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Peripheral peaking and shrinkage phenomenon in the *s* channel based on the statistical bootstrap model with spin*

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We observe that the spectrum of the statistical bootstrap model with spin can indeed give rise to peripheral peaks in two-body scattering. This further confirms the conclusion of an earlier work by Kogitz *et al.* We also present a new refined solution to the bootstrap model, which provides the shrinkage phenomenon for peripheral peaks.

In a recent paper, Kogitz, Logan, and Tanaka[1] discussed a direct-channel resonance model for two-body inelastic scattering. They saturated the inelastic amplitude with the resonance spectrum of the statistical bootstrap model with spin of Chiu and Heinmann[2] (hereafter referred to as I), which is an extension to the statistical bootstrap model proposed by Hagedorn[3] and refined by Frautschi.[4] By comparing the experimental inelastic differential cross sections in the peripheral region, Kogitz *et al.* were able to discriminate between various solutions considered in I. They demonstrated that one of the solutions gives a reasonable description to a large number of peripheral peaks. These authors also observed that if one wants to reproduce more refined energy dependence of peripheral peaks, such as the shrinkage phenomena, it will be desirable to explore a new possible solution to the bootstrap equation, which contains some additional logarithmic factor. We have investigated their proposal in some detail and find that the proposed form is indeed an asymptotic solution to this equation. In particular, for large m, where m is the mass of the bootstrap system, it satisfies the equation to the accuracy of $O((\ln m)/m)$, which is comparable to those solutions discussed in I, where the corresponding accuracy is of $O(1/m)$.

Our main purposes here are two. Firstly, we derive a statistical bootstrap amplitude similar to that in Ref. 1. We use a somewhat different language, to reiterate their conclusion, that the spectrum of the statistical bootstrap model can lead to a sharp peripheral peak. Secondly, we demonstrate that the above-mentioned possibility with additional logarithmic factor is indeed a solution to the bootstrap equation.

Within the approach of I, one explicitly takes into account the angular-momentum content of the bootstrap states. There the angular-momentum bootstrap is achieved through imposing conservation of angular momentum along some arbitrarily specified direction, say the z direction. The bootstrap is for the density of states $\sigma(m, J_z)$, where m is the mass and J_z the z-component polarization.

For simplicity, here we only exhibit the Fourier-transformed bootstrap equation. With the Fourier-transformed density of states defined by

$$F(m,\alpha)=\int_{-\infty}^{\infty} dJ_z e^{i\alpha J_z}\sigma(m,J_z)\ , \qquad (1)$$

the transformed equation is given by

$$F(m,\alpha)=\frac{1}{2h^3}\int_{\mu}^{m-\mu} dm_2 F(m_2,\alpha)\int_{\mu}^{m-m_2-\mu} dm_1 F(m_1,\alpha)\,\tilde{\phi}\,(m_1,m_2;m;\alpha)\ , \qquad (2)$$

where the kernel $\tilde{\phi}$ is specified by some finite-interaction-volume dynamics. We refer the reader to I for its form. Here we use the symbol $F(m,\alpha)$ in the place of $\tilde{\sigma}(m,\alpha)$ of I. We recall that the level density of the original statistical bootstrap equation of Hagedorn and Frautschi for large m is given by[5,6]

$$n(m)\sim m^{-3}e^{bm}\ . \qquad (3)$$

Notice from Eq. (1), at $\alpha=0$,

$$F(m,0)=\int_{-\infty}^{\infty} dJ_z\,\sigma(m,J_z)\equiv\sum_{J_z=-\infty}^{\infty}\sigma(m,J_z)$$
$$\equiv n(m)\ , \qquad (4)$$

and Eq. (2) is reduced to the precise bootstrap equation of Hagedorn and Frautschi. In general, to solve Eq. (2) one has to take into account both the m dependence and the α dependence simultaneously.

Among all the solutions discussed in I, the solution preferred[1] by the data turns out to be the one having essentially the largest width in J. It has the form

$$\sigma(m,J_z)=\frac{n(m)}{2Dm}\,\frac{1}{\cosh(\pi J_z/2Dm)}\ , \tag{5}$$

or

$$\frac{F(m,\alpha)}{F(m,0)}=\frac{1}{\cosh(Dm\alpha)}\ . \tag{5$'$}$$

From Eq. (3) one sees that for large m, the bootstrap system has high degeneracies. In its rest frame the statistical system has no preferred direction. There are equally populated $(2J+1)$-magnetic states for each level with spin J. Denote the level degeneracy for a given spin J by $\bar\rho(m,J)$. Then the total density of states at a given mass m is

$$\begin{aligned}n(m)&=\sum_{J=0}^{\infty}\sum_{J_z=-J}^{J}\bar\rho(m,J)\\&=\sum_{J=0}^{\infty}(2J+1)\,\bar\rho(m,J)\equiv\sum_{J}\rho(m,J)\,,\end{aligned} \tag{6}$$

where $\rho(m,J)$ is the density of states with spin J.

Comparing Eqs. (4) and (6), one finds that

$$\sigma(m,J_z)=\sum_{J=|J_z|}^{\infty}\bar\rho(m,J)\ . \tag{7}$$

This leads to

$$\begin{aligned}\bar\rho(m,J)&=\sigma(m,J_z)-\sigma(m,J_z+1)\big|_{J_z=J}\\&\approx-\frac{d}{dJ_z}\,\sigma(m,J_z)\big|_{J_z=J}\ .\end{aligned} \tag{8}$$

Next we turn to the model of Ref. 1. They considered the imaginary part of the amplitude for the two-body inelastic scattering. We shall ignore the spin complications. Denote the process by $i\to f$, with $p_i(0)$ the initial c.m. momentum along the z axis and $p_f(\theta)$ the final momentum along the direction with a polar angle θ and azimuthal angle $0°$. The unitarity relation is given by,

$$\begin{aligned}\mathrm{Im}A_{fi}&=\sum_n\langle f\,|A\,|n\rangle\langle n\,|A^\dagger|i\rangle\\&=\sum_n\sum_{J'M'}\sum_{JM}\langle\,\vec{\mathrm{p}}_f(\theta)\,|\,f,J'M'\rangle\langle f,J'M'\,|A\,|n\rangle\\&\qquad\times\langle n\,|A^\dagger\,|\,i,JM\rangle\langle i,JM\,|\,\vec{\mathrm{p}}_i(0)\,\rangle\ .\end{aligned} \tag{9}$$

In the first equality, we have saturated the intermediate states by the spectrum of Eq. (3), with n running through all the degeneracies. In the last step, we have inserted two identities, which involve summing over the complete sets of angular momentum states for both the initial and final two-body states.

Now we make use of the usual relations,[7]

$$\begin{aligned}\langle i,JM\,|\,\vec{\mathrm{p}}_i(0)\rangle&=\left(\frac{2J+1}{4\pi}\right)^{1/2}d^J_{M0}(0)\\&=\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M0}\end{aligned} \tag{10}$$

and

$$\langle\vec{\mathrm{p}}_f(\theta)\,|\,J'M'\,\rangle=\left(\frac{2J+1}{4\pi}\right)^{1/2}d^{J'}_{M'0}(\theta)\ ,$$

where $d^J_{ij}(\theta)$ is the rotation matrix. For a moment let us fix the initial angular momentum to be at J. Among all the angular momentum states of $|n\rangle$ and $|f\rangle$, owing to conservation of angular momentum only those states with spin J and $J_z=0$ (they are, respectively, $|n'\rangle\equiv|n,J\,0\,\rangle$ and $|\,f,J\,0\rangle$) will contribute. So

$$\begin{aligned}&\sum_n\langle f,J'M'\,|A\,|n\rangle\langle n\,|A^\dagger\,|\,i,J\,0\,\rangle\\&\quad=\sum_{n'}\langle f,J\,0\,|A\,|n,J\,0\,\rangle\langle n,J\,0\,|A^\dagger\,|\,i,J\,0\,\rangle\\&\quad\equiv\sum_{n'}\gamma_f^{(n')}(m)\gamma_i^{*(n')}(m)\\&\quad=\bar\rho(m,J)\gamma_f(m)\gamma_i^*(m)\ .\end{aligned} \tag{11}$$

In the second step we have introduced the couplings $\gamma_i^{(n')}$ and $\gamma_f^{(n')}$. In the last step we have made the statistical assumption that the couplings are independent of their label n'. Notice that the degeneracy for these $|n'\rangle$ states is $\bar\rho(m,J)$. Substituting Eqs. (10) and (11) into Eq. (9), we arrive at

$$\begin{aligned}\mathrm{Im}A_{fi}(m,\cos\theta)&=\sum_J(2J+1)\gamma_f(m)\gamma_i(m)\\&\qquad\times\bar\rho(m,J)\,P_J(\cos\theta)\ ,\end{aligned} \tag{12}$$

where $d^J_{00}(\theta)=P_J(\cos\theta)$ was used. Apparently from Eqs. (5) and (6) and Eq. (12) with the appropriate choice of the parameter D, one predicts a sharp peripheral peak,[1,8] in general agreement with the data.

We mentioned earlier that, since the bootstrap system is a statistical system, in its rest system there is no preferred direction. At this point it is instructive to contrast the angular distributions between the decay of this statistical system and the two-body scattering. First consider the decay process of $n\to f$, where the initial state is unpolarized and f is one specific final state with two spinless particles. Averaging over all the magnetic states of n, the normalized decay distribution is given by

$$\frac{1}{n(m)}\sum_n |\langle \vec{p}_f(\theta)|n\rangle|^2$$
$$\equiv \frac{1}{n(m)}\sum_{J,M}\sum_{n'} |\langle \vec{p}_f(\theta)|n, JM\rangle|^2$$
$$= \frac{1}{n(m)}\sum_J \frac{2J+1}{4\pi}\sum_{n'}\left[\sum_M d^J_{0M}(\theta)d^J_{M0}(\theta)\right]$$
$$= \frac{1}{4\pi n(m)}\sum_J (2J+1)\bar{\rho}(m,J) = 1/4\pi \ . \tag{13}$$

In the second step we used Eq. (10), and in the third step the crucial identity

$$\sum_M d^J_{iM}(\theta)\, d^J_{Mj}(\theta) = \delta_{ij} \ .$$

For the last step, Eq. (6) was used. As expected, the distribution in Eq. (13) is isotropic. On the other hand, for the two-body scattering problem, the situation is quite different. Owing to the conservation of angular momentum, only $J_z = 0$ states contribute. Thus the intermediate state is, in fact, highly polarized. This leads to the possibility of peripheral peaking.

To conclude this part of the discussion, let us now locate the position of the nearest t-channel singularity in terms of the parameter D of Eq. (5). This singularity is of particular interest,[9] since it is responsible for the presence of a sharp peripheral peak near $t=0$. We recall, that in general, the high-J behavior of a partial-wave amplitude is controlled by the nearest singularity. In particular, if this singularity is at $t=t_0$ or $z = z_0 \approx 1 + 2t_0/m^2$, for large J and large m, the partial-wave amplitude[10]

$$A(J,m) \sim \exp\{-J\ln[z_0 + (z_0^2-1)^{1/2}]\}$$
$$\sim \exp\left(-\frac{2\sqrt{t_0}}{m}J\right). \tag{14}$$

On the other hand, from Eqs. (5) and (8), for $J \gg (2D/\pi)m$,

$$\bar{\rho}(J,m) \sim \exp(-\pi J/2Dm) \ . \tag{15}$$

So the nearest singularity is at

$$\sqrt{t_0} = \frac{\pi}{4D} \ . \tag{16}$$

For the solution of Ref. 1, where $D = 1.5\ \text{GeV}^{-1}$, correspondingly $\sqrt{t_0} \sim 0.5$ GeV. This is a typical mean mass value for low-lying mesons. So it appears to be not unreasonable.

Now we come to the second part of our discussion. To allow shrinkage phenomena, we consider a trial solution essentially the same as the one suggested in Ref. 1,

$$\frac{F(m,\alpha)}{F(m,0)} = \frac{f_1(\alpha)}{\cosh[\alpha(D+D'\ln m)m]} \ . \tag{17}$$

The machinery for the bootstrap model and technique in analyzing trial solutions have been discussed extensively in I. To save space, we will not reproduce them here. We will use the notations of I and directly quote equations and arguments from there. We refer the reader to I for necessary information.

Analogous to discussion in Sec. VIB of I, we set $d=0$ and insert into the integrand of Eq. (3.10) of I the additional factor,

$$A = \frac{1}{\cosh G(m_1,\alpha)\cosh G(m_2,\alpha)} \ . \tag{18}$$

With the substitution $m_1 \to m-\mu$ due to the peaking effect, ignoring terms of the order of m^{-1}, we have

$$\cosh G(m-\mu,\alpha) \sim \cosh G(m,\alpha)\cosh H(m,\alpha)$$
$$\times[1-\tanh G(m,\alpha)\tanh H(m,\alpha)] \ , \tag{19}$$

where

$$H(m,\alpha) = \alpha\mu(D+D'+D'\ln m) \ .$$

The extra factor in the right-hand side of the bootstrap equation now becomes

$$A \sim \frac{1}{\cosh G(m,\alpha)\cosh G(\mu,\alpha)\cosh H(m,\alpha)}$$
$$\times \frac{1}{1-\tanh G(m,\alpha)\tanh H(m,\alpha)} \ . \tag{20}$$

The situation here differs somewhat from that in I. To see this difference, it is instructive to consider first the α region, where

$$|H(m,\alpha)| = \mu(D+D'+D'\ln m)|\alpha| \gg 1 \ . \tag{21}$$

Since we are interested in the case $m \gg \mu$, from Eq. (21) we have $|G(m,\alpha)| \gg |H(m,\alpha)| \gg 1$. In turn,

$$\frac{1}{1-\tanh G(m,\alpha)\tanh H(m,\alpha)}$$
$$\to \frac{1}{1-\tanh|H(m,\alpha)|} \sim m^{2\mu D'|\alpha|} \ . \tag{22}$$

In I, one investigates trial functions for a special combination of the α and m dependence [see Eq. (3.6) of I]. A glance at Eq. (22) shows that, in contrast to the situation of Sec. VIB of I, for the present trial solution in the α region of interest it is not possible to choose $f_1(\alpha)$ to give an exact match for the bootstrap of $F(m,\alpha)$. However, this is not serious, since essentially we will be concerned with the quantity $\sigma(m, J_z)$, which involves an integration over α. It turns out that the α region where Eq. (22) is valid gives a negligible contribution because of the strong damping of the $[\cosh G(m,\alpha)]^{-1}$ factor in Eq. (20).

We proceed now to give a quantitative analysis. We make the choice

$$f_1(\alpha) = \cosh G(\mu,\alpha)\left(1+\frac{2\mu R^2}{b}C(\alpha)\right) \tag{23}$$

and will demonstrate that this does the job. After an integration similar to Eq. (3.12) of I, the right-hand side of the bootstrap equation becomes

$$\mathrm{RHS} = \left(\frac{1}{\cosh H(m,\alpha)[1-\tanh G(m,\alpha)\tanh H(m,\alpha)]}\right) \times \frac{f_1(\alpha)}{\cosh G(m,\alpha)} . \tag{24}$$

This is to be compared to the left-hand side,

$$\mathrm{LHS} = \frac{f_1(\alpha)}{\cosh G(m,\alpha)} . \tag{25}$$

So the mismatched factor is the quantity in large square brackets in Eq. (24).

The corresponding final Fourier-transform spectrum is given by

$$\sigma(m,J_z) = \frac{n(m)}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha J_z}\{\cdots\} , \tag{26}$$

where the quantity in curly brackets is either the RHS or the LHS expressions. Since these expressions are even in α, Eq. (26) can be rewritten as

$$\sigma(m,J_z) = \frac{n(m)}{\pi}\int_0^{\infty} d\alpha \cos\alpha J_z\{\cdots\} . \tag{27}$$

Divide the α range of integration in Eq. (27) into two regions:

$$\begin{aligned} &\mathrm{I:}\ \ 0 \le \alpha \lesssim \alpha_M = c/m, \\ &\mathrm{II:}\ \ \alpha > \alpha_M , \end{aligned} \tag{28}$$

where c is a parameter. The quantity α_M has been chosen in such a way, as we shall see, that with c properly restricted, the contribution from II may be negligible.

In region I, $H(m,\alpha) \sim \ln m/m \ll 1$. We have

$$\begin{aligned} &\cosh H(m,\alpha) = 1 + O\left[\left(\frac{\ln m}{m}\right)^2\right], \\ &1-\tanh G(m,\alpha)\tanh H(m,\alpha) = 1 + O\left(\frac{\ln m}{m}\right), \end{aligned} \tag{29}$$

and

$$f_1(\alpha) = 1 + O(1/m^2) .$$

Taking into account these expressions, the mismatched factor in Eq. (24) is of the form $1 + O((\ln m)/m)$ and is thus essentially unity for present considerations. Furthermore, in this region one may also replace $f_1(\alpha)$ by unity.

In region II for large m, $G(m,\alpha) \gg 1$, and

$$\frac{1}{\cosh G(m,\alpha)} \sim \tfrac{1}{2}\exp[-m(D+D'\ln m)\alpha] . \tag{30}$$

This implies that the magnitude of the envelope of the integrand in Eq. (27) peaks at the lower limit of the integration. This is important. We will make use of it below.

For present purposes it is sufficient to give an upper-bound estimate for the ratio of the contribution to Eq. (27) due to II to that due to I. For this purpose we will suppress the factor $\cos\alpha J_z$ in Eq. (27). Since the cosine factor has more oscillations in region II, this suppression tends to overestimate the contribution in II more than that in I. This is in accord with the upper-bound estimation.

Suppressing the cosine factor, from Eqs. (24) and (29) for region I, we have

$$\begin{aligned} \int_0^{\alpha_M} d\alpha\{\mathrm{RHS}\} &= \int_0^{\alpha_M} d\alpha\left[1+O\left(\frac{\ln m}{m}\right)\right]\frac{1}{\cosh G(m,\alpha)} \\ &> \left[1+O\left(\frac{\ln m}{m}\right)\right]\int_0^{\alpha_M} d\alpha\, e^{-G(m,\alpha)} \\ &\sim \frac{1-e^{-G(m,\alpha_M)}}{m(D+D'\ln m)} . \end{aligned} \tag{31}$$

In the second step we have used $(\cosh x)^{-1} > e^{-x}$, for $x > 0$, and in the last step the expression $G(m,\alpha_M) = c(D+D'\ln m)$. On the other hand for II, from Eqs. (24) and (30),

$$\begin{aligned} \int_{\alpha_M}^{\infty} d\alpha\{\mathrm{RHS}\} &\sim \int_{\alpha_M}^{\infty} d\alpha \frac{f_1(\alpha)\exp[-m(D+D'\ln m)]}{2[1-\tanh G(m,\alpha)\tanh H(m,\alpha)]} \\ &\sim \frac{f_1(\alpha_M)}{2[1-\tanh G(m,\alpha_M)\tanh H(m,\alpha_M)]}\,\frac{1}{m(D+D'\ln m)}\exp[-G(m,\alpha_M)] \\ &\sim \left[1+O\left(\frac{\ln m}{m}\right)\right]\exp(-cD)\frac{m^{-cD'}}{m(D+D'\ln m)} . \end{aligned} \tag{32}$$

In the second step in evaluating the integral we have made use of the peaking effect at $\alpha = \alpha_M$. Comparison between Eqs. (31) and (32) shows that if we choose c such that

$$cD' > 1 \,, \tag{33}$$

the contribution from II is to be suppressed as compared to I by more than the order $O(m^{-1})$. So II is now negligible. Inserting back the cosine factor, one finally arrives at

$$\begin{aligned}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha J_z}\{\text{RHS}\} &= 2\left[1 + O\left(\frac{\ln m}{m}\right)\right]\int_0^{\alpha_M} d\alpha \cos\alpha J_z \frac{1}{\cosh G(m,\alpha)} \\ &= \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha J_z}\{\text{LHS}\}\left[1 + O\left(\frac{\ln m}{m}\right)\right].\end{aligned} \tag{34}$$

So we see that Eq. (17) with $f_1(\alpha) = 1$ indeed leads to a bootstrap solution for $\sigma(m, J_z)$ to the stipulated accuracy of $O((\ln m)/m)$. Analogous to Sec. VI B of I, we finally arrive at the new spectrum,

$$\begin{aligned}\sigma(m, J_z) &= n(m)\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\alpha\, e^{i\alpha J_z}}{\cosh[m(D + D'\ln m)\alpha]} \\ &= \frac{n(m)}{2(D + D'\ln m)m}\,\frac{1}{\cosh\{\pi J_z/[2m(D + D'\ln m)]\}}\,.\end{aligned} \tag{35}$$

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[7]We follow the convention given in M. Jacob and G. F. Chew, *Strong Interaction Physics* (Benjamin, New York, 1964), Chap. 2. The normalization for the angular momentum states is $\langle JM|J'M'\rangle = \delta_{JJ'}\delta_{MM'}$.

[8]Technically speaking Eq. (12) differs from the corresponding expression in Ref. 1 in that there $\bar{\rho}$ has been replaced by ρ. Despite this discrepancy, the general conclusion of Ref. 1 on the existence of the peripheral peaking is still valid, since the peak prediction is insensitive to this difference. On the other hand, with Eq. (12) their specific parameter D may have to be readjusted slightly.

[9]I thank A. Gleeson for calling my attention to looking at the predictions of Ref. 1 from the point of view of t-channel singularities.

[10]See for example, P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer, Berlin, 1968), p. 52.