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Ponicaré group, $V-A$, and SU(3) as spectrum-generating group in leptonic decays

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From as few assumptions as possible about the relations among the Poincaré group, the particle-classifying SU(3), and $V-A$ we derive properties of the K_{l3} and K_{l2} decays. From the assumed relation between SU(3) and the Poincaré group and the first-class condition it follows that the form-factor ratio ξ of K_{l3} decay is $\xi = -0.57$, and that a value of $\xi = 0$ is in disagreement with very general and well-accepted theoretical assumptions. Assuming universality of $V-A$, the Cabibbo suppression is derived from the relations between SU(3) and $V-A$ as a consequence of the brokenness of SU(3).

I. INTRODUCTION

The description of weak and electromagnetic decays of hadrons is usually given in a language using more complicated notions than those which are used in the actual calculation. The field-theoretic properties of the currents and axial-vector currents are not really made use of, and the fields serve only as a vehicle for the formulation of symmetry and transformation properties. We will try to extract here all those properties from the usual formulation that are really essential for the calculation of numbers and put them into a consistent framework. The danger of such a procedure is that the framework might be too narrow and therefore might provide only a description of a limited domain; e.g., it may describe only one-hadron states and their properties in decays involving leptons. On the other hand, such a simplified formulation will display more clearly the connection between a theoretical assumption and a predicted value and will allow more easily the modification of a theoretical assumption according to the experimental situation. This framework can later—after the assumptions leading to correct experimental values have been found—be generalized to a theory for a larger domain of particle physics, and may therefore be considered as a possible approximation of a future theory.

The framework that we will use is quantum mechanics, so the topics of this paper concern the description of the structure of a hadron in principally the same way as nonrelativistic quantum mechanics describes the structure of an atom. We chose the leptonic and semileptonic decay of mesons to illustrate this description because recent experimental results on K_{l3} decay[1,2] initiated this investigation.

For a quantum-mechanical description we have to make assumptions about the algebra of observables[3] for the one-hadron system. We make the following assumptions.

($\mathcal{P}$): The algebra of the Poincaré group, $\mathcal{P}$, extended by parity, time inversion, and charge conjugation, is a subalgebra of the algebra of observables.

(SU(3)): The charges[4] E_α ($\alpha=\pm1,\pm2,\pm3,0,8$) that generate an SU(3) which classifies particles are observables.

($V-A$): There are eight Lorentz-vector operators V_μ^α ($\mu=0,1,2,3$; $\alpha=\pm1,\pm2,\pm3,0,8$) and eight Lorentz-axial-vector operators A_μ^α in the algebra of observables which describe the transitions between various hadron states.

These three assumptions, ($\mathcal{P}$), (SU(3)), and ($V-A$), are rather obvious for a quantum mechanics that is to describe hadrons; the question is how these three assumptions can be combined with each other,[5] i.e., what is the relation between the Poincaré group and SU(3), between SU(3) and $V-A$, between the extended Poincaré group and $V-A$, etc. We will in the following sections make some precise assumptions about these relations and calculate their consequences; here we give a brief summary.

Some relations between the particle-classifying $\mathrm{SU(3)}_E$— which is treated as a spectrum-generating group—and the V_μ^α and A_μ^α are given by (2.6) and (2.7). V_μ^α and A_μ^α correspond to the vector and axial-vector currents in the usual formulation. Therefore the first guess would be that (2.7) holds, for $P=0$ which means that these quantities are $\mathrm{SU(3)}_E$ octet operators. It is this relation between the spectrum-generating $\mathrm{SU(3)}_E$ and $V-A$ that will predict the ratio between K_{l3} and π_{l3} decay and K_{l2} and π_{l2} decay (Cabibbo suppression). It will be seen that, though (2.7) for $P=0$ and $P=-1$ will predict approximately the right magnitude of the suppression of K_{l3} decay [(3.25), and (3.26) with $P=-1$, respectively], the best value is obtained if one takes $P=-\frac{1}{2}$ for the V_μ^α, whereas the suppression of K_{l2} decay is reproduced very well if one takes $P'=-1$ for the A_μ^α. Relations between V_μ^β, A_μ^β and $\mathrm{SU(3)}_E$, (2.7) will probably have to be revised and should here be considered as some model assumptions that give results which lie in the right direction.

The relation (2.7) (for any P), where E_α belong to the spectrum-generating $SU(3)_E$, together with the assumption of a definite T-transformation property of the V_μ^β, (2.9), constitutes the core of the assumption, that V_μ^β be of "first class, with respect to the spectrum-generating $SU(3)_E$."[6]

As $SU(3)_E$ is not a symmetry group, we assume that the relation between the Poincaré group $\mathcal{P}$ and $SU(3)_E$ is given by (2.4), in addition to a suitable relation between the mass operator and the E_α that leads to the mass formula. The relation (2, 4) together with the assumption that V_μ^β be of first class will lead to the value (3, 22) for the form-factor ratio ξ, which is in agreement with most of the experimental values[8] except the value obtained in one high-statistics experiment, which is considered to be the best and which gave $\xi=0$. Besides giving the value (3, 22) our calculation shows that it is quite impossible to predict $\xi=0$, except in the symmetry limit (2, 3) or if one allows for peculiar cancellations between genuine second-class contributions and those induced by the symmetry breaking. Thus, an experimental value of $\xi=0$ would be hard to understand.

In Sec. II of this paper we state and discuss the assumptions of our approach. The calculation of the physical properties is done in Sec. III for K_{l3} and π_{l3} decay and in Sec. IV for K_{l2} and π_{l2} decay. The Appendix gives a detailed derivation of (3, 17) from very general assumptions.

II. COMBINATION OF $\mathcal{P}$, SU(3), AND $V-A$

A. Relation between $\mathcal{P}$ and SU(3)

We assume, as is always done, that the space of physical states is a representation space of the Poincaré group and of some SU(3). As basis vectors in this space one can then take

$$|pss_3, \eta; II_3Y\rangle.$$

η here denotes any possible additional quantum numbers besides the quantum numbers I, I_3, Y and the space-time quantum numbers p, s, s_3. If we ignore these additional quantum numbers; i.e., if we consider only states for which these additional quantum numbers have a fixed value, we have for the basis of the space of states of this physical system

$$\begin{aligned} |pss_3; II_3Y\rangle &= |pss_3\rangle \otimes |II_3Y\rangle, \\ |\hat{\alpha}\rangle &= |II_3Y\rangle \end{aligned} \tag{2.1}$$

where $|pss_3\rangle$ is the canonical basis for the Poincaré group representation and $|II_3Y\rangle$ is the usual basis in an SU(3) multiplet.

If one assumes in addition that inside the multiplet under consideration the mass is a function of I, I_3, Y (and of s, if s does not have a fixed value, and the other possible additional quantum numbers η), then one of the quantum numbers in (2.1) is redundant, and instead of the 4-vector p one can take the 3-vector $\vec{p}$.

Instead of taking the momentum p to label the states one could as well have taken the 4-velocity $\hat{p}=p/m$ ($\vec{\hat{p}}^2=-\hat{p}^2+\hat{p}_0{}^2=1$) and use as the basis system

$$\begin{aligned} |\hat{p}ss_3II_3Y\rangle &= |\hat{p}ss_3\rangle \otimes |II_3Y\rangle, \\ |\hat{\alpha}\rangle &= |II_3Y\rangle \end{aligned} \tag{2.2}$$

($|p\hat{\alpha}\rangle$ and $|\hat{p}\hat{\alpha}\rangle$ may or may not be normalized such that $|p\hat{\alpha}\rangle=|\hat{p}\hat{\alpha}\rangle$; cf. Sec. III). If SU(3) is a symmetry group there is no relevant distinction between (2.1) and (2.2), except that (2.2) is highly unfamiliar as even the starting formulas in particle-physics textbooks are given in terms of momentum eigenvectors.

In general, if the SU(3) is not a symmetry group, an SU(3) transformation U will act on both factors of (2.1) and (2.2), i.e.,

$$U=U^{\text{ex}}\otimes U^{\text{int}}, \tag{2.1$'$}$$

where U^{ex} acts on $|pss_3\rangle$ and U^{int} acts on $|\alpha\rangle$,

$$U=\hat{U}^{\text{ex}}\otimes \hat{U}^{\text{int}}, \tag{2.2$'$}$$

where $\hat{U}^{\text{ex}}$ acts on $|\hat{p}ss_3\rangle$ and $\hat{U}^{\text{int}}$ acts on $|\alpha\rangle$. In general, $U^{\text{ex}}\neq 1$, as well as $\hat{U}^{\text{ex}}\neq 1$. If SU(3) is a symmetry group, in particular if

$$[P_\mu, SU(3)_S]=0, \quad [\text{spin}, SU(3)_S]=0, \tag{2.3}$$

then $U^{\text{ex}}=1$.

For the SU(3) that classifies the elementary particles (2.3) is clearly a highly unrealistic assumption, as the masses in an SU(3) multiplet (in particular the pseudoscalar meson octet under consideration) are far from being equal.

A more realistic approximation appears to be the assumption[7] that the particle-classifying $SU(3)_E$ satisfies

$$\left[\frac{P_\mu}{M}, SU(3)_E\right]=0, \quad [L_{\mu\nu}, SU(3)_E]=0 \tag{2.4}$$

though $[P_\mu, SU(3)_E]\neq 0$ and $[M, SU(3)_E]\neq 0$ ($M=$ mass operator).

With assumption (2.4) $\hat{U}^{\text{ex}}$ in (2.2$'$) is equal to 1:

$$\hat{U}^{\text{ex}}=1. \tag{2.5}$$

Also, the SU(3) transformation acts only upon the SU(3) basis $|II_3Y\rangle$ and not upon the $|\hat{p}, ss_3\rangle$.

(2.3) and (2.4) are two particular cases of possible algebraic relations that may exist between the generators of the Poincaré group and an SU(3).

However, under the assumption $[M, SU(3)_E] \neq 0$ (2.4) is the only simple possibility,[9] and there seems to be something fundamental to this relation. We will see in the following to what extent experimental data support (2.4).

B. Relation between SU(3) and V-A

We assume that we have an octet of Lorentz-vector operators V_μ^α and Lorentz-axial-vector operators A_μ^α ($\alpha = \pm 1, \pm 2, \pm 3, 0, 8$) that describe the transition between the hadron states, as they occur, e.g., in the weak leptonic and electromagnetic decays.

This assumption resembles the current-algebra assumption only in a very vague sense, as it is not even assumed that the V_μ^α and A_μ^α are local currents.

The simplest feasible assumption about the relation of the $V_\mu^\alpha, A_\mu^\alpha$ and the $SU(3)_E$ is that V_μ^α and A_μ^α are $SU(3)_E$-octet operators, i.e.,

$$[E_\alpha, V_\mu^\beta] = f^{\alpha\beta\gamma} V_\mu^\gamma, \quad [E_\alpha, A_\mu^\beta] = f^{\alpha\beta\gamma} A_\mu^\gamma \tag{2.6}$$

[where $f^{\alpha\beta\gamma}$ are the SU(3) structure constants].

However, it may well be that instead of V_μ^α and A_μ^α some functions of $V_\mu^\alpha, A_\mu^\alpha$ are $SU(3)_E$-octet operators, e.g., that $\{M^P, V_\mu^\alpha\}$ and $\{M^{P'}, A_\mu^\alpha\}$ are $SU(3)_E$-octet operators, i.e., that more generally

$$\begin{aligned}&[E_\alpha, \{M^P, V_\mu^\beta\}] = f^{\alpha\beta\gamma}\{M^P, V_\mu^\beta\},\\ &[E_\alpha, \{M^{P'}, V_\mu^\beta\}] = f^{\alpha\beta\gamma}\{M^{P'}, A_\mu^\beta\}\end{aligned} \tag{2.7}$$

where $P', P = 0, \pm 1, \pm 2 \ldots$ or even noninteger.

If the E_α are symmetry operators then assumption (2.7) follows from (2.6); however, for $[M, E] \neq 0$ (2.6) and (2.7) are different, and (2.6), (2.7) as well as generalizations of (2.7) can be considered as precise formulations of the vague assumptions that V_μ^β are octet operators of some "broken" SU(3). We will perform the derivations under assumption (2.6), and will state the modifications that arise as consequences of (2.7) at the end.

C. Relation between the CPT extended Poincaré group and $V-A$

Contained in the $V-A$ hypothesis is the assumption that V_μ^α are Lorentz vector operators and A_μ^α are Lorentz axial-vector operators. In addition to this we assume about the relation of $V-A$ to the extended Poincaré group that V_μ^α and A_μ^α are of "first class with respect to the spectrum-generating $SU(3)_E$." For the V_μ^α this assumption follows from the following well-accepted theoretical principles:

(a) invariance of the transition operator under time inversion [as given in its general form by Eq. (A8) of Appendix],

(b) hermiticity of the transition operator (unitarity of the S matrix, as we can neglect final-state interactions);

(c) conservation of vector current in the form that if $\{\Theta, V_\mu^{\pm 2}\}$ is the component of the $SU(3)_E$ octet operator that describes K_{l3} decay then $\{\Theta', V_\mu^0 + (1/\sqrt{3})V_\mu^8\}$ describes the electromagnetic interactions where Θ and Θ' are Hermitian operators that commute with $I, I_3, Y, \hat{P} = P_\mu M^{-1}$.

A more familiar form of the first-class condition is the following:

(a′) V_μ^β (and A_μ^β) have a definite transformation property under time reversal A_T:

$$A_T V_\mu^\beta A_T = \epsilon(\mu) V_\mu^\beta \tag{2.8}$$

$$[\text{and } A_T A_\mu^\beta A_T = \epsilon(\mu) A_\mu^\beta] \tag{2.8′}$$

where

$$\epsilon(\mu) = \begin{cases} -1 \text{ for } \mu = 0 \\ +1 \text{ for } \mu = 1, 2, 3. \end{cases}$$

(b′) (see Refs. 10 and 6) V_μ^β (and A_μ^β) are Hermitian $SU(3)_E$ octet operators,[11] i.e.,

$$V_\mu^{\beta+} = V_\mu^{-\beta} \tag{2.9}$$

$$(\text{and } A_\mu^{\beta+} = A_\mu^{-\beta}). \tag{2.9′}$$

However, this is not sufficient; in addition one has to assume that

(c′) the usual phase convention for the SU(3) Clebsch-Gordan coefficients holds.

For V_μ^β (a′), (b′), (c′) are equivalent to (a), (b), (c), and we will use this general formulation for the derivation in the Appendix. For A_μ^β there is no analog of the assumption (c), and therefore the requirement that A_μ^β be of first class does not have such a solid theoretical foundation as the same requirement for V_μ^β.[12]

III. SEMILEPTONIC DECAY OF PSEUDOSCALAR MESONS

We will apply the assumptions of Sec. II for the decay of a hadron state into a hadron state plus leptons. Processes of this kind are

$$P \to P' l\nu, \quad P \to l\nu,$$

$$V \to e\bar{e}, \quad V \to P\gamma,$$

$$B \to B' l\nu,$$

where P = pseudoscalar meson, V = vector meson, B = baryon. We will restrict ourselves in this paper to the leptonic and semileptonic decay of pseudoscalar mesons; application of the same formalism to the other processes is under investigation. We shall treat these decays in the framework of quantum mechanics, and the particular assumptions that we shall make will be given where needed and are numbered (1), (2), and (3a), (3b), (3c).

(1) We assume that these processes are weak in the sense that the part of the Hamiltonian which is responsible for the decay T causes a negligible level shift.[13] The initial decay rate is then given by (lowest-order perturbation theory)

$$\Gamma = 2\pi \sum_b \sum_{\alpha\alpha'} \delta(E_{\alpha'} - E_b) T_{b\alpha} \overline{T}_{b\alpha'} \langle \alpha | W(0) | \alpha' \rangle. \tag{3.1}$$

Here $W(t)$ is the statistical operator that describes the ensemble of decaying hadrons.

$$T_{b\alpha} = \langle b | T | \alpha \rangle, \quad \overline{T}_{b\alpha'} = \langle \alpha | T | b \rangle,$$

where $|\alpha\rangle$, $|\beta\rangle$, denote the eigenvectors of a complete system of commuting observables (CSCO) (note that the symbol α here has a different meaning than in Sec. II). $|a\rangle$, $|b\rangle, \ldots$ denote the corresponding free eigenvectors, i.e., $|a\rangle, \ldots$ denotes the eigenvector of the CSCO for $T = 0$ that has the same eigenvalues as $|\alpha\rangle$ (which is always possible when the level shift can be neglected). $\Sigma_{\alpha\alpha'}$ means summation over all the eigenvalues of the CSCO. Σ_b means summation over all those eigenvalues whose eigenvectors span the space of final states that are observed. The sums in (3.1) are discrete if one chooses as the CSCO a set of operators with discrete spectra. It is often more convenient to use observables with a continuous spectrum for the description of the experimental situation. In particular, the momentum or velocity is an observable which is easily accessible in an experiment (though a physical system cannot be in an eigenstate of this observable). If we choose the vectors $|\alpha\rangle$ to be the generalized eigenvectors of the momentum operator,

$$|\alpha\rangle = |p(\alpha), \eta_\alpha\rangle, \tag{3.2}$$

with the normalization

$$\begin{aligned} \langle p' \eta'_\alpha | p \eta_\alpha \rangle &= 2E(p) \delta^3(p - p') \delta_{\eta'_\alpha \eta_\alpha}, \\ E(p) &= (m^2 + p^2)^{1/2}, \end{aligned} \tag{3.2$'$}$$

where p is the three-momentum and η_α denotes the eigenvalues of the other observables that together with the $P_\mu^{(\alpha)}$ make up the CSCO, then

$$\sum_\alpha = \sum_{\eta_\alpha} \int \frac{d^3 p}{2E(p)}. \tag{3.3}$$

For reasons which have been given in Sec. II and which will again be discussed below we may want to choose generalized eigenvectors of the velocity operator P_μ / M rather than the momentum operator, i.e., choose

$$|\alpha\rangle = |\hat{p}_\alpha \hat{\eta}_\alpha\rangle \tag{3.4}$$

with the normalization

$$\langle \hat{\eta}' \hat{p}' | \hat{p} \hat{\eta} \rangle = 2\hat{E}(\hat{p}) \delta^3(\hat{p} - \hat{p}') \delta_{\hat{\eta}\hat{\eta}'}, \tag{3.4$'$}$$

where

$$\hat{E}(\hat{p}) = \left(1 + \frac{p^2}{m^2}\right)^{1/2}.$$

Then

$$\sum_\alpha = \sum_{\hat{\eta}} \int \frac{d^3 \hat{p}}{2\hat{E}}. \tag{3.5}$$

The decaying system may be prepared to have definite internal quantum numbers $\hat{\beta}$.

(2) We assume that the $SU(3)_E$ quantum numbers are the aforementioned internal quantum numbers (or part of them). The hadron has a definite mass, which we assume to be a function of the internal quantum numbers and which is therefore not needed for the labeling of the state. The statistical operator of the decaying system is then given by

$$W(0) = \int \int \frac{d^3 \hat{p}}{2\hat{E}} \frac{d^3 \hat{p}'}{2\hat{E}'} |\hat{p}\hat{\beta}\rangle \langle \hat{\beta} \hat{p}' | \phi(\hat{p}) \overline{\phi}(\hat{p}') \tag{3.6}$$

(if there would be some additional quantum numbers, e.g., spin, then we would have $|\hat{p}, \hat{\beta}, s\rangle$ and would have to average over these if they were not measured in the preparation). Here $|\hat{p}, \hat{\beta}\rangle$ denote the generalized eigenvectors of the CSCO:

$$\begin{aligned} &\{\hat{P}_i = P_i / M, SU(3)_E \text{ operators}, \\ &\quad \text{perhaps additional charges}\}. \end{aligned} \tag{3.7}$$

As the $SU(3)_E$ quantum numbers one may, e.g., take isospin I, I_3, hypercharge Y, and the Casimir operators of SU(3). The $|\hat{p}, \hat{\alpha}\rangle$ exist because of (2.4). Instead of the generalized eigenvectors $|\hat{p}\hat{\alpha}\rangle$ one usually uses the generalized eigenvectors of the momentum operator P_μ, $|p, \hat{\alpha}\rangle$, and writes instead of (3.6)

$$W = \int \int \frac{d^3 p}{dE} \frac{d^3 p'}{2E'} |p\hat{\alpha}\rangle \langle \hat{\alpha} p | \phi(p) \overline{\phi}(p'). \tag{3.6$'$}$$

However, the $|p, \hat{\alpha}\rangle$ only exist if the observables whose eigenvalues are $\hat{\alpha} = (I, I_3, Y, \ldots)$ commute with the momentum. If this is the case (3.6) and (3.6$'$) are equivalent. However, to assume that the Casimir operators of SU(3) commute with P_μ appears already to be a poor approximation.

If we use for the generalized eigenvectors the normalization (3.4$'$) and the measure as given in (3.5) then the requirement that W be normalized, i.e., that

$$\operatorname{Tr} W = 1, \tag{3.8}$$

leads to

$$\operatorname{Tr} W=\sum_{\hat{\beta}}\int\frac{d^3k}{2\hat{E}(k)}\langle\hat{\beta}\hat{k}|W|\hat{k}\hat{\beta}\rangle$$
$$=\sum_{\hat{\beta}}\int\frac{d^3\hat{k}}{2\hat{E}(\hat{k})}\int\frac{d^3\hat{p}}{2\hat{E}(\hat{p})}\int\frac{d^3p'}{2\hat{E}'(\hat{p}')}\langle\hat{\beta}\hat{k}|\hat{p}\hat{\alpha}\rangle\langle\hat{p}'\hat{\alpha}|\hat{\beta}\hat{k}\rangle\phi(\hat{p})\overline{\phi}(\hat{p}'), \tag{3.9}$$
$$\int\frac{d^3\hat{k}}{2\hat{E}(\hat{k})}|\phi(\hat{k})|^2=1.$$

The physical meaning of the distribution function $\phi(\hat{p})$ is that $|\phi(\hat{p})|^2$ is the probability that the momentum value of the decaying particle is $p=m\hat{p}$. For the idealized case that the system of decaying particles is prepared to have the exact momentum p_A the probability $|\phi(\hat{p})|^2$ will be zero except if $\hat{p}=\hat{p}_A$, which in the normalization (3.9) is given by

$$|\phi(\hat{p})|^2=2\hat{E}(\hat{p}_A)\delta^3(\hat{p}-\hat{p}_A). \tag{3.10}$$

If we insert W of (3.6) into (3.1) and choose for $|\alpha\rangle$ the generalized eigenvectors of the CSCO (3.7), i.e., the generalized eigenvectors (3.4) with $\hat{\eta}_\alpha=\hat{\alpha}=$[SU(3) quantum numbers and possibly further charges], then we obtain using (3.4′)

$$\Gamma=2\pi\sum_b\frac{d^3\hat{p}_\alpha}{2\hat{E}_\alpha}\frac{d^3\hat{p}'_\alpha}{2\hat{E}_\alpha}\delta(E_b-E_\alpha)\phi(\hat{p}_\alpha)\overline{\phi}(\hat{p}'_\alpha)\langle b|T|\hat{p}_\alpha\hat{\alpha}\rangle\langle\hat{p}'_\alpha\hat{\alpha}|T|b\rangle. \tag{3.11}$$

So far we have only specified that the decaying state is a one-hadron state with spin $S=0$ and definite internal quantum numbers $\hat{\alpha}$. We will now specify the final state to be π^0 plus a lepton pair; in particular we consider the process $\alpha\to\pi^0 l\nu$.

Then we take for the vectors $|b\rangle$

$$|b\rangle=|\hat{p}^\pi\pi^0,lp_l,\nu p_\nu\rangle=|\hat{p}_\pi\pi^0\rangle\otimes|p_l l\rangle\otimes|p_\nu\nu\rangle$$

and correspondingly for Σ_b

$$\sum_b\sum_{\text{pol}}\int\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_l}{2E_l}\frac{d^3\hat{p}_\pi}{2\hat{E}_\pi}.$$

[The generalized vectors for the leptons $|p_l l\rangle$ have been normalized according to (3.2′).] With this (3.11) reads

$$\Gamma=2\pi\int\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_l}{2E_l}\frac{d^3\hat{p}_{\pi^0}}{2\hat{E}_{\pi^0}}$$
$$\times\int\frac{d^3\hat{p}_\alpha}{2\hat{E}_\alpha}\frac{d^3\hat{p}'_\alpha}{2\hat{E}_\alpha}\delta(E_\nu+E_l+E_{\pi^0}-E_\alpha)\phi(\hat{p}_\alpha)\overline{\phi}(\hat{p}'_\alpha)\sum_{\text{pol}}\langle\nu lp_\nu p_l\hat{p}_{\pi^0}|T|\hat{p}_\alpha\alpha\rangle\langle\hat{p}'_\alpha\alpha|T|\hat{p}_{\pi^0}p_lp_\nu l\nu\rangle. \tag{3.12}$$

The calculations leading to (3.12) contain only well-accepted principles of quantum mechanics. They have been given here in order to show that there is absolutely nothing mysterious about the velocity eigenvectors $|\hat{p}\alpha\rangle$ and that these can be used instead of the momentum eigenvectors $|p\eta_\alpha\rangle$, and in fact have to be used if one wants to use as the additional quantum numbers eigenvalues of observables that do not commute with the momenta but commute with the velocity operator. It is easy to see that if these additional quantum numbers η_α commute with the momenta then the use of the usual $|p\eta_\alpha\rangle$ or the $|\hat{p}\eta_\alpha\rangle$ is equivalent.

Before we proceed further we have to specify the operator T that describes the transition for the process

hadron $\alpha\to$ hadron $\beta+l+\nu$.

(3a) We assume that T conserves total momenta.[14] Then its matrix element can be written as a product of the momentum δ function and a reduced matrix element,

$$\langle p_lp_\nu\hat{p}_\pi|T|\hat{p}_\alpha\rangle=\delta^3(p_l+p_\nu+p_\pi-p_\alpha)\langle\langle l\nu\pi|T|\alpha\rangle\rangle. \tag{3.13}$$

If we insert this into (3.12) we obtain [using $\delta^3(\hat{p})=m^{-3}\delta^3(p/m)$]

$$\Gamma=2\pi\int\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_l}{2E_l}\frac{d^3\hat{p}_\pi}{2\hat{E}_\pi}\sum_{\text{pol}}\delta(E_\alpha-E_\pi-E_l-E_\nu)\left|\phi\left(\frac{p_l+p_\nu+p_\pi}{m_\alpha}\right)\right|^2$$
$$\times\frac{1}{m_\alpha{}^6}\left(\frac{1}{2\hat{E}_\alpha}\right)^2\left|\left\langle\left\langle l\nu\pi p_lp_\nu\hat{p}_\pi|T|\alpha\frac{p_l+p_\nu+p_\pi}{m_\alpha}\right\rangle\right\rangle\right|^2,$$

where

$$|\langle\langle l\nu\pi|T|\alpha\rangle\rangle|^2=\langle\langle l\nu\pi|T|\alpha\rangle\rangle\langle\langle\alpha|T|l\nu\pi\rangle\rangle.$$

This can be brought into a quite familiar form if one makes the idealized assumption that the ensemble of decaying hadrons has a definite momentum p_A. Then $|\phi|^2$ is given by (3.10) and we obtain

$$\Gamma=2\pi\int\frac{d^3p_l}{2E_l}\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_\pi}{2E_\pi}\sum_{\text{pol}}\delta^4(p_\alpha-p_\pi-p_l-p_\nu)\frac{1}{m_\pi{}^2m_\alpha{}^2}\frac{1}{2E_\alpha}|\langle\langle l\nu\pi p_l p_\nu\hat{p}_\pi|T|\alpha\hat{p}_A\rangle\rangle|^2, \tag{3.14}$$

where $p_\alpha=(E_\alpha,p_A)$ is the 4-momentum of the decaying hadron.

The distinction between this and the conventional expression (except for irrelevant normalization factors of 2π to some power and a different normalization for the lepton measure) is that there the invariant matrix element is a function of the momenta of the hadrons whereas here it is a function of the velocities, and therefore we have a factor of $1/m_\pi m_\alpha$.

Before we can proceed we have to specify the operator T further.

(3b) We assume that the matrix element of T is the product of a leptonic part and a hadronic part. The leptonic part is given by the usual $V-A$ matrix element for leptons, i.e., the lepton pair is treated as noninteracting particles (lowest-order perturbation):

$$\langle\langle\bar{l}\,\nu\pi p_l p_\nu\hat{p}_\pi|T|\hat{p}_\alpha\alpha\rangle\rangle$$
$$=\bar{u}(p_\nu)\gamma^\lambda(1-\gamma_5)v(p_l)\langle\langle\pi\hat{p}_\pi|H_\lambda|\hat{p}_\alpha\hat{\alpha}\rangle\rangle. \tag{3.15}$$

(3c) We assume that the transition operator in the hadron space H_λ is the sum of a vector and an axial vector and each of these is the sum of the V^α_μ and A^α_μ ($\alpha=\pm1,\pm2$) of Sec. II with equal weight:

$$H_\lambda=g\sum_\alpha(V^\alpha_\mu+A^\alpha_\mu). \tag{3.16}$$

If we take the $V^\alpha_\mu,A^\alpha_\mu$ dimensionless, which will be the natural choice if (2.6) or (2.7) with $P=0$ holds, then the constant g, which expresses the strength of the interaction, has the dimension of mass as the reduced matrix element $\langle\langle|T|\rangle\rangle$ has the dimension of mass square.[15]

The assumptions (3b), (3c) are analogous to the usual $V-A$ theory for leptonic weak interactions. The distinction is that we assume a higher universality using only one and the same constant g for the strangeness-changing $V^{\pm2}_\mu$ and for the strangeness-nonchanging $V^{\pm1}_\mu$ instead of a $(g\tan\theta)$ for $V^{\pm2}_\mu$ and g for $V^{\pm1}_\mu$. However, in distinction to the Cabibbo model, our V^α_μ are octet operators of a spectrum-generating group $\mathrm{SU(3)}_E$ that changes the mass. The idea behind this is that the relation between this spectrum-generating $\mathrm{SU(3)}_E$ and the transition operator H_λ will result in the Cabibbo suppression.

As mentioned already in Sec. II, the assumption (2.6) together with (3.16) would be the first guess for the relation between $V-A$ and the spectrum-generating $\mathrm{SU(3)}_E$, from the analogy with the assumption for the hadronic vector and axial-vector current. However, if the SU(3) noninvariance is taken into account, (2.7) with any value for P or other assumptions which in the limit of SU(3) symmetry lead back to (2.6) are equally good candidates and only the experimental data can decide this question.

Instead of assuming (2.7) together with (2.6) we can take the SU(3) noninvariance into account in a different way:

One requires always (2.6) for the V^α_μ and A^α_μ but assumes instead of (3.16) a relation between the transition operator H_μ and the $\mathrm{SU(3)}_E$ octet operators V^α_μ and A^α_μ which involves those operators which are not $\mathrm{SU(3)}_E$ invariants, i.e., P_μ and M. I.e., one assumes (2.6), and for the transition operator one assumes, e.g.,

$$H_\mu=g\sum_\alpha(\{V^\alpha_\mu,M^q\}+\{A^\alpha_\mu,M^{q'}\}), \tag{3.17}$$

where $q,q'=0,\pm1,\pm2\ldots$ or even noninteger.

Either the assumption (2.7) with (3.16) or the assumption (2.6) with (3.17) will lead to correction factors for the decay amplitude that are functions of the hadron masses depending upon the values of P and q. However, the prediction for the form-factor ratio $\xi=f_-/f_+$ is independent of the value of q or P and only a consequence of the assumption (2.4).

We will in the derivation discuss the case that V^β_μ is the octet operator, (2.6), and that H_μ is given by (3.16); modifications due to values of P,P' or q,q' different from zero will be mentioned at the end.

We shall now discuss the matrix element of the $\mathrm{SU(3)}_E$ octet operator V^β_μ which enters in (3.15). By the same arguments which are used in the conventional formulation to write the matrix element of a V^β_μ between momentum vectors as

$$\langle\langle\pi p_\pi|V^\beta_\mu|p_\alpha\alpha\rangle\rangle$$
$$=\langle\pi|V^\beta|\alpha\rangle[f_+(p_\alpha+p_\pi)_\mu+f_-(p_\alpha-p_\pi)_\mu.] \tag{3.18}$$

one can write the matrix elements of the octet operator V^β_μ between velocity vectors as

$$\langle\langle \pi \hat{p}_\pi | V_\mu^\beta | \hat{p}_\alpha \alpha \rangle\rangle$$

$$= \langle \hat{\pi} | \hat{V}^\beta | \hat{\alpha} \rangle [F_+(\hat{p}_\alpha + \hat{p}_\pi)_\mu + F_-(\hat{p}_\alpha - \hat{p}_\pi)_\mu], \quad (3.18')$$

where $\langle \hat{\pi} | \hat{V}^\beta | \hat{\alpha} \rangle$ are the F-type SU(3) matrix elements. The essential difference between (3.18) and (3.18′) and the reason for using velocity eigenvectors rather than momentum eigenvectors is that the $f_\pm$ in (3.18) are functions of the momenta and therewith masses and are only SU(3) scalars if SU(3) is a symmetry group, i.e., if (2.3) holds, whereas the $F_\pm$ are $\mathrm{SU(3)}_E$ invariants as a consequence of the assumption (2.4). This can easily be seen from (2.5).

In analogy to the proof that $f_- = 0$ for the case of SU(3) symmetry as a consequence of the first-class condition, one proves that as a consequence of the condition that V_μ^β be first class with respect to the spectrum-generating $\mathrm{SU(3)}_E$ it follows that

$$F_- = 0. \quad (3.19)$$

The derivation of this from the very general assumptions stated in Sec. II is given in the Appendix.

The reduced matrix element of T can now be written using (3.19), (3.18′), and (3.16):

$$\langle\langle \bar{l} \nu p_l p_\nu \hat{p}_\pi \pi | T | \hat{p}_\alpha \alpha \rangle\rangle = \bar{u}(p_\nu) \gamma^\lambda (1-\gamma_5) \nu(p_l) g \langle \hat{\pi} | \hat{V}^\beta | \hat{\alpha} \rangle \times F_+ \left(\frac{p_\alpha}{m_\alpha} + \frac{p_\pi}{m_\pi} \right)_\mu, \quad (3.20)$$

where $\langle \hat{\pi} | \hat{V}^\beta | \hat{\alpha} \rangle$ is the SU(3) matrix element of the only component V^β of the octet operator that contributes to the particular decay ($\beta = -2$ for $\alpha = K^+$, $\beta = -1$ for $\alpha = \pi^+$, etc.). F_+ is a function of the invariant formed with $\hat{p}^\alpha$ and $\hat{p}^\pi$ or $t = m_\alpha^2 + m_\pi^2 - 2m_\alpha m_\pi \hat{p}_\alpha \cdot \hat{p}_\pi$ whose functional dependence we cannot predict without any further assumption. Inserting (3.20) into (3.14) we obtain

$$\Gamma = 2\pi \left| \frac{\langle \hat{\pi} | \hat{V}^\beta | \hat{\alpha} \rangle g}{m_\pi m_\alpha} \right|^2 \int \frac{d^3 p_l}{2E_l} \frac{d^3 p_\nu}{2E_\nu} \frac{d^3 p_\pi}{2E_\pi} \times \sum_{\mathrm{pol}} \delta^4(p_\alpha - p_\pi - p_\nu - p_l) \frac{1}{2E_\alpha} \left| \bar{u} \gamma^\lambda (1-\gamma_5) \nu [p_{\alpha\lambda} - \tfrac{1}{2}(1 - \xi^{(\pi\alpha)})(p_{\alpha\lambda} - p_{\pi\lambda})] f_+^{(\pi\alpha)} \right|^2, \quad (3.21a)$$

where we have introduced the quantities

$$f_+^{(\pi\alpha)} = \frac{m_\alpha + m_\pi}{m_\pi m_\alpha} F_+, \quad (3.21b)$$

$$\xi^{(\pi\alpha)} = \frac{m_\pi - m_\alpha}{m_\pi + m_\alpha} \quad (3.21c)$$

in order that the amplitude in the integral (3.21) has the familiar form expressed in terms of the form factor f_+ and the form factor ratio $\xi = f_-/f_+$ in which the K_{l3} experiments are analyzed.

Thus for the K_{l3} decay we predict[16]

$$\xi^{(K_{l3})} = \frac{m_\pi - m_K}{m_\pi + m_K} = -0.57. \quad (3.22)$$

Integration of (3.19) (for α_{e3} decay) leads to

$$\Gamma(\alpha_{e3}) = \frac{\pi^3}{24} \left| \langle \hat{\pi} | V^\beta | \hat{\alpha} \rangle \frac{(m_\alpha + m_\pi)}{m_\pi^2 m_\alpha^2} \right|^2 |g|^2 |F_+|^2 m_\alpha^5 C\left(\frac{m_\pi}{m_\alpha} \right), \quad (3.23)$$

where $C(x)$ is the familiar

$$C(x) = 1 - 8x^2 + 8x^6 - x^8 - 24x^4 \ln x$$

and where we have assumed that $F_+(t) = F_+(0) = \text{constant}$. If we assume $f_+(t) = f_+(0)(1 + \lambda_+ t/m^2)$ for the K_{l3} decay then the right-hand side of (3.23) has to be multiplied by the correction factor $1 + 0.27\,\lambda_+(m_K/m_\pi)^2$.

In order to facilitate comparison with familiar quantities we introduce the suppression factor S_{l3}, which we define as the ratio of the amplitudes for K_{l3} and π_{l3} decay divided by the ratio of the SU(3) Clebsch-Gordan coefficients, i.e.,[17]

$$S_{l3} = \frac{\langle \pi^0 | V^{-1} | \pi^+ \rangle}{\langle \pi^0 | V^{-2} | K^+ \rangle} \left(\frac{\Gamma(K_{l3}) \times \text{phase space } (\pi_{l3})}{\Gamma(\pi_{l3}) \times \text{phase space } (K_{l3})} \right)^{1/2}. \quad (3.24)$$

S_{l3} corresponds in the usual treatment to $f_+(0) \tan \theta_V$. From (3.23) we find that our prediction for S_{l3} is

$$S_{l3} = \frac{m_\pi^2}{m_K^2} \frac{m_{K^+} + m_\pi}{(m_{\pi^+}) + m_\pi)} = 0.183. \quad (3.25)$$

The latest experimental value for S_{l3} is[18] 0.224.

Before proceeding let us analyze the derivation of the results (3.22) and (3.25). We had assumed that the SU(3) that classifies the hadrons (in this particular case the pseudoscalar mesons), the $SU(3)_E$, has the property (2.4) but is not a symmetry group, and that the operators V_μ^β that describe the transition between the hadrons in the initial and final states are octet operators with respect to this $SU(3)_E$, i.e., fulfill (2.6) and are connected with H_μ by (3.16). Though we have not completely fixed the algebra of observables that governs the physics of one-hadron states, this is a fairly precise assumption and leads to a fairly precise result, e.g., (3.22), (3.25).

If one had assumed (2.3) instead of (2.4) then one would have arrived by the same arguments as above at the results

$$S_{l3}=1, \tag{3.25$'$}$$

$$f_-=0. \tag{3.19$'$}$$

(3.25′) is usually remedied by making (3.16) unsymmetrical and introducing a suppression factor (Cabibbo angle) there, which will not affect the result (3.19′).

In spite of the fact that the assumption (2.3) is rejected by the huge mass differences, (3.19′) is still considered possible. From the derivation given here (including the Appendix) it is clear that (3.19′) can only be derived from these very general assumptions if $SU(3)_E$ is a symmetry group. Thus an experimental result of $f_-=0$ will require that we change many well-accepted theoretical concepts, the nearest at hand of which would probably be the assumption that the transition matrix element is given by the product of a leptonic part and a hadronic part as given in (3.15) (which follows, e.g., from a local phenomenological current interaction). It is, therefore, comforting that most experiments on K_{l3} decay[8] agree with (3.19) rather than (3.19′).

In Sec. II it was mentioned that if the particle-classifying SU(3) fulfills (2.4) and one has an octet of Lorentz-vector operators V_μ^α and axial-vector operators A_μ^β then the assumption that V_μ^α is an octet operator with regard to this $SU(3)_E$, i.e., that (2.6) holds, and that the transition operator H_μ is given by (3.16), is one of many possibilities of specifying what octet operator with regard to a "broken SU(3)" might mean. Whereas (3.19) [and therewith (3.22)] is independent of this particular assumption and also fulfilled if (2.7) or (3.17) or even more general relations hold, (3.25) is a direct consequence of (2.6) and (3.16) and any other assumption would have led to a different result. E.g., if we assume (2.7) instead of (2.6) we obtain that ξ is again given by (3.22) but instead of (3.25) we obtain

$$S_{l3}=S_{l3}^0\,\frac{m_\pi{}^P+m_\pi{}^P}{m_K{}^P+m_\pi{}^P}\,, \tag{3.26}$$

where S_{l3}^0 is given by (3.25).

(2.7) for $P=-1$ is a very attractive assumption from a theoretical point of view. First, (2.7) for $P=-1$ is the relation required for the A_μ^α, as will be shown in the following section; however, there is of course no reason that the A_μ^α should deviate from an $SU(3)_E$ octet operator in exactly the same way as the V_μ^α. Further, with the assumption that $\{M^{-1}, V_\mu^\beta\}$ and $\{M^{-1}, A_\mu^\beta\}$ are dimensionless octet operators the strength constant g in (3.16) will be dimensionless and the SU(3) invariants F_+ as well as the F in Eq. (4.7) will also be dimensionless.

For $P=-1$ the suppression factor is

$$S_{l3}=\frac{m_\pi}{m_K}=0.283, \tag{3.26$'$}$$

which certainly does not compare more favorably with the experimental value than the value of (3.25). A result in agreement with the experimental value is obtained if one admits also noninteger values for P; $P=-\frac{1}{2}$ gives

$$S_{l3}=0.238. \tag{3.26$''$}$$

No theoretical reason exists for choosing $P=-\frac{1}{2}$. It is quite likely that neither (2.6) nor (2.7) is the experimentally correct relation between $V-A$ and SU(3); therefore all results (3.25) and (3.26) should only serve as illustrations of how relations between $V-A$ and a spectrum-generating SU(3) can lead to the Cabibbo suppression.

If the symmetry breaking for the transition operator is taken into account in the form (3.17) with V_μ^α satisfying (2.6), then the suppression factor is given by

$$S_{l3}^{(q)}=S_{l3}^{(0)}\,\frac{m_K{}^q+m_\pi{}^q}{m_\pi{}^q+m_\pi{}^q}\,. \tag{3.27}$$

IV. LEPTONIC DECAYS OF PSEUDOSCALAR MESONS

The K_{l2} and π_{l2} decays are described by the Lorentz axial-vector operators A_μ^β. We assume that the A_μ^β fulfill (2.7) with $P'=-1$, i.e., that

$$[E_\alpha,\{M^{-1},A_\mu^\beta\}]=f^{\alpha\beta\gamma}\{M^{-1},A_\mu^\gamma\}\,, \tag{4.1}$$

and we calculate the consequences of (4.1).

By the same arguments as given in the first part of Sec. III for the α_{l3} decay one obtains for the α_{l2} decay

$$\Gamma = 2\pi \sum_{\text{pol}} \int \frac{d^3p_l}{2E_l}\frac{d^3p_\nu}{2E_\nu}\delta^4(p_\alpha - p_\nu - p_l) \times \frac{1}{2E_\alpha m_\alpha{}^2} |\langle\langle l\nu p_l p_\nu|T|\alpha, \hat{p}_\alpha\rangle\rangle|^2 . \tag{4.2}$$

In accordance with assumption (3b) of Sec. III the reduced matrix element $\langle\langle l\nu|T|\alpha\rangle\rangle$ is given, in analogy to (3.15), by the product of a leptonic part and a hadronic part:

$$\langle\langle \bar{l}\nu p_l p_\nu|T|\alpha\hat{p}_\alpha\rangle\rangle = \bar{u}(p_\nu)\gamma^\lambda(1-\gamma_5)v(p_l)\langle\langle\sigma|H_\lambda|\hat{p}_\alpha\alpha\rangle\rangle , \tag{4.3}$$

where $|\sigma\rangle$ is the "hadronic vacuum state," for which, as usual, we take as the internal quantum numbers the quantum numbers of an SU(3) singlet:

$$|\sigma\rangle = |I=0, I_3=0, Y=0(\lambda=0,\mu=0)\rangle . \tag{4.4}$$

The hadronic transition operator is given by (3.16); consequently only the A^β_μ contribute to (4.3). (Their strength g is assumed to be the same as for the V^β_μ; this, however, does not lead to an observable effect for the meson decays as it only fixes the relative value of the reduced matrix elements for V^α and $\{M^{-1}, A^\alpha\}$.)

As by (4.1) not A^β_μ but $\{M^{-1}, A^\beta_\mu\}$ is an $SU(3)_E$ octet operator, one has instead of (3.18)

$$\begin{aligned}\langle\sigma|\{A^\beta_\mu, M^{-1}\}|\alpha\hat{p}_\alpha\rangle &= \langle\sigma|\tilde{A}^\beta|\alpha\rangle\langle\sigma|\tilde{A}_\mu|\hat{p}_\alpha\rangle \\ &= \langle\sigma|\tilde{A}^\beta|\alpha\rangle\,\mathfrak{F}\,\frac{p^{(\alpha)}_\mu}{m_\alpha} ,\end{aligned} \tag{4.5}$$

where $\langle\sigma|\tilde{A}_\mu|\hat{p}_\alpha\rangle$ is the reduced matrix element of $SU(3)_E$ which depends upon $\hat{p}_\alpha$ and $\mathfrak{F}$ is a constant of the dimension of mass^{-1}. The matrix element A^β_μ is therefore

$$\langle\sigma|A^\beta_\mu|\alpha\hat{p}_\alpha\rangle = \langle\sigma|\tilde{A}^\beta|\alpha\rangle\,\mathfrak{F}p^{(\alpha)}_\mu . \tag{4.6}$$

For the reduced matrix element in (4.3) one obtains therewith

$$\langle\langle\sigma|H_\lambda|\alpha\rangle\rangle = g\langle\sigma|\tilde{A}^\beta|\alpha\rangle F p^{(\alpha)}_\mu . \tag{4.7}$$

The suppression factor for the K_{l2} decay is defined by the ratio of the amplitudes for K_{l2} decay and π_{l2} decay, i.e., by[17]

$$S_{l2} = \frac{\langle 0|A^{-1}|\pi^+\rangle}{\langle 0|A^{-2}|K^+\rangle}\left[\frac{\Gamma(K_{l2})}{\Gamma(\pi_{l2})}\,\frac{\text{phase space }(\pi_{l2})}{\text{phase space }(K_{l2})}\right]^{1/2} . \tag{4.8}$$

S_{l2} corresponds in the usual treatment to $(f_K/f_\pi)\tan\theta$. If we insert (4.7) for $\alpha = K^+$ and $\alpha = \pi^+$ into (4.2) and the results into (4.8) we obtain

$$S_{l2} = \frac{m_\pi}{m_K} = 0.283 , \tag{4.9}$$

which is in very good agreement with the experimental value[18] of $(f_K/f_\pi)\tan\theta = 0.276$.

Instead of (3.16) and (2.7) with $P=-1$ one can again use (2.6) and describe the deviation of the transition operator from an octet operator by (3.17) with $q=1$; the result is again (4.9).

V. SUMMARY

If we assume that the particle-classifying $SU(3)_E$ is a spectrum-generating group that fulfills (2.4), we obtain the value (3.22) for the form-factor ratio. This value is largely independent of the relation between the particle-classifying $SU(3)_E$ and the operators V^β_μ and A^β_μ. It depends upon (2.4), which gives the particular value (3.22). And it depends upon the assumption that V^β_μ be of "first class with respect to a spectrum-generating $SU(3)_E$." This assumption for the V^β_μ is based upon principles so well accepted (neglecting the small effect of time-inversion noninvariance) that it is hard not to believe in it. Under this assumption the value $\xi=0$ can only be obtained if $SU(3)_E$ is a symmetry group, which makes the result of the experiment of Ref. 1 very remarkable. If one assumes (2.7) with $P=0$ for V^β_μ, i.e., that V^β_μ is an $SU(3)_E$-octet operator, then the vector Cabibbo suppression is given by (3.25). The assumption (2.7) with $P=-\frac{1}{2}$ gives a value in perfect agreement with experimental data. For the octet of axial-vector operators one has to assume (2.7) with $P'=-1$ in order to obtain perfect agreement with the experimental value. Though there is no reason that V^β_μ and A^β_μ should deviate from the octet property in exactly the same way when SU(3) breaking is taken into account, (2.7) should probably not be taken as more than an illustration of relations between $V^\alpha_\mu, A^\alpha_\mu$ and the $SU(3)_E$ generators, when SU(3) is considered a spectrum-generating group.

Note added in proof. In (3.23) and (3.24) the t dependence, or more precisely the dependence of F_+ upon $\tau = (p_\alpha/m_\alpha - p_\pi/m_\pi)^2$ has not properly been taken into account. Doing this leads to multiplication of the above predicted values for S_{l_3} by the factor

$$F_+\left(-\frac{(m_K - m_\pi)^2}{m_K m_\pi}\right)\Big/F_+\left(-\frac{(m_{\pi^+} - m_{\pi^0})^2}{m_{\pi^+} m_{\pi^0}}\right) = 0.82$$

[A. Böhm, J. Werle, and M. Igarashi, in proceedings of the International Symposium on Mathematical Physics, Mexico City, 1976 (unpublished)]. Consequently (3.25) leads to a value 0.183×0.82, which is too small. However, the value now predicted for S_{l_3} from the assumption (2.7) with $P=-1$ becomes [cf. (3.26′)] $0.283\times 0.83=0.23$, which is in very good agreement with the value determined from the experimental data. This is very nice, as the experimental data for the α_{l_2} decays also require $P=-1$ for the A_μ^α.

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APPENDIX

In this appendix we give the derivation of (3.19) from the assumptions (a), (b), and (c) of Sec. II.

The reduced matrix element of T is written as

$$\begin{aligned}\langle\langle \nu e^+ p_\nu p_l \pi^0 \hat{p}_\pi | T | \alpha \hat{p}_\alpha\rangle\rangle &= \bar{u}(p_\nu)\gamma^\mu(1-\gamma_5)v(p_l)\langle\langle \pi^0 \hat{p}_{\pi^0} | H_\mu | \alpha\hat{p}_\alpha\rangle\rangle \\ &= \bar{u}(p_\nu)\gamma^\mu(1-\gamma_5)v(p_l)\langle\hat{\pi}|\hat{V}^\beta|\hat{\alpha}\rangle(F_+\hat{p}_\mu + F_-\hat{q}_\mu),\end{aligned} \tag{A1}$$

where we have used (3.18′) and where we have introduced the abbreviations

$$\hat{p}=\hat{p}_\alpha+\hat{p}_\pi, \quad \hat{q}=\hat{p}_\alpha-\hat{p}_\pi, \tag{A2}$$

and where V^β is the component of the octet operator between the states $|\pi^0\rangle$ and $|\alpha\rangle$ that does not vanish.

For the matrix element of the time inversion of T, $A_T^{-1}TA_T$, one obtains

$$\langle p_\nu \sigma_\nu p_l \sigma_l \hat{p}_\pi \pi^0 | A_T^{-1}TA_T | \alpha\hat{p}_\alpha\rangle = (-1)(-1)^{\sigma_\nu+\sigma_l}\alpha_\nu^*\alpha_l^*\langle -p_\nu -\sigma_\nu -\hat{p}_l -\sigma_l -\hat{p}_\pi \pi | T | \alpha -\hat{p}_\alpha\rangle^*,$$

where σ_ν, σ_l denotes the component of the spin of the leptons, where the asterisk indicates the complex conjugate, and where we have used

$$A_T|\hat{p}_\alpha \alpha\rangle = \eta|\alpha -\hat{p}_\alpha\rangle \quad (\alpha \text{ a pseudoscalar meson})$$

and

$$A_T|\hat{p}\sigma\rangle = (-1)^{1/2+\sigma}\alpha_l|-\hat{p}-\sigma\rangle \text{ for lepton with } \eta,\ \alpha_l \text{ phase factors.}$$

Thus the reduced matrix element of $A_T^{-1}TA_T$ written in the form (A1) is given by

$$\begin{aligned}&\langle\langle p_\nu\sigma_\nu p_l\sigma_l\hat{p}_\pi\pi^0|A_T^{-1}TA_T|\alpha\hat{p}_\alpha\rangle\rangle \\ &\quad = (-1)(-1)^{\sigma_\nu+\sigma_l}\alpha_\nu^*\alpha_l^*\langle\langle -p_\nu -\sigma_\nu -p_l -\sigma_l -\hat{p}_\pi\pi|T|\alpha -\hat{p}_\alpha\rangle\rangle^* \\ &\quad = (-1)(-1)^{\sigma_\nu+\sigma_l}[\alpha_\nu\alpha_l\bar{u}(-p_\nu,-\sigma_\nu)\gamma^\mu(1-\gamma_5)v(-p_l-\sigma_l)\langle\langle\pi -\hat{p}_\pi|H_\lambda|\alpha -\hat{p}_\alpha\rangle\rangle]^* \\ &\quad = (-1)(-1)\epsilon(\mu)\alpha_\nu^*\alpha_l^*\bar{u}(p_\nu,\sigma_\nu)\gamma^\mu(1-\gamma_5)v(p_l,\sigma_l)\langle\langle\pi -\hat{p}_\pi|H_\mu|-\hat{p}_\alpha \propto\rangle\rangle^* \qquad (\text{A3}) \\ &\quad = \alpha_\nu^*\alpha_l^*\bar{u}(p_\nu\sigma_\nu)\gamma^\mu(1-\gamma_5)v(p_l,\sigma_l)\langle\hat{\pi}|\hat{V}^\beta|\alpha\rangle^*\{F_+^*(\hat{p}^{(\alpha)}+\hat{p}^{(\pi)})_\mu + F_-^*(\hat{p}^{(\alpha)}-\hat{p}^{(\pi)})_\mu\}, \qquad (\text{A4})\end{aligned}$$

where we have used the property of the Dirac spinor:

$$\begin{aligned}&\bar{u}(-p_\nu-\sigma_\nu)\gamma^\mu(1-\gamma_5)v(-p_l-\sigma_l)(-1)^{\sigma_\nu+\sigma_l} = -\epsilon(\mu)[\bar{u}(p_\nu\sigma_\nu)\gamma^\mu(1-\gamma_5)v(p_l,\sigma_l)]^*, \\ &\epsilon(\mu)=\begin{cases}+1 & \text{for } \mu=0\\ -1 & \text{for } \mu=1,2,3.\end{cases}\end{aligned} \tag{A5}$$

The observable quantity is

$$|\langle\langle \bar{l}\nu\pi|T|\alpha\rangle\rangle|^2 = \langle\langle\nu\bar{l}p_\nu p_l\pi\hat{p}_\pi|T|\alpha\hat{p}_\alpha\rangle\rangle\langle\langle\alpha\hat{p}_\alpha|T|\hat{p}_\pi\pi p_\nu p_l\bar{l}\nu\rangle\rangle,$$

where $\langle\langle\alpha|T|\pi l\nu\rangle\rangle$ is the adjoint reducd matrix:

$$\langle\langle\alpha|T|\pi l\nu\rangle\rangle = \bar{v}(p_l)\gamma^\lambda(1-\gamma_5)u(p_\nu)\langle\langle\alpha|H_\lambda|\pi^0\rangle\rangle. \tag{A6}$$

According to (A3) one has for the adjoint matrix of $A_T^{-1}TA_T$:

$$\langle\langle\alpha|A_T^{-1}TA_T|\pi l\nu\rangle\rangle = \epsilon(\lambda)\alpha_\nu\alpha_l\bar{v}(p_l)\gamma^\lambda(1-\gamma_5)u(p_\nu)\langle\langle\alpha -\hat{p}_\alpha|H_\lambda|\pi -\hat{p}_\pi\rangle\rangle^*. \tag{A7}$$

If time-reversal invariance holds, i.e., if the assumption (a) of Sec. II C is fulfilled, then

$$|\langle\langle \bar{l}\nu\pi|T|\alpha\rangle\rangle|^2 = |\langle\langle \bar{l}\nu\pi|A_T^{-1}TA_T|\alpha\rangle\rangle|^2. \tag{A8}$$

Inserting (A1), (A6), (A3), and (A7) into (A8) one obtains

$$\langle\langle \pi\hat{p}_\pi|H_\mu|\hat{p}_\alpha\alpha\rangle\rangle\langle\langle \alpha\hat{p}_\alpha|H_\lambda|\pi\hat{p}_\pi\rangle\rangle = \langle\langle \pi-\hat{p}_\pi|H_\mu|\alpha-\hat{p}_\alpha\rangle\rangle^*\epsilon(\mu)[\langle\langle \alpha-\hat{p}_\alpha|H_\lambda|\pi-\hat{p}_\pi\rangle\rangle\epsilon(\lambda)]^*, \tag{A9}$$

or

$$\eta(F_+\hat{p}_\mu + F_-\hat{q}_\mu)(F_+\hat{p}_\lambda - F_-\hat{q}_\lambda) = \eta^*(F_+^*\hat{p}_\mu + F_-^*\hat{q}_\mu)(F_+^*\hat{p}_\lambda - F_-^*\hat{q}_\lambda), \tag{A10}$$

where

$$\eta = \langle\pi|V^\beta|\alpha\rangle\langle\alpha|V^{-\beta}|\pi\rangle. \tag{A11}$$

If H_μ is Hermitian, i.e. if we use the assumption (b) of Sec. II C, then

$$H_\mu^\dagger = H_\mu \tag{A12}$$

and we can write the left-hand side of (A9) also as

$$\langle\langle \pi p_\pi|H_\mu|\alpha p_\alpha\rangle\rangle\langle\langle \pi p_\pi|H_\lambda|\alpha p_\alpha\rangle\rangle^* = \langle\pi|V^\beta|\alpha\rangle\langle\pi|V^\beta|\alpha\rangle^*\{F_+\hat{p}_\mu + F_-\hat{q}_\mu\}\{F_+^*\hat{p}_\lambda + F_-^*\hat{q}_\lambda\}. \tag{A13}$$

Comparing this with the left-hand side of (A10), i.e. using the assumption (A6), one obtains

$$F_+^2 = F_+F_+^*\phi,\quad F_+F_- = -F_+F_-^*\phi,\quad F_-F_+ = F_-F_+^*\phi,$$
$$F_-^2 = -F_-^*F_-\phi, \tag{A14}$$

where $\phi = \langle\pi|V^\beta|\alpha\rangle^*/\langle\alpha|V^{-\beta}|\pi\rangle$.

From (A10), i.e. from the assumption (a), one obtains

$$F_+^2 = (F_+^*)^2\psi,\quad F_+F_- = F_+^*F_-^*\psi,\quad F_-F_+ = F_-^*F_+^*\psi,$$
$$F_-^2 = (F_-^*)^2\psi,$$
$$\psi = \eta^*/\eta. \tag{A15}$$

From (A14) it follows that

$$F_-F_+^* = -F_-^*F_+. \tag{A16}$$

If we assume that

$$F_+ \neq 0$$

then from (A15) it follows that

$$F_- = F_-^*\frac{F_+^*F_+^2}{F_+(F_+^*)^2} = F_-^*\frac{F_+}{F_+^*}, \tag{A17}$$

and (A16) and (A17) lead to

$$F_- = 0. \tag{A18}$$

If we now assume

$$F_- \neq 0,$$

then from (A15) it follows that

$$F_+ = F_+^*\frac{F_-^*}{F_-}\frac{F_-^2}{(F_-^*)^2} = F_+^*\frac{F_-}{F_-^*}, \tag{A19}$$

and (A16) and (A19) lead to

$$F_+ = 0. \tag{A20}$$

Thus, from the assumptions (a) and (b) of Sec. II C it follows that *either* $F_+ = 0$ *or* $F_- = 0$.

In order to see that $F_- = 0$ and $F_+ \neq 0$ we have to use the assumption (c) of Sec. II C. According to assumption (c),

$$\langle\langle \pi^+\hat{p}'|V_\mu^Q|\hat{p}\pi^+\rangle\rangle = \langle\pi^+|V^Q|\pi^+\rangle[F_+(\hat{p}+\hat{p}') + F_-(\hat{p}-\hat{p}')]$$

(where V_μ^Q is the charge component of the octet operator V_μ^β) must be proportional to the charge of the π^+ for all values of $\hat{p}$ and $\hat{p}'$, i.e., $\neq 0$ for $\hat{p} = \hat{p}'$. Consequently $F_+ = 0$ is impossible and therewith it follows from the results of assumptions (a) and (b) that $F_- = 0$.

In the usual convention in which V^Q is proportional to the Hermitian charge operator and $\langle\alpha|V^{-\beta}|\pi\rangle$ to the SU(3) Clebsch-Gordon coefficient with the usual convention $\langle\alpha|V^{-\beta}|\pi\rangle = \langle\pi|V^\beta|\alpha\rangle^*$ it follows from (A14) that

$$F_+ = F_+^*.$$

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[1]G. Donaldson *et al.*, Phys. Rev. Lett. 31, 337 (1973).

[2]J. Sandweiss *et al.*, Phys. Rev. Lett. 30, 1002 (1973).

[3]We will use the term "observable" in a slightly broader sense than usual (a physical quantity represented by an essentially self-adjoint operator). E.g., the 3 com-

ponents J_i of angular momentum are observables in the usual sense. We will also call the $J_\pm = \pm(J_1 \pm iJ_2)$ observables.

[4]We use the Cartan-Weyl basis for the SU(3) Lie algebra in which the commutation relations are given by

$$[H_i, H_j] = 0,$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_\gamma,$$

$$[H_j, E_\alpha] = r_j(\alpha) E_\alpha,$$

$$[E_\alpha, E_{-\alpha}] = r^i(\alpha) H_i.$$

The root vectors are (in the normalization we use)

$$r_i(1) = (1/\sqrt{3})(1,0), \quad r_i(2) = (1/2\sqrt{3})(1,\sqrt{3}),$$

$$r_i(-\alpha) = -r_i(\alpha), \quad r_i(3) = (1/2\sqrt{3})(-1,\sqrt{3}),$$

and

$$N_{\alpha\beta} = \pm(\tfrac{1}{6})^{1/2} \text{ if } r(\alpha) + r(\beta) = r(\gamma) \text{ is also a nonvanishing root vector}$$

$$= 0 \text{ otherwise;}$$

in particular,

$$N_{1,3} = -N_{3,1} = N_{-3,-1} = -N_{-1,-3} = N_{3,-2}$$

$$= -N_{-2,3} = N_{-2,1} = -N_{1,-2} = N_{2,-3}$$

$$= -N_{-3,2} = N_{-1,2} = -N_{2,-1} = (\tfrac{1}{6})^{1/2}.$$

In the normalization we have used here, the hypercharge is

$$Y = 2H_2$$

and the isospin is

$$I_3 = \sqrt{3}\, H_1, \quad (I_1 \pm iI_2) = (\sqrt{6}) E_{\pm 1}.$$

We also call $H_1 = E_0$, $H_2 = E_8 = E_{-8}$.
Our notation differs from the one conventionally used in physics literature. The connection is as follows:

$V_\mu^{\pm 1}$ corresponds to $\mathfrak{F}_{1\mu} \pm i\mathfrak{F}_{2\mu}$,

$A_\mu^{\pm 1}$ corresponds to $\mathfrak{F}^5_{1\mu} \pm i\,\mathfrak{F}^5_{2\mu}$,

$V_\mu^{\pm 2}$ corresponds to $\mathfrak{F}_{4\mu} \pm i\,\mathfrak{F}_{5\mu}$,

$A_\mu^{\pm 2}$ corresponds to $\mathfrak{F}^5_{4\mu} \pm i\,\mathfrak{F}^5_{5\mu}$,

etc.

[5]That the algebra of observables cannot be an envoloping algebra of a group if one wants a nontrivial mass spectrum follows from the O'Raifeartaigh theorem [L. O'Raifeartaigh, Phys. Rev. 171, 1698 (1968)].

[6]Even if instead of (2.7) one has a more general relation between E_α and V_μ^β, e.g., one which one obtains from (2.7) by replacing M^P with any (Hermitian) operator $\mathcal{O}$ that commutes with P_i, I, I_3, Y, one still has the condition of being first class.

[7]First suggested by J. Werle [ICTP Report, Trieste, 1965 (unpublished)], this approximation has been used previously to simplify calculations [see, e.g., A. Böhm, Phys. Rev. D 7, 2101 (1973) and references therein]. Its consequences for the form factors in a relativistic quantum mechanics have been explored here for the first time.

[8]S. Merlan *et al*. [BNL Report No. BNL-18076, 1973 (unpublished)] find $\xi(0) = -0.57 \pm 0.24$ and $\lambda = 0.027 \pm 0.019$ from Dalitz-plot analysis and $\xi(0) = -0.64 \pm 0.27$ from μ polarization analysis; this is an experiment with reasonable statistics whose advantage is that Dalitz-plot analysis and polarization analysis give results consistent with each other. Reference 2 gives from μ polarization analysis $\mathrm{Re}\,\xi(0) = -0.655 \pm 0.127$. The primary purpose of this experiment was the test of T invariance, and it has high statistics, comparable with the statistics of the experiment in Ref. 1, which gives from Dalitz-plot analysis values of ξ quoted as $\xi = 0.00 \pm 0.04$ (unparametrized fit) and $\xi = -0.11 \pm 0.02$ (parametrized fit); even if one wants to ignore all previous (low-statistics) results, the discrepancies between these experiments have to be explained. P. Haidt *et al*. (X_2) [Phys. Rev. D 3, 10 (1972)] find by all three methods values in agreement with (3.22); however, their results favor a linear t dependence of ξ. The $K_{\mu 3}/K_{e3}$ branching ratio result of G. W. Brandenburg *et al*. [Phys. Rev. D 8, 1978 (1973)] is in disagreement with (2.22) and also in disagreement with the $\Delta I = \frac{1}{2}$ rule. Reference 1 disagrees with (3.22). I. H. Chiang *et al*. [Phys. Rev. D 6, 1254 (1972)] and C. Ankenbrandt *et al*. [Phys. Rev. Lett. 28, 1472 (1972)] are in agreement with (3.22). Preliminary results of the Aachen-Bari-Brussels-CERN Collaboration [Particle Physics Meeting, Hamburg, 1974 (unpublished)] are also in agreement with (3.22). The K^0_{l3} Dalitz-plot analysis experiments [M. G. Albrow *et al*., Nucl. Phys. B44, 1 (1972); E. Dally *et al*., Phys. Lett. 41B, 647 (1972)] are in agreement with (3.22). The average values of the older data [L. M. Chounet and J. M. Gaillard, Phys. Rep. 4C, 199 (1972)] are also consistent with (3.22).

[9]It says that the observables that have the dimension of MeV do not commute, but the observables that have the same dimension as the charges commute with $\mathrm{SU}(3)_E$ so that the strength of the noncommutativity of $\mathcal{P}$ and $\mathrm{SU}(3)_E$, (i.e., of the symmetry breaking) is described by a parameter (or several parameters) of dimension MeV (inverse of an elementary length).

[10]If (2.7) with $P \neq 0$ holds one has to replace everywhere V_μ^β by $\{M^P, V_\mu^\beta\}$ (or $\mathcal{O}$; see footnote 6.) This will not affect the result for ξ but will alter the prediction for S_{l3} as discussed at the end of Sec. III.

[11]We will call an octet operator Hermitian if its components have the same Hermiticity property as the corresponding generators in a unitary representation.

[12]Therefore, besides the "normal" operators fulfilling (2.8′) one may also have to consider the "abnormal" operators fulfilling $A_T A_\mu^\beta A_T = -\epsilon(\mu) A_\mu^\beta$ and besides the "regular" operators fulfilling (2.9′) one may have to consider the "irregular" operators fulfilling $A_\mu^{\beta+} = -A_\mu^{-\beta}$. See C. W. Kim and H. Primakoff, Phys. Rev. 180, 1502 (1969).

[13]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[14]Note that the operator of total momentum is not P_μ. P_μ is the momentum operator in the hadron subspace.

[15]The u and v have dimension $(\text{mass})^{1/2}$, as they are normalized such that $\sum_{\text{pol}} u\bar{u} = \gamma_\mu p^\mu + m$ in accordance with the normalization of the measure in (3.14)).

[16]The same number has been predicted in the framework of Kemmer currents by E. Fischbach, F. Iachello, A. Lande, M. M. Nieto, M. Primakoff, and C. K. Scott,

Phys. Rev. Lett. 26, 1200 (1971); E. Fischbach, M. M. Nieto, M. Primakoff, C. K. Scott, and J. Smith, *ibid.* 27, 1403 (1971); Phys. Rev. D 6, 726 (1972); B. Nagel and H. Snellman, Phys. Rev. Lett. 27, 761 (1971); Phys. Rev. D 6, 731 (1972); A. O. Barut and Z. Z. Aydin, *ibid.* 6, 3340 (1972). Fischbach *et al.* also obtain a Cabibbo angle which is different from the conventional value and numerically very close to the value given by (3.25). However, their $\tan\theta_V$ is an entirely different entity from the suppression factor S_{l3}. In the present paper a higher universality is assumed and the Cabibbo suppression is obtained as a consequence of the nonsymmetry of SU(3). In contrast to this, Fischbach *et al.* consider θ_V in the conventional spirit as an independent parameter which, however, takes different values for the Kemmer equation than for the Klein-Gordon equation.

[17]Phase space (α_{l_3}) denotes the integral of that part of the amplitude square from which the dynamical quantities have been factored off. Thus, phase space (α_{l_3}) $\sim m_\alpha{}^5 C(m_{\pi^0}/m_\alpha) C_1(\alpha)$, where $C_1(K^\pm)$ is the correction factor for the t dependence of f_+ and $C_1(\pi^\pm)$ is the correction factor from $m_e \neq 0$, which is ≈ 1.04.

[18]M. Roos, Phys. Lett. 36B, 130 (1971).