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Some predictions of the Dashen–Gell-Mann program concerning the asymptotic behavior of form factors and the threshold behavior of the scaling functions νW_2. I. General considerations and the case of mesons

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It is shown, within a specific model, that the quark-model representations of current algebra at infinite momentum can be reconciled with the relativity requirement in the restricted kinematical domain of large momentum transfers (in the configuration space this corresponds to the values of the particle wave function in the neighborhood of the origin). We conjecture that this feature holds true in general and then make a number of specific predictions concerning the asymptotic behavior of the form factors and the threshold behavior of the scaling functions νW_2. The mesons are treated as usual, namely as quark-antiquark pairs, while for baryons it is assumed that two of the three quarks are glued together (diquark), leaving only one space-time coordinate free. In this way the dynamically complicated issue of three bodies is reduced to a more manageable case of two bodies.

INTRODUCTION

Recent work of Melosh[1] has caused revival of interest in the Dashen–Gell-Mann program[2] of relativistic quark-model representations of the current algebra at infinite momentum (the algebra of form factors). The currents constructed by Melosh, however, correspond to the picture of free noninteracting quarks, leaving the issue of forces between quarks essentially untouched. The present work can also be regarded as a modest attempt to shed some light on this issue of obvious physical importance.

The central objects in the Dashen–Gell-Mann program are the currents (quark-model realizations of the local current algebra at infinite momentum)

$$F_i(\vec{q}_\perp)=\sum_{\sigma=1}^{N}\frac{\lambda_{i,\sigma}}{2}\exp(i\vec{q}_\perp\cdot\vec{X}_{\perp,\sigma}),\qquad i=0,1,2,\ldots,8.$$

$\lambda_{i,\sigma}/2$ are the quark charges, actually the 3×3 Gell-Mann matrices, $\vec{X}_{\perp,\sigma}$ are the quark "coordinates," and N is the number of quarks in the particle. More specifically, if mesons are visualized as the bound states of $q\bar{q}$ pairs with the "distance" $\vec{X}_\perp$ between q and $\bar{q}$ as the only space-time degree of freedom, the associated currents will read

$$F_i(\vec{q}_\perp)=\frac{\lambda_{i,1}}{2}\exp(i\vec{q}_\perp\cdot\vec{X}_\perp/2)+\frac{\lambda_{i,2}}{2}\exp(-i\vec{q}_\perp\cdot\vec{X}_\perp/2). \qquad (1)$$

All indications are[3] that in any model except the free-quark model, the described currents involving only a finite number of quark terms are incompatible with the Dashen–Gell-Mann angular condition.[2,4] Indeed, in the model that we discuss in the present article the clash between relativity and the current algebra becomes quite apparent: The mesonic currents (1) fail to satisfy the equations of the angular condition. However, not everything is lost. It turns out that the described clash is a function of the kinematical domain under consideration. In some domains it may become very pronounced, while in others it is for all practical purposes negligible. We will show that in at least one nontrivial model the requirement of relativity can be imposed on the mesonic currents (1) in the limit of large momentum transfers $\vec{q}_\perp$ or, equivalently, for small values of $\vec{X}_\perp$. It is reasonable to assume that this phenomenon is not isolated, but occurs in most models of physical interest. This is our conjecture: At least in the case of mesonic currents the predictions of the Dashen–Gell-Mann program can be trusted in those experimental circumstances which place emphasis on the neighborhood of the origin of the particle wave function. In other words, according to this conjecture the asymptotic behavior of the elastic form factors, or the threshold behavior of the scaling functions νW_2, will be predicted accurately, while the predictions concerning the mean square radius of the particle, for example, should be taken with reservations because an evaluation of this quantity requires the knowledge of the particle wave function away from the origin.

What about the baryonic currents? They involve three quark terms instead of two as in the mesonic case. This causes a problem since we do not know how to use the angular condition in this case. The difficulty can be circumvented, however, if we make the assumption that two of the quarks in the baryon are glued together (diquark). This way the difficult case of the three-body problem is reduced to the tractable case of two bodies. We set aside the issue whether such a picture of baryons is consistent with other aspects of particle phenomenology such as the mass spectra, magnetic moments, decay rates, etc., and make a few specific predictions concerning the asymptotic behavior of the form factors and the threshold behavior of the structure functions νW_2. We discuss the diquark model at length in the second paper of this two-article series.

THE MODEL

We consider the model due to Leutwyler,[5] whose $(\text{mass})^2$ operator is given by the expression

$$M^2 = \tfrac{1}{2}\{\alpha, \vec{\pi}_\perp^{\,2}\} + \beta . \tag{2}$$

α and β are specific functions of $\vec{x}_\perp$, the same "coordinate" that appears in the current operator. Their explicit form is

$$\alpha = \frac{a|\vec{x}_\perp|}{1-a|\vec{x}_\perp|}$$

and

$$\beta = b^2(1+\alpha) + \frac{\alpha}{4|\vec{x}_\perp|^2}(1+\alpha)^2,$$

where a and b are constants.

$\vec{\pi}_\perp$ is the conjugate "momentum" variable satisfying the canonical commutation relations

$$[x_i, \pi_j] = i\delta_{ij}, \quad i,j = 1,2.$$

First of all, the model possesses the nontrivial and physically interesting mass spectrum (there are actually two branches; one is ascending and physical, while the other is decreasing and presumably of no significance for the physics of hadrons)

$$M_J = aJ \pm (a^2J^2 + b^2)^{1/2}, \quad J = 0,1,2,\ldots .$$

Equally important, the model gives rapidly decreasing form factors whose shape is not too far from the shape of the observed form factors (of π mesons and nucleons). Finally, it leads to a nontrivial scaling function νW_2. In a way the model can be looked upon as a crude but complete description of the world of hadrons, regarded as composite objects. The mass operator (2) satisfies the equations of the angular condition ($\vec{J}$ is the angular momentum—the spin of the particle)

$$\begin{aligned} &[[[M^2, x_+], x_+], x_+] = 0, \\ &[M^2, x_+^2] + 4i[MJ_+, x_+] = 0, \end{aligned} \tag{3}$$

with

$$x_\pm = x_1 \pm i x_2, \quad J_\pm = J_1 \pm i J_2 .$$

It is this fact that makes the model acceptable as far as the requirement of relativistic invariance is concerned. Now, strictly speaking, the angular condition is satisfied only if it is imposed on a single quark term

$$\exp(i\vec{q}_\perp \cdot \vec{x}_\perp).$$

If we want to subject the complete mesonic currents (1) to the requirement of the relativistic invariance, the angular condition (3) should be applied [in the limit of SU(3) symmetry] to each of the two quark terms in $F_i(\vec{q}_\perp)$ separately. As a consequence we obtain

$$[M^2, x_+^2] = 0. \tag{4}$$

Clearly, the mass operator (2) does not meet the additional requirement (4), and we see explicitly in which form the clash develops between the relativity and the current algebra. If we evaluate the commutator $[M^2, x_+]$ (note that $[M^2, x_+^2] = \{[M^2, x_+], x_+\}$), we obtain

$$[M^2, x_+] = -i\left\{\frac{a|\vec{x}_\perp|}{1-a|\vec{x}_\perp|}, \pi_+\right\}, \quad \pi_\pm = \pi_1 \pm i\pi_2 .$$

The commutator $[M^2, x_+^2]$ vanishes for $a|\vec{x}_\perp| \ll 1$. In other words, for $a|\vec{x}_\perp| \ll 1$ the angular condition can be reconciled with the form of the mesonic currents (1). For finite values of $\vec{x}_\perp$, however, this is no longer true.

An alternative way of looking at this problem is to expand the commutator $[M^2, x_+^2]$ in the power series in a:

$$[M^2, x_+^2] = -i\{\{|\vec{x}_\perp|, \pi_+\}, x_+\}a + \text{higher powers in } a.$$

For $a = 0$ the angular condition is satisfied exactly. For small values of a the angular condition is violated with the intensity proportional to a, in the first approximation. Higher powers in a become important for larger values of a. Note that the physical significance of a is that of an average mass splitting, divided by 2:

$$\Delta M_J \simeq a + \frac{a^2J + a^2/2}{(a^2J^2 + b^2)^{1/2}} .$$

For J large, $\Delta M_J \simeq 2a$.

FORM FACTORS

Even without knowledge of a mass operator, some useful information about the behavior of the form factors can be obtained just on the basis of the very special representation (1). In particular, we show that the matrix elements

$$\langle\lambda'|\exp(i\vec{q}_\perp\cdot\vec{X}_\perp)|\lambda\rangle$$

decrease for $|\vec{q}_\perp|\to\infty$ not slower than $1/|\vec{q}_\perp|^2$, a result which leads to an interesting upper bound for the form factors. To show this we only need some natural assumptions concerning the behavior of the (infinite momentum) particle wave functions $\psi_\lambda(\vec{x}_\perp)$ at the origin. Let us consider first the easier case of diagonal matrix elements. The individual quark piece of the particle form factor is equal to $[r=|\vec{x}_\perp|,\ \rho(r)=|\psi_\lambda(\vec{x}_\perp)|^2$, and $J_0(z)$ is the Bessel function of zeroth order$]$

$$\langle\lambda|\exp(i\vec{q}_\perp\cdot\vec{X}_\perp)|\lambda\rangle=\frac{1}{2\pi}\int\psi^*_\lambda(\vec{x}_\perp)\exp(i\vec{q}_\perp\cdot\vec{x}_\perp)\psi_\lambda(\vec{x}_\perp)d^2x_\perp$$
$$=\int_0^\infty J_0(|\vec{q}_\perp|r)\rho(r)r\,dr.$$

We now assume that $\rho(r)$ is regular at the origin, i.e.,

$$\rho(r)\underset{r\to0}{\simeq}r^m\exp(-\kappa r),\quad m=0,1,2,\ldots.$$

The only role of the exponential function is to ensure the convergence of the integrals under consideration. Besides, such exponentially decreasing behavior has been observed in models, and is presumably a general feature of all theories at infinite momentum that employ the *unitary* representations of the Lorentz group.

For $|\vec{q}_\perp|\to\infty$ we will have

$$\langle\lambda|\exp(i\vec{q}_\perp\cdot\vec{X}_\perp)|\lambda\rangle\underset{|\vec{q}_\perp|\to\infty}{\sim}\int_0^\infty J_0(|\vec{q}_\perp|r)r^{m+1}\exp(-\kappa r)dr$$
$$\underset{|\vec{q}_\perp|\to\infty}{\sim}(-1)^{m+1}\frac{d^{m+1}}{d\kappa^{m+1}}\times\int_0^\infty J_0(|\vec{q}_\perp|r)\exp(-\kappa r)dr$$
$$\underset{|\vec{q}_\perp|\to\infty}{\sim}(-1)^{m+1}\frac{d^{m+1}}{d\kappa^{m+1}}(\kappa^2+\vec{q}_\perp^{\,2})^{-1/2},$$

and we note that the slowest possible asymptotic decrease is $1/|\vec{q}_\perp|^3$. This upper bound is attained for $m=0$, i.e., when the particle wave function is finite at the origin. For higher values of m the decrease of the form factors gets faster.

For comparison it is instructive to mention that in the nonrelativistic framework of the Schrödinger theory, the regularity of the wave function at the origin implies $1/|\vec{q}|^4$ as the slowest possible asymptotic decrease of the spin-0 elastic form factors.[6] This is by one power of $1/|\vec{q}|$ faster than predicted in the described relativistic approach to composite systems ($\vec{q}$ is the three-momentum transfer).

Investigation of the off-diagonal current matrix elements leads to the expressions

$$\langle\lambda'|\exp(i\vec{q}_\perp\cdot\vec{X}_\perp)|\lambda\rangle=i^{\alpha-\beta}\int_0^\infty J_{\alpha-\beta}(|\vec{q}_\perp|r)\psi^*_{\lambda'}(r)\psi_\lambda(r)r\,dr,$$

where α and β are the helicities of the particle states λ' and λ, respectively $(\alpha,\beta=0,\pm\frac{1}{2},\pm1,\ldots)$, while $J_{\alpha-\beta}(z)$ are the Bessel functions of integral index. Using similar arguments as in the previous case of the diagonal matrix elements one can easily see that, asymptotically, the lower bound for the excitation form factors is $1/|\vec{q}_\perp|^2$. This is by one power of $1/|\vec{q}_\perp|$ slower than in the case of diagonal matrix elements. Among the immediate consequences of the above considerations is the conclusion that the elastic form factor for the pion (or for any spin-0 particle, for that matter) cannot decrease slower than $(t=-\vec{q}_\perp^{\,2})$

$$G_\pi(t)\underset{-t\to\infty}{\sim}(1/-t)^{3/2}.$$

(Clearly, the evaluation of the elastic form factors of spin-0 particles involves only the diagonal current matrix elements.) If we adopt the diquark model for baryons, then we can reach similar conclusions about the asymptotic behavior of the Sachs magnetic form factors of nucleons (whose evaluation involves off-diagonal matrix elements), namely that the magnetic form factors cannot decrease slower than

$$G_M(t)\underset{-t\to\infty}{\sim}1/-t.$$

In view of the experimentally observed rapid decrease of the magnetic form factors

$$G^{\text{experiment}}_{M,\text{proton}}(t)\underset{-t\to\infty}{\sim}(1/-t)^2,$$

the obtained upper bound does not appear particularly stringent.

One frequently encounters in the literature the statement that the asymptotic behavior of the excitation form factors for higher-spin resonances is similar if not identical to the asymptotic behavior of the elastic form factors. Within the framework of the Dashen–Gell-Mann program this statement can be proved rigorously, provided that one makes the assumption that *the radial dependence of the wave functions of the higher excited states is the same as the radial dependence of the ground-state wave function*. This assumption leads to the following expressions for the asymptotic limit of the excitation form factors:

$$i^n\int_0^\infty J_n(|\vec{q}_\perp|r)\rho(r)r\,dr\underset{|\vec{q}_\perp|\to\infty}{\sim}\int_0^\infty J_n(|\vec{q}_\perp|r)r^{m+1}\exp(-\kappa r)dr$$
$$\underset{|\vec{q}_\perp|\to\infty}{\sim}(-1)^{m+1}\frac{d^{m+1}}{d\kappa^{m+1}}\left\{(\kappa^2+|\vec{q}_\perp|^2)^{-1/2}\left[\left(1+\frac{\kappa^2}{|\vec{q}_\perp|^2}\right)^{1/2}-\frac{\kappa}{|\vec{q}_\perp|}\right]^n\right\}.$$

It is a simple matter to verify that the rate of the decrease of the obtained expressions for $|\vec{q}_\perp| \to \infty$ does not depend upon n.

For an illustration take m to be 0. In this case we deal with the limit

$$\underset{|\vec{q}_\perp|\to\infty}{\sim} (\kappa^2+|\vec{q}_\perp|^2)^{-1/2}\left[\left(1+\frac{\kappa^2}{|\vec{q}_\perp|^2}\right)^{1/2}-\frac{\kappa}{|\vec{q}_\perp|}\right]^n \left\{\kappa(\kappa^2+|\vec{q}_\perp|^2)^{-1}-n\left[\left(1+\frac{\kappa^2}{|\vec{q}_\perp|^2}\right)^{1/2}+\frac{\kappa}{|\vec{q}_\perp|}\right]\left[\frac{\kappa}{|\vec{q}|^2}\left(1+\frac{\kappa^2}{|\vec{q}_\perp|^2}\right)^{-1/2}-\frac{1}{|\vec{q}_\perp|}\right]\right\}$$

$$\underset{|\vec{q}_\perp|\to\infty}{\sim} \begin{cases} \dfrac{1}{|\vec{q}_\perp|^2} \text{ for } n=1,2,3,\ldots \\ \dfrac{1}{|\vec{q}_\perp|^3} \text{ for } n=0, \end{cases}$$

which obviously does not depend upon n except at the transition point from $n=0$ to $n=1$. This example clearly illustrates why the asymptotic behavior of the excitation form factors should be independent of the excited state itself, and also why the case of the ground-state non-spin-flip elastic form factor is exceptional. It should be added that the derived conclusions do not cover isolated cases where the excitation form factor vanishes identically, or is very small because of the mismatch between the quantum numbers of the resonance and the ground state. Examples of such quantum numbers and/or selection rules are the isospin and the charge strangeness.

THRESHOLD BEHAVIOR OF THE SCALING FUNCTIONS

There exists another area of considerable experimental activity where the neighborhood of the origin of the particle wave function is being explored. This is the threshold behavior of the scaling functions νW_2.[7] Since both the asymptotic behavior of the elastic form factors and the threshold behavior of νW_2 are determined by the values of the particle wave function around the origin, it is natural to expect a quantitative relation between these two seemingly unrelated aspects of hadron phenomenology. This quantitative connection is supplied by the Drell-Yan formula[8] $n=p+1$ that links the rate of decrease of the *spin-averaged* elastic form factors of the nucleon (the same formula can be applied to spin-0 targets as well)

$$G(t) \underset{-t\to\infty}{\sim} (1/-t)^{n/2}$$

to the curvature of νW_2 at the origin

$$\nu W_2 \underset{\xi\to 1}{\sim} \xi(1-\xi)^p.$$

($\xi=-t/2M_{\text{nucleon}}\nu$, t is the invariant momentum transfer, and $\nu=p\cdot q/M_{\text{nucleon}}$ can be interpreted as the energy of the virtual photon in the rest frame of the target.) Since we have shown in the preceding section that n cannot be smaller than 2, we conclude that p must necessarily be greater or equal to unity. Of course, when reaching this conclusion we must remember that the Dashen–Gell-Mann program allows for a proof of the Drell-Yan formula.[9]

MESONS

We regard mesons as $q\bar{q}$ excitations with the currents given by (1). Since the charges of quarks and antiquarks that make π^+, π^-, K^+, and K^- mesons are the same, we expect the following predictions to hold, at least in the asymptotic region $-t\to\infty$:

$$F_{\pi^+}(t)=F_{\pi^-}(t)=F_{K^+}(t)=F_{K^-}(t).$$

The F's are, of course, the elastic form factors of the particles in question. Analogously, the scaling functions of π^+, π^-, K^+, and K^- should be the same, at least at the threshold $\xi\to 1$:

$$(\nu W_2)_{\pi^+}=(\nu W_2)_{\pi^-}=(\nu W_2)_{K^+}=(\nu W_2)_{K^-}.$$

Similar relations can be established for the elastic form factors and the scaling functions of other mesons.

The predictions concerning the asymptotic domain $|\vec{q}_\perp|\to\infty$ can be made about the excitation form factors as well. Consider, for example, the transition matrix element between a spin-0 and a spin-1 meson (for definiteness, we can visualize π meson and ω meson, respectively). If we assume that the radial dependence of the wave functions of the two systems is the same, and the only difference is in the angular dependence (appropriate to the spin assignments of the particles in question), we can make definite predictions about the rate of decrease of the transition matrix elements when compared with the rate of decrease of the diagonal matrix elements (for the spin-0 particle).

The described assumption about the wave functions of the particle states under consideration leads to the following expression for the excitation form factor:

$$i\int_0^\infty J_1(|\vec{q}_\perp|r)|\psi_{\mathrm{spin}\,0}(r)|^2 r\,dr.$$

By acknowledging the validity of the well-known formula $J_1(z)=-dJ_0(z)/dz$, and subsequently performing integration by parts, we arrive at the prediction

$$\frac{\langle \mathrm{spin}\ 0|F_i(\vec{q}_\perp)|\mathrm{spin}\ 0\rangle}{\langle \mathrm{spin}\ 0|F_i(\vec{q}_\perp)|\mathrm{spin}\ 1,\ \mathrm{helicity}\pm 1\rangle}\underset{|\vec{q}_\perp|\to\infty}{\sim}\frac{1}{|\vec{q}_\perp|}.$$

The same conclusion was reached by Ravndal[10] on the basis of arguments taken from the parton model.

Another prediction which is characteristic for the Dashen–Gell-Mann program is the relationship between the value of the particle wave function (in the infinite momentum Lorentz frame) at the origin, and the integral of the particle elastic form factor over the complete range of variation of the invariant momentum transfer $t=-|\vec{q}_\perp|^2$. In the case of π^+ mesons, for example, we will have[11]

$$\begin{aligned}\frac{1}{8\pi}\int_{-\infty}^0 F_{\pi^+}(t)dt &= \frac{1}{2(2\pi)^2}\int\langle\pi^+|F(\vec{q}_\perp)|\pi^+\rangle d^2q_\perp\\ &= \frac{1}{2(2\pi)^2}\int\langle\pi^+|e_1\exp(i\vec{q}_\perp\cdot\vec{X}_\perp/2)\\ &\qquad + e_2\exp(-i\vec{q}_\perp\cdot\vec{X}_\perp/2)|\pi^+\rangle d^2q_\perp\\ &= \int\psi^*_{\pi^+}(\vec{x}_\perp)\delta(\vec{x}_\perp)\psi_{\pi^+}(\vec{x}_\perp)d^2x_\perp\\ &= |\psi_{\pi^+}(0)|^2.\end{aligned}$$

We know that the obtained relationship cannot be exact, for the current algebra and the relativity requirement cannot be reconciled everywhere on the t axis. Nevertheless, the measurements of deviations from the exact equality in this sum rule may serve the useful purpose of providing an indication of the magnitude of violation of relativity in the Dashen–Gell-Mann program.

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[11] Note that the analogous nonrelativistic prediction differs from the relativistic one in the factor 2π, namely

$$\begin{aligned}\frac{1}{8\pi}\int_{-\infty}^0 F_{\pi^+}^{\mathrm{nonrel}}(t)dt &= \cdots = \int\langle\pi^+|e_1\exp(i\vec{q}\cdot\vec{X}/2)\\ &\qquad + e_2\exp(-i\vec{q}\cdot\vec{X}/2)|\pi^+\rangle d^3q\\ &= \cdots = 2\pi|\psi_{\pi^+}^{\mathrm{nonrel}}(0)|^2.\end{aligned}$$