac{\delta_H G}{\delta_T G} \sim \left(\frac{GM^2}{\hbar}\right)^{1/2} \simeq M/10^{-5}\,\mathrm{g} \tag{1.5}$$

is extremely large for any astronomical object and the quantum uncertainties of the metric will be much larger than the change induced by the radiation. This means that in the region where the actual radiation is taking place, it is not possible to determine the location of the event horizon with the precision required to specify exactly whence the radiation comes. A calculation of the quantum-gravity corrections to the process would be of great interest in elucidating these points.

The thin shell is the physical system considered in this paper. It is assumed that the shell is static, maintaining itself at some fixed radius, prior to some time t_0; after that time the shell is assumed to be a dust shell with no internal stress.

Before the shell begins to collapse, the metric is static and the modes of the quantum field and the associated eigenfunctions are well defined. The Green's function of the collapsing system,

$$\bar{G}(x,x') \equiv i\langle 0-|T(\phi(x)\phi(x'))|0-\rangle , \tag{1.6}$$

where $\langle 0-|$ is the vacuum state of the initial system, may be expressed in terms of the wave functions which are, in the past when the shell is static, true eigenfunctions of the past modes (although they have not such simple interpretation once the collapse has begun). This Green's function is appropriate for calculating expectation values of physical quantities.

In order to discuss more detailed questions of correlations and single-particle matrix elements, it is convenient to use the matrix element of the fields between the initial vacuum, $|0-\rangle$, and the future vacuum, $\langle 0+|$. The fields may then be used to construct any desired state from the final vacuum and the individual amplitudes and correlations may be studied. This Green's function,

$$G(x,x') = \frac{i\langle 0+|T(\phi(x)\phi(x'))|0-\rangle}{\langle 0+|0-\rangle} , \tag{1.7}$$

is constructed in Sec. II, using the vacuum state which was found earlier[11,12] by considering the complete analytic extension of the Schwarzschild spacetime. The resultant Green's functions, for spin 0 and spin $\frac{1}{2}$, have only negative frequencies at large negative times and only positive frequencies (defined relative to the indicated vacuum) in the future at spatial infinity and near the singularity. The solution to the homogeneous wave equation which is of positive frequency in the past develops both positive- and negative-frequency components in the future with the amount of the latter being a measure of the particle production. The Green's function, G, is then expressed in terms of the respective coefficients, α and β, of the positive- and negative-frequency components which develop in the future.

This Green's function is used in Sec. III to calculate the amplitude for the emission of various numbers of particles. Because the only coupling is taken to be through the metric,

$$W = \int d^4x\sqrt{-g}\,(-\tfrac{1}{2})(\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi + \mu^2\phi^2) , \tag{1.8}$$

the effective source is of the form $\frac{1}{2}\eta\phi^2$ and can only scatter a particle (without emission or absorption) or emit (or absorb) a pair of particles. The emission, of course, occurs all along the world line of the collapsing shell, but only near the event horizon does the emission become large. There the characteristically thermal, Hawking, radiation appears.

The emission mechanism is a kind of barrier penetration induced by the shell. The role of the shell (or other matter) is crucial because the Kruskal spacetime appropriate to no matter does not exhibit the Hawking radiation. Because of the red-shift the wave of the particle, which is to reach some large finite distance at late times within a time interval δt and with an average energy ω, must be concentrated along the shell very near the event horizon with extremely rapid variation with respect to the proper time of the shell and within a very narrow proper-time interval. The interior of the shell is flat spacetime and a positive-frequency wave function of the appropriate form cannot be constructed. There is a wave on the other side of the event horizon, hence the radiation.

For comparison with the results found in Sec. III, it is useful, to present a reminder of the basic quantum mechanics for the uncorrelated emission of pairs. In the case at hand, there are an infinite number of modes which do not interfere, therefore only one mode will be considered here.

Consider a system which is in a state Ψ_0. In the future, the state will contain some superposition of a pair of particles,

$$\Psi_0=\sum_n |np\rangle C_n\,, \tag{1.9}$$

where $|np\rangle$ is the state with n pairs. If the emission of the pairs is uncorrelated: Ψ_0 will be the result of acting on the no-particle state with an operator, e^{iV}, or, after normal ordering,

$$\Psi_0=Ne^{a^\dagger b^\dagger \lambda}|0\rangle$$
$$=\left[\sum_{n=0}^{\infty}|np\rangle\lambda^n\right]N\,, \tag{1.10}$$

where $a^\dagger$ and $b^\dagger$ are the creation operators for the members of the pair and the sum terminates at $n=1$ for fermions. The normalization factor, N, is

$$N=(1\mp|\lambda|^2)^{\pm 1/2}\,, \tag{1.11}$$

where the upper (lower) sign refers to bosons (fermions).

The expectation value for the number of pairs is

$$\langle n\rangle=\sum n|\lambda|^{2n}N^2$$
$$=\Psi_0^\dagger a^\dagger a\Psi_0$$
$$=\Psi_0^\dagger b^\dagger b\Psi_0$$
$$=\frac{|\lambda|^2}{1\mp|\lambda|^2} \tag{1.12}$$

and the off-diagonal matrix element is

$$\langle a^\dagger b^\dagger\rangle=\sum(1\pm n)|\lambda|^{2n}\lambda^* N^2$$
$$=\frac{1}{\lambda}\langle n\rangle\,, \tag{1.13}$$

and in Sec. III these matrix elements are shown to be precisely of this form with $|\lambda|=e^{-4\pi M\omega}$, where ω is the energy of the mode in question.

This is, of course, not a thermal distribution despite the thermodynamic aspect of $\langle n\rangle$, because the system is in a definite state. However, if the particles in the b state cannot be observed, the density operator for the system becomes[9,15]

$$\rho=\mathrm{tr}_b\Psi_0\Psi_0^\dagger$$
$$=\sum_n |na\rangle|\lambda|^{2n}N^2\langle na|\,, \tag{1.14}$$

which, for $|\lambda|^2=e^{-8\pi M\omega}$, is that of a thermal distribution of particles characterized by the temperature $T=\hbar c^3/8\pi GMk$.

The problem, then, is to understand why $|\lambda|^2=e^{-8\pi M\omega}$. Mathematically, it is straightforward (see Refs. 1, 5 and Sec. III) but, at least to this author, the derivation is not intuitive, and no physical argument has been given which provides an *intuitive* understanding of either the thermodynamic character of the result or the specific value of the temperature [on dimensional grounds, the temperature can only be of the form $(\hbar c^3/kGM)f(GM^2/\hbar c)$; but M only appears in the form GM, hence f must be a constant: $1/8\pi$, of course.]

In Sec. IV, the Green's functions are used to calculate various matrix elements of the stress-energy tensors. Both in the past and in the future the Green's functions may be written as a zeroth-order term which, when used to calculate the stress-energy tensor, contains all the divergences: In the past, it is just the Green's function for the shell, while in the future it is the Green's function for the Kruskal spacetime. These terms, when renormalized, yield the matrix elements of the stress-energy tensor for the corresponding spacetime and are of no interest for the problem at hand since there is no radiation present in either case.

The remaining terms yield finite well-defined matrix elements for $T^{\mu\nu}$ which are associated with the radiation. Both in the past and in the future there are rapidly oscillating terms which average to zero. In addition, there are terms which, in the future, do not average to zero: There is a well-defined flux of energy-momentum along the inside and outside of the event horizon extending from the shell to, respectively, infinity and the $r=0$ singularity.

II. THE GREEN'S FUNCTIONS

In the case of a static shell it is easy to obtain the Green's function, and therefore, the quantum field theory; there is no event horizon, the metric is static, eigenfunctions can be obtained, and the Green's function constructed just as was done for the complete analytic extension of the Schwarzschild spacetime for spin-0 (Ref. 11) and spin-$\frac{1}{2}$ (Ref. 12) fields. There are then no problems of interpretation due to event horizons, time dependence of the metric, or any other pathologies.

The problem of a collapsing shell is more complicated; the matrix element between the initial state with no particles and a final state with only a few (two in this instance) particles is required to study the correlations in full detail. The final states may be constructed by creating particles in the final vacuum state with the aid of the reduction formulas[11,12] suitably modified to reflect the changed geometry. In order to do this, the matrix elements of the field between the final and initial vacua are required. It is technically somewhat complicated to construct the required Green's function, Eq. (1.7) directly; instead, first consider the Green's function, Eq. (1.6), which is the expectation value in the initial vacuum of the time-ordered product of two fields.

In the case at hand, there is a spherical shell whose Schwarzschild radius is $R(\tau)$, where τ is the proper time of the shell. Outside the shell, the metric is the Schwarzschild metric (units in which $\hbar = c = G = 1$ are used)

$$\begin{aligned} ds^2 &= -[1-2\Phi(r)]dt^2 + dr^2/[1-2\Phi(r)] \\ &\quad + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ \Phi(r) &= M/r\,, \end{aligned} \tag{2.1}$$

while inside the shell, the time coordinate will be denoted by T instead of t and $M \to 0$. The shell naturally divides spacetime into two regions which are exhibited in a Kruskal diagram in Fig. 1. Then the scalar Green's function must satisfy

$$(-\partial_\mu g^{\mu\nu}\sqrt{-g}\,\partial_\nu + \mu^2\sqrt{-g})G(x,x') = \delta(x-x') \tag{2.2}$$

and the spin-$\frac{1}{2}$ Green's function must satisfy

$$\beta\left(\mu + \gamma^a \frac{1}{i}\nabla_a\right) S(x,x') = \frac{\delta(x-x')}{\sqrt{-g}}\,, \tag{2.3}$$

where the covariant derivatives employed in the second equation are defined in Ref. 12.

Before the shell begins its collapse, the metric is static and well-defined positive-frequency modes exist. The positive-frequency solution to the homogeneous equation corresponding to Eq. (2.2) is

$$\begin{aligned} \psi^{l,m}(x;q) &= Y_l^m(\theta,\phi)\psi^l(r,t;q)\,, \\ \psi^l(r,t;q) &\underset{\substack{r\sim\infty \\ \text{early times}}}{\sim} \left(\frac{2}{\pi}\right)^{1/2} \frac{\cos[q\bar{r} - \eta_l(q)]}{qr}\, e^{-i\omega(q)t}\,, \end{aligned} \tag{2.4}$$

while the positive-frequency solution to the homogeneous Dirac equation, (2.3), is

$$\begin{aligned} \psi^{k,m}(x;q) &= \mathcal{Y}_m^k(\theta,\phi)\psi^k(r,t;q)\,, \\ \psi^k(r,t;q) &\underset{\substack{r\sim\infty \\ \text{early times}}}{\sim} \frac{1}{(2\pi)^{1/2}qr} \\ &\quad \times \left[\begin{pmatrix} (\omega+q)^{1/2} \\ i(\omega-q)^{1/2} \end{pmatrix} e^{i[q\bar{r}-\eta_k(q)]} + \begin{pmatrix} (\omega-q)^{1/2} \\ i(\omega+q)^{1/2} \end{pmatrix} e^{-i[q\bar{r}-\eta_k(q)]}\right] \\ &\quad \times e^{-i\omega(q)t}\,, \end{aligned} \tag{2.5}$$

where

$$\bar{r} \equiv r^* + [(\omega^2 - q^2)/q^2]M\ln(r/M)\,,$$

$$r^* \equiv r + 2M\ln(|r-2M|/2M)\,,$$

and

$$\omega^2 - q^2 \equiv \mu^2\,.$$

The states are normalized so that the sum over states is $\int_0^\infty q^2 dq/2\omega$. There are, in addition, bound-state wave functions for $\mu \neq 0$; in the sums over initial states which occur below, only the high-q behavior is important and the bound states

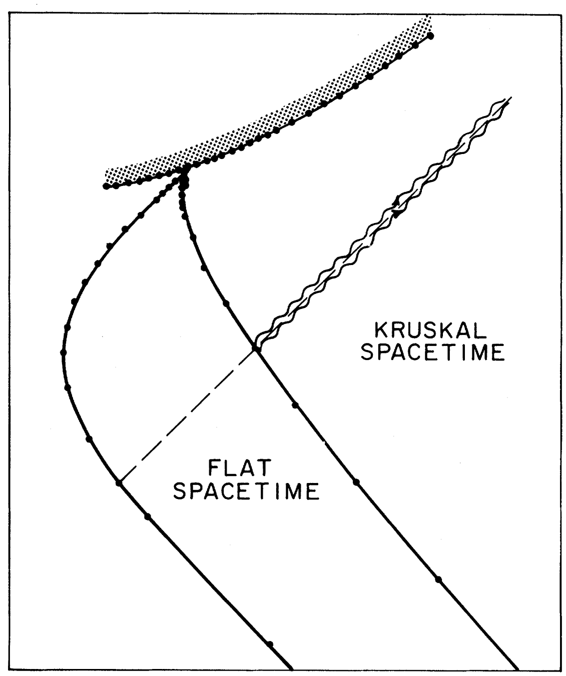

FIG. 1. Spacetime diagram for the collapsing shell. The plot is in terms of Kruskal coordinates exterior to the shell and the coordinates in the interior are chosen so that radial light rays move along the 45° line. The wavy lines represent the particles associated with the Hawking radiation. The left boundary of the flat spacetime region is the center of the shell. The other solid line is the shell. The dotted line denotes the event horizon and the stippled band denotes the singularity at $r=0$.

need not be explicitly considered. With the normalizations given in Eqs. (2.4) and (2.5), the Green's functions, $\overline{G}$, may be written down immediately. For spin 0,

$$\overline{G}(x,x') \equiv \sum_{l,m} Y_l^m(\theta,\phi)\overline{G}^l(r,t;r',t')Y_l^{m*}(\theta',\phi'),$$

and for spin $\frac{1}{2}$, (2.6)

$$\overline{G}(x,x') = \sum_{k,m} \mathcal{Y}_k^m(\theta,\phi)\overline{G}^k(r,t;r',t')\mathcal{Y}_k^{m\dagger}(\theta',\phi'),$$

where, for either spin 0 or spin $\frac{1}{2}$,

$$\overline{G}^k(r,t;r',t') = i\int_0^\infty \frac{q^2dq}{2\omega(q)}\psi^k(x_>,q)\psi^k(x_<,q)^\dagger\epsilon(x,x'). \quad (2.7)$$

In the latter equation $x_>$ ($x_<$) is the later (earlier) of (r,t) and (r',t'), and $\epsilon(x,x')=1$ for bosons, while for fermions $\epsilon(x,x')=1$ (-1) if $x=x_>$ $(x=x_<)$. If the separation (for $d\theta=0=d\phi$) is spacelike, then either assignment may be made since the value of $\overline{G}^k$ is independent of the assignment.

The wave functions $\psi^{k,l}$ may be integrated forward in time using the appropriate wave equation; because of the time dependence of the shell, the wave function develops negative-frequency components. There are two regions of interest in the future: spatial infinity and the singularity. At spatial infinity in the future the wave must be a superposition of positive- and negative-frequency outgoing waves. The wave functions for the mode which describes outgoing particles at infinity but *no* particle crossing the (future) event horizon are, for spin 0,[11]

$$\psi_l^1(x',\omega) \sim \begin{cases} \dfrac{1}{(2\pi)^{1/2}}\left(\dfrac{\omega}{q}\right)^{1/2}\dfrac{1}{r}(e^{iq\bar{r}}+S_{11}^*e^{-iq\bar{r}})e^{-i\omega t}, & r\sim\infty \\[2ex] \dfrac{1}{(2\pi)^{1/2}r}S_{12}^*e^{i\omega(r^*-t)}, & r\sim 2M \end{cases} \quad (2.8)$$

and, for spin $\frac{1}{2}$,[11]

$$\psi_k^1(x',\omega) \sim \begin{cases} \dfrac{1}{(2\pi)^{1/2}}\left(\dfrac{\omega}{q}\right)^{1/2}\dfrac{1}{r}\left[\begin{pmatrix}(\omega+q)^{1/2}\\ i(\omega-q)^{1/2}\end{pmatrix}e^{iq\bar{r}}+S_{11}^*\begin{pmatrix}(\omega-q)^{1/2}\\ i(\omega+q)^{1/2}\end{pmatrix}e^{-iq\bar{r}}\right]e^{-i\omega t}, & r\sim\infty \\[2ex] \dfrac{1}{(2\pi)^{1/2}r}S_{12}^*\begin{pmatrix}(2\omega)^{1/2}\\ 0\end{pmatrix}\dfrac{e^{i\omega(r^*-t)}}{w^{1/2}(r)}, & r\sim 2M \end{cases} \quad (2.9)$$

where S_{1j} is the S-matrix element [the dependence on $l(k)$ and ω is suppressed] for a particle escaping to infinity after starting at infinity ($j=1$) or the past event horizon ($j=2$) and $w(r)\equiv[1-2\Phi(r)]^{1/2}$. The states are normalized so that the sum over states is $\int_\mu^\infty d\omega/2\omega$.

Then, the asymptotic behavior for late times and large r is given by, for either spin 0 or spin $\frac{1}{2}$,

$$\psi^k(x',q) \underset{\substack{r\sim\infty \\ \text{late times}}}{\sim} \int_\mu^\infty \frac{d\omega'}{2\omega'}[\psi_k^1(x',\omega')\alpha_k^1(\omega',q) + \psi_k^1(x',\omega')^*\beta_k^1(\omega',q)]. \quad (2.10)$$

There will be particles which end up inside the event horizon; for them the identification of the positive frequency is in terms of the increase with time (r) of the phase of the wave function. The complete orthonormal set of wave functions on the spacelike surfaces $r=$ constant, $-\infty<t<\infty$, are the solutions to the homogeneous wave equations which behave as, for spin 0,

$$\psi_l^2(x',\omega) \underset{r\sim 2M^-}{\sim} \frac{1}{(2\pi)^{1/2}r}e^{-i(\omega t+|\omega|r^*)}$$

and, for spin $\frac{1}{2}$, (1.11)

$$\psi_k^2(x',\omega) \underset{r\sim 2M^-}{\sim} \frac{1}{(2\pi)^{1/2}r}\begin{pmatrix}(-i)(-\omega+|\omega|)^{1/2}\\ i(\omega+|\omega|)^{1/2}\end{pmatrix} \times e^{-i(\omega t+|\omega|r^*)},$$

where the wave functions are normalized with the sum over states, $\int_{-\infty}^\infty d\omega'/2|\omega'|$. As is apparent from Fig. 1, only half the space on which these functions are complete is available in the case of the collapsing shell. If the shell is taken to collapse to zero radius at $t=t_0$, the functions are overcomplete on the available region, $t>t_0$, but they may describe particles localized in the region

$t<t_0$. Although it is straightforward to construct a complete orthonormal set on the half space $t>t_0$, the use of that set entails, especially for spin $\frac{1}{2}$, a number of technical complications which may be avoided by using the overcomplete set of Eqs. (2.11).

Then, for $r\sim 0+$, the wave functions $\psi^{l,k}(x',q)$ may be written as a superposition of the positive-frequency solutions, $\psi^2_{l,k}(x',q)$, and their complex conjugates, the negative-frequency solutions,

$$\psi^k(x',q) \underset{r\sim 0+}{\sim} \int_{-\infty}^{\infty} \frac{d\omega'}{2|\omega'|}[\psi^2_k(x',\omega')\alpha^2_k(\omega';q) + \psi^2_k(x',\omega')^*\beta^2_k(\omega';q)]. \tag{2.12}$$

Then, Eqs. (2.11) and (2.12) may be combined to read

$$\begin{aligned}\psi^k &\simeq \psi_k\alpha + \psi_k^*\beta \\ &\equiv \sum_{i=1}^{3}\int_0^\infty \frac{d\omega'}{2\omega'}[\psi^i_k(x',\omega')\alpha^i_k(\omega';q) + \psi^i_k(x',\omega')^*\beta^i_k(\omega';q)],\end{aligned}$$

where

$$\beta^1_k,\ \alpha^1_k(\omega',q)=0, \quad \omega'<\mu$$

and

$$(\psi^3,\alpha^3,\beta^3)(\omega') \equiv (\psi^2,\alpha^2,\beta^2)(-\omega').$$

The wave functions and the functions α and β form vectors,

$$\psi=(\psi^1,\psi^2,\psi^3)$$

and

$$\alpha=\begin{pmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\end{pmatrix}, \quad \beta=\begin{pmatrix}\beta^1\\ \beta^2\\ \beta^3\end{pmatrix}.$$

the first components, α^1 and β^1, are, essentially, those of Hawking[1] and DeWitt[5] which were used by them to calculate the expectation value of the number of particles emitted and of operators such as the stress-energy tensor.

As an illustration, to calculate the expected number of particles, note that a particle may be created in a state acting to the right with, for spin 0,

$$\lim_{x\to \text{future}} \int d\sigma_\mu\, \phi(x)\frac{1}{i}\overleftrightarrow{\nabla}^\mu\psi_a(x) \equiv \mathfrak{A}_a^\dagger .$$

Then, the expectation value is

$$\begin{aligned}\langle N_a\rangle &= \sum_{\{n\}} n_a|\langle\{n\}+|0-\rangle|^2 \\ &= \sum_{\{n\}} |\langle\{n\}|\mathfrak{A}_a|0-\rangle|^2 \\ &= \langle 0-|\mathfrak{A}_a^\dagger\mathfrak{A}_a|0-\rangle \\ &= \lim_{x,x'\to\text{future}} \int d\sigma_\mu d\sigma'_\nu \psi_a^T(x)\frac{1}{i}\overleftrightarrow{\nabla}^\mu\langle -0|\phi(x)\phi(x')|0-\rangle\frac{1}{i}\overleftrightarrow{\nabla}^\nu\psi_a^*(x'),\end{aligned} \tag{2.13}$$

but in the future,

$$\langle 0-|\phi(x)\phi(x')|0-\rangle = (\psi,\psi^*)\begin{pmatrix}\alpha\alpha^\dagger & \alpha\beta^\dagger\\ \beta\alpha^\dagger & \beta\beta^\dagger\end{pmatrix}\begin{pmatrix}\psi^\dagger\\ \psi^T\end{pmatrix}$$

and

$$\int d\sigma_\mu\psi_a^\dagger\frac{1}{i}\overleftrightarrow{\nabla}^\mu\psi_{a'}=\delta_{aa'},$$

hence, the expectation value of the number of particles in state a at late times is

$$\begin{aligned}\langle N_a\rangle &= \langle a|\beta\beta^\dagger|a\rangle \\ &= \int_0^\infty \frac{q^2dq}{2\omega}|\beta(a,q)|^2,\end{aligned} \tag{2.14}$$

where a denotes both the energy, ω_a, and the other quantum numbers.

We also note that the requirement that the Green's function satisfy the correct inhomogeneous equation at late times [the fields must still satisfy the correct (anti-) commutation relations] implies that

$$\alpha\alpha^\dagger \mp \beta^*\beta^T = 1$$

and (2.15)

$$\alpha\beta^\dagger \mp \beta^*\alpha^T = 0,$$

with the $-$ (+) sign holding for bosons (fermions).

The remaining task is to construct the Green's function,

$$G(x,x')=i\,\frac{\langle 0+|T(\psi(x)\psi(x'))|0-\rangle}{\langle 0+|0-\rangle}$$
$$\equiv \sum \mathcal{Y}^k G^k \mathcal{Y}^{k\dagger}\,, \tag{2.16}$$

where $\langle 0+|$ is the future vacuum in which there are no particles. This Green's function is also a solution of the same inhomogeneous wave as $\overline{G}(x,x')$. Thus, G may be written as

$$G=\overline{G}+\text{solution to homogeneous wave equation}.$$

Both G and $\overline{G}$ are (anti-) symmetric in $x \leftrightarrow x'$, and $\overline{G}$ already satisfies the appropriate boundary conditions as $x' \to$ past (it has only negative frequencies in t'), thus no positive-frequency solutions may be added. Then,

$$G^k(x,x')=i\int_0^\infty \frac{q^2dq}{2\omega(q)}\left[\psi^k(x_>,q)\psi^k(x_<,q)^\dagger\epsilon+\int_0^\infty\frac{q'^2dq'}{2\omega(q')}\psi^k(x,q)^*C(q,q')\psi^k(x',q)^\dagger\right], \tag{2.17}$$

where $\epsilon=1$ for bosons and $+1$ (-1) for fermions if $x_>$ $(x_<)=x$. The function $C(q,q')$ may be viewed as a matrix

$$C(q,q')=\langle q|C|q'\rangle,$$

which may be constructed by using $\overline{G}$ to calculate the projection operator for no particles in the final state; this leads to the explicit form

$$C=-(\pm\beta^T \quad \alpha^T)A^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

where A is the 6×6 matrix (2.18)

$$A=\begin{pmatrix}\alpha\alpha^\dagger & \beta^*\alpha^T\\ \beta\alpha^\dagger & \alpha^*\alpha^T\end{pmatrix}$$

and the $\pm$ refers to bosons or fermions.

The required (anti-) symmetry of C follows from

$$\begin{pmatrix}0 & 1\\ \pm 1 & 0\end{pmatrix}A^T\begin{pmatrix}0 & 1\\ \pm 1 & 0\end{pmatrix}=\pm A,$$

which, in turn, is a consequence of the symmetry properties of $\alpha\beta^\dagger$ exhibited in Eq. (2.15).

The original construction of G is not important: if G satisfies the boundary conditions, it must be the correct solution. To show that G is the correct solution, observe that it manifestly satisfies the wave equation: the first term satisfies the inhomogeneous equation and the second term is a solution to the homogeneous equation in both x and x'. Further, at large negative times, it is purely negative frequency in the earlier coordinate. It remains to be shown that G is purely positive frequency in the later coordinate for $t\to\infty$ or $r\to 0$ inside the event horizon. In that limit, the negative-frequency components of ψ^k and ψ^{k*} are

$$\psi^k \sim \psi_k^*\beta\,, \quad \psi^{k*}\sim\psi_k^*\alpha^*\,,$$

hence

$$G\sim\text{positive frequency}$$
$$+i\psi_k^*\left[\beta-(\pm\alpha^*\beta^T \quad \alpha^*\alpha^T)A^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]\psi^{k\dagger}$$

and the coefficient of ψ_k^* vanishes.

This construction fails for fermions if $\alpha=0$ [from Eq. (2.14), α cannot vanish for bosons]; then A^{-1} does not exist. In that case, the Green's function satisfying the inhomogeneous equation cannot be constructed because the solution which is negative frequency in the past is also the positive-frequency solution in the future. The physical situation being described is one in which the state which starts with no particles evolves into the state with at least one mode having a unit probability of being occupied. In that case, the matrix element of the commutator vanishes. It should be emphasized that this is different from the situation obtained for the collapsing shell in which, although no mode has a unit probability of being occupied, the amplitude for *no* mode being occupied, $\langle 0+|0-\rangle$, vanishes because there are an infinite number of modes which have less than unit probability of being not occupied; the product of these probabilities vanishes. In this latter case, the Green's function may still be constructed because, although

$$\langle 0+|iT(\psi(x)\psi(x'))|0-\rangle \text{ and } \langle 0+|0-\rangle$$

separately vanish, their quotient, G, exists.

A difficulty similar to the vanishing of α may arise in the boson case, if the past vacuum evolves into a future state with zero probability of having no particles in a given mode. However, because the model considered here has no correlations beyond the two-body correlations imposed by the requirement that the particles be created in pairs, this would require that the expected number of particles in that mode be infinite. Such "infrared" divergencies do not occur with the Hawking radiation.

III. THE EMISSION AMPLITUDES

In this section an approximation to the coefficients α and β defined in Sec. II will be given and the emission amplitudes and correlations discussed. Hawking[1] and DeWitt[5] have given approximations to the coefficients corresponding to α^1 and β^1 for the spin-0 field and DeWitt[5] has given the coefficients corresponding to $\alpha^2(\omega, q)$ and $\beta^2(\omega, q)$. The same arguments will be used here to find the coefficients for spin $\frac{1}{2}$ and the remaining coefficients for spin 0.

The resulting expressions are only valid for particles emerging from the hole at late times; for such particles, the only paths by which the waves can reach infinity are shown in Fig. 2. Along the path marked "I" only the incoming waves of ψ^k contribute and the shell itself plays no role; thus, the contributing terms are

$$\psi^l(r,t;q) \underset{\substack{\text{past}\\ \text{incoming}}}{\sim} \frac{e^{i\eta_l(q)}}{(2\pi)^{1/2}qr} e^{-i[q\bar{r}+\omega(q)t]}$$

$$\underset{t\to+\infty}{\sim} \begin{cases} \dfrac{e^{i\eta_l(q)}}{(2\pi)^{1/2}qr} S_{11} e^{i[q\bar{r}-\omega(q)t]}, & r\sim\infty \\[2ex] \dfrac{e^{i\eta_l(q)}}{(2\pi\omega q)^{1/2}r} S_{21} e^{-i\omega(q)(r^*+t)}, & r\sim 2M \end{cases} \tag{3.1}$$

while the spin-$\frac{1}{2}$ wave function is

$$\psi^k(r,t;q) \underset{\substack{\text{past}\\ \text{incoming}}}{\sim} \begin{pmatrix} (\omega-q)^{1/2} \\ i(\omega+q)^{1/2} \end{pmatrix} \frac{e^{-i[q\bar{r}+\omega(q)t]}}{(2\pi)^{1/2}qr} e^{i\eta_k(q)}$$

$$\underset{t\to\infty}{\sim} \begin{cases} \begin{pmatrix} (\omega+q)^{1/2} \\ i(\omega-q)^{1/2} \end{pmatrix} \dfrac{e^{i[q\bar{r}-\omega(q)t]}}{(2\pi)^{1/2}qr} S_{11} e^{i\eta_k(q)}, & r\sim\infty \\[2ex] \begin{pmatrix} 0 \\ i(2\omega)^{1/2} \end{pmatrix} \dfrac{e^{-i\,\omega(q)[r^*+t]}}{(2\pi q\omega)^{1/2} r w^{1/2}(r)} S_{21} e^{i\eta_k(q)}, & r\sim 2M. \end{cases} \tag{3.1$'$}$$

From these expressions, the contributions to α and β may be read off immediately,

$$\begin{aligned} \alpha_{\mathrm{I}}^1(\omega,q') &= 2\omega(q)\frac{\delta(\omega-\omega(q'))}{(q'\omega)^{1/2}} S_{11} e^{i\eta_k(q')}, \\ \beta_{\mathrm{I}}^1(\omega,q') &= 0, \end{aligned} \tag{3.2}$$

and the solutions near $r=2M$ may be continued across the event horizon to yield the corresponding solutions there, with the result

$$\begin{aligned} \alpha_{\mathrm{I}}^2(\omega,q') &= \frac{2\omega\delta(\omega-\omega(q'))}{(\omega q')^{1/2}} S_{21} e^{i\eta_k(q')}, \\ \beta_{\mathrm{I}}^2(\omega,q') &= 0. \end{aligned} \tag{3.2$'$}$$

These expressions hold for both spin $\frac{1}{2}$ and spin 0 but are restricted in that the q' integration may only be taken over wave functions which do not cross the shell. The factor of $(\omega q)^{1/2}$ is a result of the different normalizations of the initial and final states.

The remaining contributions come from waves which pass through the shell. In order to emerge at late times, they must leave the shell immediately before it passes the event horizon. Thus, the waves of interest are those which follow path II in Fig. 2. These pass into the shell after having been Doppler- and blue-shifted by an amount appropriate to how far in the shell is when the waves cross it. (For a shell falling freely from a distance large compared to its Schwarzschild radius, the wave enters when the shell is at about $R=6M$.) It turns out that only high-frequency, $\omega(q)$, waves contribute so the probability of the wave being reflected is small and, along the shell,

$$\psi^l(R(\tau),t(\tau);q) \simeq \frac{e^{-i[q\bar{R}(\tau)+\omega(q)t(\tau)]} e^{i\eta_l(q)}}{(2\pi)^{1/2}qR(\tau)}.$$

If the wave is well localized compared to the rate at which R changes, the argument of the exponential may be approximated by

$$-iq\{[\dot{R}_0/(1-2M/R_0)]+\dot{t}_0\}(\tau-\tau_0) - iq(R_0^*+t_0),$$

where τ_0 is the proper time of the shell when the wave passes and R_0 and t_0 are the values of R and t at $\tau=\tau_0$. This wave must be matched continuously and with continuous normal derivatives to a reflected wave and a transmitted wave. At these large frequencies virtually all of the wave is transmitted and the wave on the inside becomes

$$\psi^l(r,T;q)\simeq\frac{e^{-iq\lambda(r+T)}\,e^{i\bar\eta_l(q)}}{(2\pi)^{1/2}qr},$$

where

$$\bar\eta_l(q)=\eta_l(q)+q\bar\lambda(R_0+T_0)-q(R_0^*+t_0)$$

and

$$\lambda=\{[\dot R_0/(1-2M/R_0)]+\dot t_0\}/(\dot R_0+\dot T_0).$$

The wave then passes through the center, acquiring a phase $(-1)^{l+1}$, and the wave along the shell near the event horizon is

$$\psi^l(R(\tau),T(\tau);q)\simeq\frac{e^{iq\lambda(\dot R-\dot T)\tau}\,e^{i[\bar\eta_l(q)+2Mq]}(-1)^{l+1}}{(2\pi)^{1/2}Rq},$$

where T is chosen so that $T(0)=0=R(0)-2M$.

Again, the wave, because of the high frequency, must be almost entirely transmitted and the wave on the other side must be a superposition of the normal modes

$$\frac{e^{\pm i\omega'(r^*-t)}}{(2\pi)^{1/2}r}\underset{\text{along the shell}}{\sim}\frac{(|\dot R|\,|\tau|/2M)^{\pm i4M\omega'}}{(2\pi)^{1/2}R},$$

where the path of the shell near the event horizon,

$$R(\tau)\simeq 2M+\dot R\tau,$$

$$t(\tau)\simeq 2M\ln(|\dot R\tau|/2M),$$

has been used. Then, the relation

$$e^{-i\nu\tau}=\int_{-\infty}^{\infty}d\omega\left(\frac{|\dot R\tau|}{2M}\right)^{i4M\omega}e^{-\epsilon 2\pi M\omega}f(\omega,\nu),\quad(3.3)$$

where

$$\epsilon\equiv\tau/|\tau|$$

and

$$f(\omega,\nu)=\frac{\Gamma(-i4M\omega)4M}{2\pi}\left(\frac{\nu 2M}{|\dot R|}\right)^{i4M\omega},$$

allows ψ^l to be rewritten as

$$\psi^l(x;q)\simeq\int_{-\infty}^{\infty}d\omega'\frac{e^{i\omega'(r^*-t)}}{(2\pi)^{1/2}r}\times e^{\pm 2\pi M\omega'}f(\omega',q\lambda(\dot T-\dot R))\frac{e^{i\psi(q)}}{q}\quad(3.4)$$

for (r,t) near the event horizon just outside the shell. Here

$$e^{i\psi(q)}=e^{i[\bar\eta_l(q)+2Mq]}(-1)^{l+1}$$

and the $\pm$ refers to outside $(+)$ or inside $(-)$ the event horizon.

As the wave which starts as

$$e^{i\omega'(r^*-t)}/(2\pi)^{1/2}r$$

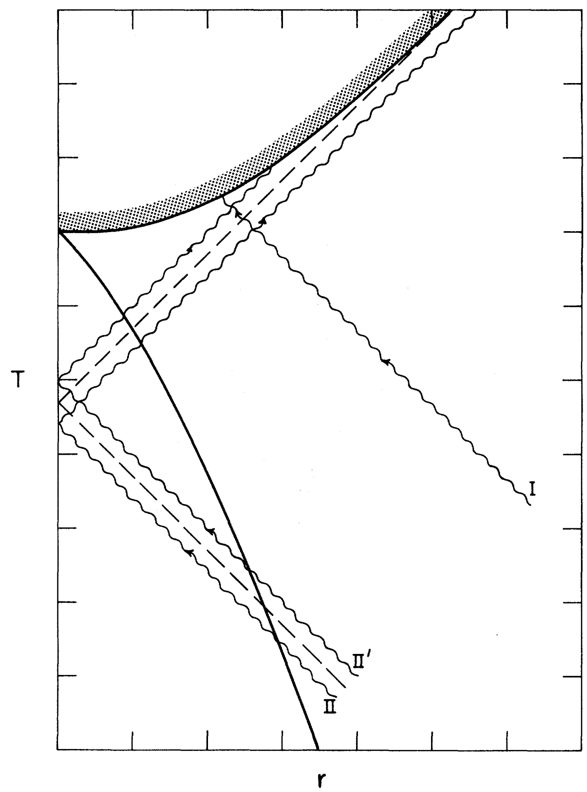

FIG. 2. The same diagram as Fig. 1 expressed in r,T coordinates in the interior of the shell with the exterior coordinates chosen so that radial light rays move along the 45° line. The wavy lines represent the virtual excitations which are studied to exhibit the Hawking radiation.

propagates to infinity, it becomes

$$\psi_l'(r,\omega')S_{12}(\omega')$$

and $\alpha^1_{\rm II}$ and $\beta^1_{\rm II}$ may be read off:

$$\alpha^1_{\rm II}(\omega',q)=2\omega' S_{12}e^{2\pi M\omega'}f(\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q},$$

$$\beta^1_{\rm II}(\omega',q)=2\omega' S_{12}^* e^{-2\pi M\omega'}f(-\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q}.\quad(3.5)$$

The waves which are reflected acquire an S-matrix factor, S_{22}, then propagate through the event horizon as

$$e^{\pm i\omega'(r^*+t)}/(2\pi)^{1/2}r,$$

hence

$$\alpha^2_{\rm II}(\omega',q)=2\omega' S_{22}e^{2\pi M\omega'}f(\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q}$$

and (3.6)

$$\beta^2_{\rm II}(\omega',q)=2\omega' S_{22}^* e^{-2\pi M\omega'}f(-\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q}.$$

The remaining coefficients are associated with the waves which pass directly from the shell to the interior of the event horizon along the path denoted II′ in Fig. 2. For these no reflection coefficients are required and

$$\alpha^3_{\rm II}(\omega',q)=2\omega' e^{2\pi M\omega'}f(-\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q},$$

$$\beta^3_{\rm II}(\omega',q)=2\omega' e^{-2\pi M\omega'}f(\omega',q\lambda(\gamma-\dot R))\frac{e^{i\psi(q)}}{q}. \tag{3.7}$$

An exactly similar argument may be carried through for the spin-$\frac{1}{2}$ case. There are no differences for the path-I contributions while the wave at the shell is

$$\psi^k(R(\tau),T(\tau);q)\simeq\frac{e^{iq\lambda(\dot R-\gamma)\tau}}{(2\pi)^{1/2}Rq}e^{i[\overline{\eta}_k(q)+2Mq]}\times(-1)^{k+1/2}\begin{pmatrix}(2q\lambda)^{1/2}\\ 0\end{pmatrix},$$

which must be expressed in terms of the outgoing solutions:

$$\frac{e^{+i\omega'(r*-t)}}{(2\pi)^{1/2}rw^{1/2}(r)}\begin{pmatrix}(2|\omega'|)^{1/2}\\ 0\end{pmatrix}\simeq\left(\frac{|\dot R\tau|}{2M}\right)^{i4M\omega'-1/4}\times\begin{pmatrix}(2|\omega'|)^{1/2}\\ 0\end{pmatrix}.$$

However, the local frames with respect to which the spinors are defined on either side of the shell are not the same. The problem is to express the spinor boundary conditions across the shell in terms of the behavior of the shell. The radius of the shell, R, varies with the proper time of the shell, and the value of the radial coordinate, r, either just outside or just inside the shell is $R(\tau)$. Then, the time coordinate $t(\tau)$ just outside the shell is determined by the requirement that the four-velocity of the shell as seen by an outside observer is a unit timelike vector,

$$u^\mu=\left(\frac{dt}{d\tau},\dot R,0,0\right),$$

hence

$$\frac{dt}{d\tau}=\frac{[1-2\Phi(R)+\dot R^2]^{1/2}}{1-2\Phi(R)}\equiv\frac{n^1}{1-2\Phi}, \tag{3.8}$$

while the outward normal to the shell is

$$n^\mu=\left(\frac{+\dot R}{1-2\Phi},n^1,0,0\right). \tag{3.9}$$

From the inside, the same formulas hold except that $\Phi=0$, or

$$u^\mu=\left(\frac{dT}{d\tau},\dot R,0,0\right)=((1+\dot R^2)^{1/2},\dot R,0,0),$$
$$n^\mu=(\dot R,(1+\dot R^2)^{1/2},0,0). \tag{3.10}$$

The four-velocity and the normal may be expressed relative to the orthonormal basis vectors,

$$n^a=e^a{}_\mu n^\mu=\begin{cases}(\dot R,n^1,0,0)/(1-2\Phi)^{1/2}, & R>2M\\ (-n^1,-\dot R,0,0,)/(2\Phi-1)^{1/2}, & R<2M\end{cases} \tag{3.11}$$

on the outside while

$$n^a=e^a{}_\mu n^\mu=(\dot R,(1+\dot R^2)^{1/2},0,0)$$

on the inside, the difference in the components reflecting the misalignment of the basis vectors; the Lorentz transformation which brings them into alignment is

$$\Lambda^a{}_b=\begin{bmatrix}\cosh\alpha & \sinh\alpha & 0 & 0\\ \sinh\alpha & \cosh\alpha & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$n^a_{\rm in}=\Lambda^a{}_b n^b_{\rm out},$$

where

$$e^\alpha=[(1+\dot R^2)^{1/2}+\dot R](n^1-\dot R)/w(R)$$

and the continuity condition is

$$e^{-i(1/2)\alpha\sigma^{01}}\psi_{\rm out}=\psi_{\rm in}=\left[e^{\alpha/2}\left(\frac{1+\alpha^1}{2}\right)+e^{-\alpha/2}\left(\frac{1-\alpha^1}{2}\right)\right]\psi_{\rm out}. \tag{3.12}$$

The required relationship is then

$$e^{-i\nu\tau}=\int_{-\infty}^{\infty}d\omega\left(\frac{|\dot R\tau|}{2M}\right)^{i4M\omega-1/2}e^{-\epsilon 2\pi M\omega}\times f(\omega,\nu)e^{i\epsilon\pi/4}, \tag{3.13}$$

where

$$f(\omega,\nu)=\frac{4M}{2\pi}\frac{\Gamma(\frac{1}{2}-i4M\omega)}{(4M\omega)^{1/2}}\left(\frac{|\dot R|}{2M\nu}\right)^{-i4M\omega}\text{ for spin }\tfrac{1}{2}.$$

Then, the expressions for α and β all hold for spin $\frac{1}{2}$ with this expression for f replacing the spin-0 expression and the factor $e^{i\psi}$ becomes

$$e^{i\psi}=e^{i[\bar{\eta}_k(q)+2Mq]}(-1)^{k+1/2}\,. \tag{3.14}$$

The only significant difference between the expressions is the appearance of $\Gamma(\frac{1}{2}-i4M\omega)$ rather than $\Gamma(-i4M\omega)$; this results directly from the spin carried by the particle and the way in which the spinning particle must transform under Lorentz boosts. Although there is no direct connection with the statistics of the particles, the respective Γ functions are appropriate to produce the particle distributions required by the spin and statistics theorem.

The complete coefficients may then be written out:

$$\alpha=\begin{pmatrix}S_{11}\\ S_{21}\\ 0\end{pmatrix}\frac{2\omega'\delta(\omega'-\omega(q))}{(\omega'q)^{1/2}}+\begin{pmatrix}S_{12}\\ S_{22}\\ 0\end{pmatrix}2\omega'e^{2\pi M\omega'}f(\omega',q\lambda(\gamma-\dot{R}))\frac{e^{i\psi(q)}}{q}e^{i\pi S/2}$$
$$+\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}2\omega'e^{-2\pi M\omega'}f(-\omega',q\lambda(\gamma-\dot{R}))\frac{e^{i\psi(q)}}{q}e^{i\pi S/2}$$

and (3.15)

$$\beta=\begin{pmatrix}S_{12}^*\\ S_{22}^*\\ 0\end{pmatrix}2\omega'e^{-2\pi M\omega'}f(-\omega',q\lambda(\gamma-\dot{R}))\frac{e^{i\psi(q)}}{q}e^{-i\pi S/2}+\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}2\omega'e^{2\pi M\omega'}f(\omega',q\lambda(\gamma-\dot{R}))\frac{e^{i\psi(q)}}{q}e^{-i\pi S/2}\,,$$

where S is the spin of the particle.

With the aid of

$$ff^\dagger(\omega,\omega')\equiv\int_0^\infty\frac{dq}{2q}f(\omega,\lambda q)f^*(\omega',\lambda q)$$
$$=[1/(e^{4\pi M|\omega|}\mp e^{-4\pi M|\omega|})]\delta(\omega-\omega')/2|\omega|\,,$$

it is straightforward to calculate the various products

$$\alpha\alpha^\dagger(\omega,\omega')=2|\omega|\delta(\omega-\omega')\left[1\pm n(\omega)\begin{pmatrix}S_{12}S_{12}^* & S_{12}S_{22}^* & 0\\ S_{22}S_{12}^* & S_{22}S_{22}^* & 0\\ 0 & 0 & 1\end{pmatrix}\right], \tag{3.16}$$

where

$$n(\omega)=\frac{1}{e^{8\pi M|\omega|}\mp 1}\,,$$
$$\beta\beta^\dagger(\omega,\omega')=2|\omega|\delta(\omega-\omega')n(\omega)\times\begin{pmatrix}S_{12}S_{12}^* & S_{12}S_{22}^* & 0\\ S_{22}S_{12}^* & S_{22}S_{22}^* & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.17}$$

and

$$\alpha\beta^\dagger(\omega,\omega')=e^{i\pi S/2}\omega\delta(\omega-\omega')e^{4\pi M\omega}n(\omega)\times\begin{pmatrix}0 & 0 & S_{12}\\ 0 & 0 & S_{22}\\ S_{12} & S_{22} & 0\end{pmatrix}.$$

The expectation value of the number of particles emitted is $\beta\beta^\dagger$; the number escaping to infinity is just a thermal Bose-Einstein distribution characterized by a temperature $kT=1/8\pi M\omega$ or Fermi-Dirac distribution modified by the probability, $|S_{12}|^2$, that a particle near the event horizon moving outward will escape. (The probability is zero for $\omega^2<m^2$.) There are also the particles reflected through the event horizon with probability $|S_{22}|^2$. The off-diagonal terms arise because the particles which first appear outside the event horizon end up in a coherent state,

$$\begin{pmatrix}S_{12}\\ S_{22}\\ 0\end{pmatrix},$$

with amplitudes both at infinity and at the singularity; hence an operator acting in either place can annihilate the state. In addition, there are the negative-frequency particles in the interior; they too are produced with the appropriate thermal distribution and, since their state is characterized in terms of their behavior near the event horizon, there is no S matrix, S_{i2}, as in the other terms.

The product $\alpha\beta^\dagger$ is the matrix element of the annihilation operators

$$\alpha_i\beta_j{}^\dagger = \langle 0- \,|a_i a_j|\,0-\rangle ,$$

hence it is the overlap of the final state with the state with two particles removed. As is apparent from Eq. (3.17), this is only nonvanishing when one particle travels along the inside of the event horizon and the other either escapes to infinity or is reflected back through the event horizon; thus, the state $|0-\rangle$, when expressed in terms of the basis states of the future, is a superposition of pairs of particles. Furthermore, because there is no reaction back on the metric from the emission, the amplitude for n pairs being emitted is just, modulo a common normalization factor, the amplitude for one pair raised to the nth power. In that case $\langle aa\rangle$ must be $(1\pm n)$ times the amplitude for a single pair, $e^{-4\pi M\omega}$.

Although these properties may be obtained from $\overline{G}$, the amplitudes and correlations are most directly found from G. First, observe that the matrix element of $2n$ fields

$$i^n\langle 0+|T(\phi(x_1)\cdots\phi(x_{2n}))|0-\rangle/\langle 0+|0-\rangle = \sum_{\text{perm}}\prod_{k=1}^{n} G(x_{2k-1}, x_{2k}),$$

hence the amplitude for the $2n$-particle state is

$$\langle a_1\cdots a_{2n}+|0-\rangle/\langle 0+|0-\rangle = \sum_{\text{perm}}\prod_{k=1}^{n}[\langle a_{2k+}|0-\rangle/\langle 0+|0-\rangle]$$

exhibiting the uncorrelated emission of pairs.

To calculate G, the inverse of the matrix A defined in Eq. (2.18) with matrix elements given by Eqs. (3.16), (3.17), is required. The result is

$$A^{-1} = \begin{pmatrix} 1 & \mp\xi \\ \xi^* & 1 \end{pmatrix},$$

where (3.18)

$$\xi = e^{i\pi S}e^{-4\pi M\omega}\begin{pmatrix} 0 & 0 & S_{12} \\ 0 & 0 & S_{22} \\ S_{12} & S_{12} & 0 \end{pmatrix}$$

and the frequency δ function, $2\omega\delta(\omega-\omega')$, has been suppressed.

If the Green's function is then used to evaluate the single-particle matrix element of the field, the result is

$$\langle a+|\phi(x)|0-\rangle/\langle 0+|0-\rangle = \langle a|\alpha\psi^\dagger \mp \langle a|\xi\beta\psi^\dagger$$

and (3.19)

$$\langle a,b+|0-\rangle/\langle 0+|0-\rangle = \langle a|\xi|b\rangle .$$

The amplitude vanishes unless the final state consists of a pair of particles, one of which travels just inside the event horizon while the other travels outside the event horizon with a probability $|S_{12}|^2$ of escaping. The angular momentum has been suppressed; the amplitude vanishes unless the pair of particles has zero total angular momentum.

With the emission amplitudes in hand, the range of validity of the expressions may be studied. Two points are to be established: First, the Hawking radiation appears as the collapse is observed, there is no delay; and second, actual as opposed to potential existence of the event horizon does not play a direct role in the production of the radiation which appears at infinity.

To study these questions, consider a wave packet rather than an energy eigenstate; then the wave function for the escaping particle is given by

$$\psi_f(x) = \int_\mu^\infty \frac{d\omega}{2\omega}\,\psi_1^f(x;\omega)f(\omega),$$

where ψ_f describes a particle which has average frequency ω_0 with an uncertainty $\delta\omega$. If the wave packet is chosen so that the particle escapes at late times, then, as in Sec. II,

$$\begin{aligned}\langle n_f\rangle &= \int d\sigma_\mu d\sigma'_\nu\,\psi_f(x)\frac{1}{i}\overleftrightarrow{\nabla}\mu\,\langle 0-|\phi(x)\phi(x')|0-\rangle \\ &\quad\times\frac{1}{i}\overleftrightarrow{\nabla}\nu'\psi_f^*(x) \\ &= \int_\mu^\infty\frac{d\omega}{2\omega}|f(\omega)|^2 n(\omega) .\end{aligned}$$

Because ψ_f is a solution to the wave equation outside the shell, this may be rewritten as an integral along the world line of the shell using

$$\begin{aligned}0 &= \int_{\text{exterior of shell}} d\sigma\,\nabla_\mu\left[\psi_f(x)\frac{1}{i}\overleftrightarrow{\nabla}\mu\,\phi(x)\right] \\ &= \int_{\text{surface}} d\sigma_\mu\,\psi_f\frac{1}{i}\overleftrightarrow{\nabla}\mu\,\phi(x)\end{aligned}$$

as

$$\langle n_f\rangle = \left|\int_{-\infty}^{0} d\tau' R^2(\tau)\psi_f(x(\tau))\frac{1}{i}\overleftrightarrow{\nabla}_n\psi^i(x(\tau);q)\right|^2 \times\frac{q^2dq}{2\omega(q)} .$$

But, along the shell, $\psi_f(x(\tau))$ becomes

$$\int_\mu^\infty \frac{d\omega}{2\omega} f(\omega)\left(\frac{\dot{R}\omega}{2M}\right)^{i4\pi M\omega} \frac{S_{12}^*}{(2\pi)^{1/2} r}$$

$$\simeq \left(\frac{\tau}{\tau_0}\right)^{i4M\omega_0} \frac{e^{-[\delta\omega_0 4M \ln(\tau/\tau_0)]^2}}{[2\pi(\delta\omega_0/\omega_0)]^{1/2} 2M} S_{12}^*$$

for a Gaussian wave packet whose center, near the shell is at τ_0. The wave packet is localized in the interval ($\tau_0 < 0$) $\tau_0 e^{M\delta\omega_0/2} \lesssim \tau \lesssim \tau_0 e^{-M\delta\omega_0/2}$ and vanishes more rapidly than any other power as τ approaches the event horizon, $P=0$. The width, $\delta\tau$, of the wave packet along the shell is determined by how far the center of the wave packet is from the event horizon. A typical wave packet is shown in Fig. 3.

This expression is valid provided that ψ_l' is, in fact, well approximated by $e^{i\omega(r^*-t)}$ at the region where the wave is incident. If we require that the corrections be less than 10%, this will be true, as may be seen from either an eikonal approximation or a power-series expansion around $r=2M$, if

$$|r-2M| \lesssim \frac{(0.1)2M|1+4i\omega M|}{l(l+1)+1+(2M_\mu)^2} ,$$

which corresponds to, for $\dot{R} \sim -\frac{1}{2}$,

$$|\tau| \lesssim \frac{(0.1)4M|1+4i\omega M|}{l(l+1)+1+(2M^\mu)^2} .$$

The ω values of primary interest are $2M \lesssim 1/4\pi$, hence

$$|\tau| \lesssim (0.1)4M/[l(l+1)+1] .$$

For larger values of l the radiation will not appear until later times, but the $l=0$ contribution will begin appearing at the time when the shell is seen from afar to have passed $R-2M \sim 0.4M$.

Note that only the behavior of the wave packet along the shell is important in producing the effect; it is irrelevant whether there is, in fact, an event horizon at $\tau=0$. Thus, if the mass of the shell is taken to decrease because of the emitted radiation, the radiation will still occur chasing an apparently ever-receding event horizon. Although the radiation will change the metric and that change will, in turn, affect the radiation, that effect will be small as long as there is no significant change over the wave packet.

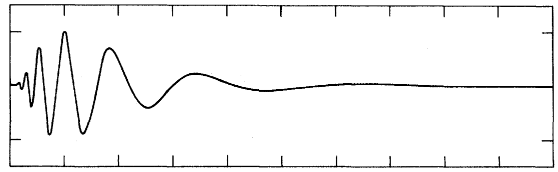

FIG. 3. Graph of a wave packet localized along the collapsing shell which will have a frequency of $\omega_0 = 5/M$ when it escapes.

IV. THE STRESS TENSOR

In order to calculate the effect of the radiation and the existence of the quantum field on the shell and on the metric, it is necessary to calculate the stress-energy tensor of the field. Knowledge of the stress-energy tensor enables one, from its discontinuity across the shell, to calculate the rate at which energy and momentum are extracted from the shell and provides the source for the perturbation of the metric by the radiation. Further, knowledge of the matrix elements of the stress-energy tensor at infinity provides an independent calculation of the energy carried by the particles as they escape; the latter calculations have been given by DeWitt[5] for the expectation value of the stress-energy tensor in the initial vacuum in agreement with the results of Hawking and those given here.

The stress-energy tensor is the product of two fields at the same spacetime point, thus it must be renormalized. In flat spacetime, the parts which must be renormalized are just the vacuum expectation values, the singular structure of which is determined by the short-distance behavior of the field Green's functions. These are solely determined by the local high-energy behavior which is, in turn, determinable from an expansion of the Green's function in Riemann normal coordinates or the short-distance expansion of DeWitt.[5] There is no interesting physics in the renormalizations; it is the deviation of the stress-energy tensor from that in an appropriately chosen reference state that is important. As the reference state, I have chosen the vacuum state of the Kruskal spacetime for which, due to time reversal invariance, there is no energy flux. It is then straightforward to express the matrix element of $T^{\mu\nu}$ between arbitrary states as the (renormalized, but uninteresting) matrix element in the Kruskal vacuum plus an interesting deviation. To start, consider the matrix element between the past and future vacua which is formally given by

$$\begin{aligned} \langle 0+|T^{\mu\nu}(x)|0-\rangle/\langle 0+|0-\rangle &\equiv {}_+\langle T^{\mu\nu}(x)\rangle_- \\ &= \langle 0+|\{\partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\tfrac{1}{2}[(\partial_\lambda\phi)(\partial^\lambda\phi)+\mu^2\phi^2]\}|0-\rangle/\langle 0+|0-\rangle \end{aligned} \tag{4.1}$$

for the scalar field and may be written in terms of the Green's function,

$$ {}_{+}\langle T^{\mu\nu}(x)\rangle_{-} = -i\left[\nabla^{\mu}\nabla'^{\nu} - g^{\mu\nu}\tfrac{1}{2}(\nabla_{\lambda}\nabla'^{\lambda} + \mu^2)\right] \times G(x,x')\big|_{x=x'} \ . \tag{4.2}$$

In the region where the wave functions, ψ^l, may be written in their asymptotic form

$$\psi^l \simeq \psi_l\alpha + \psi_l^*\beta, \tag{4.3}$$

then the Green's function, Eq. (2.17) becomes for a given angular momentum mode, l,

$$G^l(x) = i\left[\psi_l(x_>)\psi_l^{\dagger}(x_<) + \psi_l(x)\xi\psi_l^{T}(x')\right], \tag{4.4}$$

where ξ is defined in Eq. (2.18). The first term is precisely the Green's function[10] for the Kruskal spacetime; the stress-energy tensor constructed from it has all the renormalization problems characteristic of flat space plus some additional ones associated with the renormalization of the gravitational coupling constant and the renormalization of R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms in the action.[5,11] These renormalizations are not difficult in principle; however, in practice it is not known whether there is a *nonvanishing* residue to the vacuum expectation value of the stress tensor after the renormalizations have been performed because the renormalization either involves locally flat coordinates or, equivalently, a short-distance expansion which is not easily related to the mode sum used here. The correction term to the Kruskal value for $\langle T^{\mu\nu}\rangle$ vanishes unless both points are inside the event horizon; there it is the product of two positive-frequency waves traveling in opposite directions, the contribution of which is rapidly oscillating near the event horizon, the only region where it does not vanish.

Furthermore, the full expectation value, contracted with the four-velocity u of the shell or the normal n, must be continuous across the shell because both ϕ and its normal derivative are continuous and the renormalization procedure effectively only subtracts terms of the form $\delta\lambda g^{\mu\nu}$ which must be the same on both sides (in both cases the constant may be determined using Riemann normal coordinates). Thus, there is no pressure on the shell nor is any work done on the shell by the matter stress-energy tensor.

The matrix element of the stress-energy tensor between the initial vacuum and the state containing a pair of particles may be calculated from

$$\begin{aligned}\langle 0+|T(\phi(y)\phi(y')\phi(x)\phi(x'))|0-\rangle/\langle 0+|0-\rangle &= -G(y,y')G(x,x') - G(y,x)G(y',x') \\ &\quad - G(y,x')G(y',x) \ .\end{aligned} \tag{4.5}$$

If $\phi(y)$ and $\phi(y')$ are used to create the particles in the final state and $\phi(x)$ and $\phi(x')$ to form the stress-energy tensor, the first term reproduces the expression for the $\langle 0+|T^{\mu\nu}|0-\rangle/\langle 0+|0-\rangle$ matrix element times the emission amplitude. The remaining terms yield the stress energy associated with the produced particles. Again the expression for the discontinuity of $(\alpha u_\mu + \beta n_\mu)n_\nu T^{\mu\nu}$ vanishes across the shell and there is no direct reaction on the shell. The entire reaction back on the shell must come through the effect which the radiation has on the metric.

The additional terms for the stress-energy tensor do exhibit the radiation; they become

$$\begin{aligned} T^{\mu\nu}(x';a,b) = {}& \nabla^{\mu}\phi(x';a)\nabla^{\nu}\phi(x';b) \\ &+ \nabla^{\nu}\phi(x';a)\nabla^{\mu}\phi(x';b) \\ &- g^{\mu\nu}\left[\nabla_{\lambda}\phi(x';a)\nabla^{\lambda}\phi(x';b)\right. \\ &\left. + \mu^2\phi(x';a)\phi(x';b)\right], \end{aligned} \tag{4.6}$$

where

$$\phi(x;a) \equiv \langle a+|\phi(x)|0-\rangle/\langle 0+|0-\rangle \ .$$

The wave function $\phi(x;a)$ may be written in terms of the past basis functions, ϕ^l, using the reduction formula and the Green's function, Eqs. (2.17) and (3.18),

$$\phi(x;a) = \alpha_a\psi^{l\dagger} - \xi_a\beta\psi^{l\dagger} \ ,$$

and, in the regions where the future asymptotic forms apply,

$$\phi(x;a) \simeq \psi_{la}^{\dagger}(x) + \xi_a\psi_l^{T}(x) \ . \tag{4.7}$$

The $\psi_a^{\dagger}$ term comes from the field operator, ϕ, annihilating the particle in the state $\langle a+|$, while the other term describes the creation of a particle by ϕ acting to the left. The matrix ξ_{ab} is just the amplitude with which the initial state $|0-\rangle$ develops into the final state, $\langle a,b,+|$, containing particles in modes a and b.

In the region exterior to the shell, but near the event horizon or inside the event horizon, the asymptotic forms may be used to a good approximation. Then, the terms involving $\psi^{\dagger}\psi^{\dagger}$ and $\psi\psi$ oscillate "rapidly" and any average over space or time will eliminate these terms. The cross terms all involve the matrix ξ_{ac} which vanishes unless the modes a, c refer to a pair of created particles.

Consider the case where a represents a particle which escapes to infinity, b represents a particle traveling along the inside of the event horizon, and x is outside the event horizon along the path of the escaping particle. Then the stress-energy tensor becomes

$$T^{\mu\nu}(x;a,b)=e^{-4\pi M\omega_b}S_{12}(\omega_b)\{\nabla^\mu\psi_l^\dagger(x,\omega_a)\nabla^\nu\psi_l(x,\omega_b)+\nabla^\nu\psi_l^\dagger(x,\omega_a)\nabla^\mu\psi_l(x,\omega_b)$$
$$-g^{\mu\nu}[\nabla_\lambda\psi_l^\dagger(x,\omega_a)\nabla^\lambda\psi_l(x,\omega_b)+\mu^2\psi_l^\dagger(x,\omega_a)\psi_l(x,\omega_b)]\}. \tag{4.8}$$

The factor in curly brackets is $\langle a|T^{\mu\nu}|b\rangle$, the matrix element of the stress tensor where $|b\rangle$ has energy ω_b and the angular, Y_l^m, dependence is included in ψ. If a wave packet is used to specify the time at which the particle $\langle a|$ arrives at a given (large) radius, then, as discussed in Sec. III, the wave function, $\psi^\dagger$, will be localized along the world line of the particle and $T^{\mu\nu}$ will also. Thus, there is a well-defined localizable energy-momentum flux associated with the particles which escape (and, by a similar argument, with the reflected particles). The factor $e^{-4\pi M\omega_b}S_{12}$ is just the amplitude for the emission of the pair.

There is an exactly analogous expression for the flux on the inside of the event horizon which describes the flow along the inside of the event horizon and the flow of the particles which are reflected to cross the event horizon.

It is now easy to calculate the expectation value of the stress-energy-tensor in the initial state $\langle 0-|T^{\mu\nu}(x)|0-\rangle\equiv\langle T^{\mu\nu}(x)\rangle$. The Green's function, $\overline{G}$, for given angular momentum, may be written

$$\overline{G}=i\int\frac{q^2dq}{2\omega(q)}\,\phi^l(x_>,q)\phi^{l\dagger}(x_<,q)$$
$$=i\psi_l(x),\psi_l(x_<)^\dagger$$
$$+(\psi_l(x),\psi_l^*(x))\begin{pmatrix}\beta^*\beta^T & \alpha\beta^\dagger\\ \beta\alpha^\dagger & \beta\beta^\dagger\end{pmatrix}\begin{pmatrix}\psi_l^\dagger(x')\\ \psi_l^T(x')\end{pmatrix} \tag{4.9}$$

Again, the first term is the Green's function for the Kruskal spacetime and the measureable stress energy will be the deviation from that of the Kruskal spacetime. As before, the $\psi\psi^T$ and $\psi^*\psi^\dagger$ terms are rapidly oscillating and yield a vanishing average contribution. The remaining term is the stress-energy associated with the radiation,

$$\langle T^{\mu\nu}(x)\rangle_{\rm rad}=\sum_l(2l+1)\mathrm{tr}\beta^*\beta^T\{\nabla^\mu\psi^\dagger\nabla^\nu\psi+\nabla^\nu\psi^\dagger\nabla^\mu\psi-g^{\mu\nu}[\nabla_\lambda\psi^\dagger\nabla^\lambda\psi+\mu^2\psi^\dagger\psi]\}(x) \tag{4.10}$$

which is the sum over all modes of the radiation probability, $\beta^*\beta^T$, times the curly-bracketed term which is the stress-energy of the emitted mode. When the asymptotic form of ψ is used along with the expression for $\beta^*\beta^T$, Eq. (3.17), and the appropriate, $q^2dq/2\omega$, normalization for particles escaping to infinity used, the energy density at infinity, measured by an observer at fixed r becomes

$$T^{00}(r)=\frac{1}{4\pi r^2}\sum_l(2l+1)\int_0^\infty\frac{dq\,\omega\,|S_{12}|^2(\omega)}{2\pi(e^{8\pi M\omega}-1)}$$

and the energy flux is (4.11)

$$T^{0r}(r)=\frac{1}{4\pi r^2}\sum_l(2l+1)\int_0^\infty\frac{dq\,q\,|S_{12}|^2(\omega)}{2\pi(e^{8\pi M\omega}-1)},$$

where $|S_{12}|^2(\omega)$ is the probability of a particle near the shell escaping to infinity.

Near the shell, the treatment is somewhat delicate; $\phi(x)$ is a superposition of waves traveling both in and out; the cross term averages to zero, because it varies rapidly with time. If the observer follows a path $t(\tau)$, $r(\tau)$ and has four-velocity

$$u^\mu=\left(\frac{dt}{d\tau},\frac{dr}{d\tau}\right)$$
$$=\left(\frac{n^1}{1-2\phi(r)},\dot r\right)$$

then the unit spacelike vector orthogonal to u in the r-t plane is

$$n^\mu=\left(\frac{\dot r}{1-2\phi},n^1\right),$$

with

$$n^1=[1-2\phi(r)+\dot r^2]^{1/2}.$$

The derivatives of the fields are then given by

$$u\partial e^{-i\omega(t\mp r^*)}=-\frac{i\omega}{1-2\phi}(n^1\mp\dot r)+\mp n\partial e^{-i\omega(t\mp r^*)},$$

which exhibits both the gravitational blue-shift and the observer's Doppler shift. Thus, the energy density as measured by the observer with four-velocity u is

$$u_\mu T^{\mu\nu}u_\nu(x)=\sum_l\frac{(2l+1)/4\pi}{2\pi(2M)^2}\int_0^\infty\frac{d\omega\,\omega}{e^{8\pi M\omega}1}\left[\left(\frac{n^1-\dot r}{1-2\phi}\right)^2+|S_{22}(\omega)|^2\left(\frac{n^1+\dot r}{1-2\phi}\right)^2\right] \tag{4.12}$$

The first term corresponds to the outgoing particles which are strongly blue-shifted if the observer is falling in ($\dot{r}<0$) and the second term corresponds to particles which are crossing the event horizon and are red-shifted for an observer falling in so that their energy is of the order of neglected terms. In any case, each term is infinite because all angular momenta may contribute for the particles which reenter the event horizon; however, the infinite sum does not, in fact, appear because at any given spacetime point, x, only waves which emerge from the shell within the past light cone of x contribute; hence there is some $\tau(x)$ which is the last proper time contributing. For large l, the wave is well approximated by $e^{\pm i\omega r^*}$ only for

$$r-2M\lesssim\frac{2M|4M\omega i-1|}{(2M\mu)^2+l(l+1)+1}\,,$$

hence if the shell is taken to cross the event horizon at $\tau=0$, the radiation will be present at x only for those modes with

$$\tau(x)\gtrsim-\frac{2M[(4M\omega)^2+1]^{1/2}}{|\dot{R}|[2M\mu+l(l+1)+1]} \tag{4.13}$$

and, for any given x, the sum must be cut off at $l_m(x)$ where

$$l_m(x)[l_m(x)+1]\simeq\frac{2M}{|\dot{R}\tau(x)|}\simeq\frac{2M}{\delta}\,,$$

where δ is the distance measured from the event horizon along the shell. Thus, the energy density and the corresponding energy flux

$$\begin{aligned}-u_\mu T^{\mu\nu}(x)n_\nu&=\sum_{l<l_m(x)}\frac{(2l+1)/4\pi}{2\pi(2M)}\int_0^\infty\frac{d\omega\,\omega}{e^{8\pi M\omega}-1}\left[\left(\frac{n^1-\dot{r}}{1-2\phi}\right)^2-[1-p_l(\omega)]\left(\frac{n^1+\dot{r}}{1-2\phi}\right)^2\right]\\&\simeq1/M\delta^3,\quad R-2M\sim\delta\end{aligned} \tag{4.14}$$

are finite.

In the past, the expectation value of the stress-energy tensor must be renormalized as before. There, the Green's function, $\overline{G}$, is exactly what it would be if the shell never collapsed; thus, after renormalization, $\langle T^{\mu\nu}\rangle$ is exactly the same as if the shell were to remain static; there is no indication of the radiation which will appear (although $\langle T^{\mu\nu}\rangle$ will differ from that of the Kruskal spacetime by a finite amount).

On the other hand, the matrix element of $T^{\mu\nu}$ between the initial and final vacua may be calculated from G, Eqs. (3.16) and (3.17); the contribution of the first term,

$$\psi^l(x_>)\psi^{l\dagger}(x_<)\,,$$

is just the vacuum matrix element of $T^{\mu\nu}$ for a static shell. The remaining terms reflect the radiation and may be calculated with the aid of the explicit forms for α, β, and G, and the relation

$$\int_0^\infty\frac{\nu^2d\nu}{2\nu}\,\psi^{l\dagger}(x,\nu)f(\omega,\lambda\nu)\frac{e^{i\psi}}{\nu}=\frac{1}{2\omega r(2\pi)^{1/2}}\left(\frac{\lambda2M}{|\dot{R}|(\xi+i\epsilon)}\right)^{i4M\omega}\frac{e^{2\pi M\omega}}{e^{8\pi M\omega}-1}\,, \tag{4.15}$$

where

$$\xi\equiv\overline{r}+t-\xi_0$$

is equal to zero along the null geodesic which, after passing through the interior of the shell, becomes the event horizon. The result is, in the past,

$$[\langle0+|T^{\mu\nu}(x)|0-\rangle/\langle0+|0-\rangle]-\langle0-|T^{\mu\nu}(x)|0-\rangle=-\frac{k^\mu k^\nu}{4\pi r^2}\left(\frac{4M}{\xi+i\epsilon}\right)^2\sum_l\int_0^\infty\frac{d\omega\,\omega(2l+1)}{(e^{8\pi M\omega}-1)2\pi}\,,$$

where

$$k^\mu=\nabla^\mu\xi\,.$$

For $\xi\neq0$, the result is negative definite, reflecting the absence of the stress energy which ultimately makes up the radiation. However, the expression comes from $\psi^\dagger\psi^\dagger$ terms in G which must average to zero and, indeed, because of the negative-frequency conditions, ξ appears as $\xi+i\epsilon$ and the integration over all space vanishes; there is, effectively, a δ function with an infinite coefficient along the $\xi=0$ line. As with the expression for $\langle T^{\mu\nu}\rangle$ near the shell and near the event horizon, the divergent sum over l is illusory and for given x near the $\xi=0$ line the matrix element is finite. Both the δ function at $\xi=0$ and the divergent sum arise from the lack of reaction by the metric.

The contribution of a single emitted pair may be found from Eq. (4.6) evaluated at early times where

$$\phi(x;a)\simeq\begin{pmatrix} S_{12}\\ S_{22}\\ 0\end{pmatrix}\frac{e^{-4\pi M\omega}}{\omega r(2\pi)^{1/2}}\frac{1}{2}\left(\frac{\lambda 2M}{|\dot{R}|(\xi+i\epsilon)}\right)^{i4M\omega}+\begin{pmatrix}0\\0\\1\end{pmatrix}\frac{1}{\omega r(2\pi)^{1/2}}\frac{1}{2}\left(\frac{\lambda 2M}{|\dot{R}|(\xi+i\epsilon)}\right)^{-i4M\omega}.$$

If a wave function, $f(\omega)$, such that

$$\int_0^\infty \frac{d\omega}{2\omega}\frac{e^{-i\omega\eta}}{(2\pi)^{1/2}}f(\omega)=e^{-i\omega\eta}\frac{e^{-(\eta-a)/4\delta^2}}{(2\pi)^{1/2}}$$

is used, then

$$\nabla^\mu\phi(x;f_a)=\frac{-i2Mk^\mu}{\xi+i\epsilon}\left[\begin{pmatrix} S_{12}\\ S_{22}\\ 0\end{pmatrix}\left(\frac{\lambda 2M}{|\dot{R}|e^{-i\pi}(\xi+i\epsilon)}\right)^{i4M\omega_0}\frac{e^{-(\eta_--a)^2/4\delta^2}}{(2\pi)^{1/2}\delta}-\begin{pmatrix}0\\0\\1\end{pmatrix}\left(\frac{\lambda 2M}{|\dot{R}|(\xi+i\epsilon)}\right)^{-i4M\omega_0}\frac{e^{-(\eta-a)^2/4\delta^2}}{(2\pi)^{1/2}\delta}\right],$$

where

$$\eta=0.4M\ln[\lambda 2M/|\dot{R}|(\xi+i\epsilon)],$$

$$\eta_-=-\eta-i\pi.$$

The stress-energy tensor, $T^{\mu\nu}(x;f_a,f_b)$, calculated from this wave function is well localized away from the $\xi=0$ line; however, because the product of factors, one of which contains η while the other contains η_-, yields a non-positive-definite $T^{\mu\nu}$ whose integral vanishes. In the past, there is no net energy-momentum associated with the produced particles.

A similar treatment just inside the shell and near the event horizon also yields the result of no net energy-momentum flux associated with the produced particles.

In conclusion there is, both inside and outside the shell, a well-defined energy-momentum flux associated with each pair of emitted particles. This flux is covariantly conserved and (except for the flux associated with the particles which are reflected back through the event horizon) vanishes at the event horizon. In the past, before the shell has collapsed, the stress energy associated with the emitted pair is not zero, but it has vanishing total energy and momentum. Inside the event horizon there is a well-defined, locally positive, energy density associated with the particles, but their Killing "energy" is negative and opposite to that of the particles escaping to infinity. Near and inside the shell, stress-energy density is nonvanishing but the total energy-momentum vanishes. The quantity calculated is a matrix element between an initial state and a final state, thus the matrix element reflects in the past the emission of the pair which is to occur: It gives the stress energy *if* one pair is emitted.

The expectation value in the initial state $\langle T^{\mu\nu}\rangle$ behaves somewhat differently. Before the shell begins its collapse, $\langle T^{\mu\nu}\rangle-\langle T^{\mu\nu}\rangle_{\text{Kruskal}}$ has a static nonvanishing value, the value in the presence of a static shell; this arises because the modes are changed by the presence of the shell. In the future, the expectation value differs from the Kruskal value by the radiation *and*, along the inside of the event horizon, by the oscillating terms which must average to zero (the latter terms force the full stress-energy tensor to violate the dominant energy condition: There must exist states in which the full energy density has negative values[16]; in a free field as here the negative contributions arise when $T^{\mu\nu}u_\mu u_\nu$ connects states of different particle number). Because of the nonvanishing of $\langle T^{\mu\nu}\rangle-\langle T^{\mu\nu}\rangle_{\text{Kruskal}}$ in the past and the violation of the dominant energy condition, the conservation of the stress-energy tensor, in addition to being locally satisfied in the renormalization process, is consistent with the particle creation.

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