# Electromagnetic polarizabilities of nucleons

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A general theory of electromagnetic polarizabilities (and higher-order effects) of composite systems described by infinite-component wave equations is developed. Such wave equations were previously used to predict form factors, mass spectrum, and structure functions of the proton. As a special case the polarizabilities of the relativistic H atom, proton, and neutron are evaluated explicitly and compared with experiments.

## I. INTRODUCTION

The electromagnetic polarizabilities of a composite system, such as the H atom, are determined by the wave function of the system and the minimal-coupling principle to the electromagnetic field. One of the main tasks of particle physics in recent years has been to determine, more and more precisely, the wave function of the proton (and other hadrons) as a composite system in order to calculate and predict all the *intrinsic* properties of the proton. These intrinsic properties are the elastic and inelastic form factors, quantum numbers and masses of the excited states, the decay rates of the excited states, the structure and scaling functions in inelastic lepton scattering, and electromagnetic polarizabilities. We believe the O(4, 2) infinite-multiplet model of the proton, developed in analogy to the H atom, provides a reliable wave function of the proton from which all the above-mentioned properties associated with the internal structure of the proton can be and have been evaluated in closed form, and are consistent with experiments.[1-10] The purpose of the present paper is to complete the above list by calculating the electromagnetic polarizabilities.

Because the relativistic H atom can also be described completely by an O(4, 2) infinite-component wave equation,[11] there is similarly a physical "atomic" picture of the proton underlying the wave equation, with its discrete and continuous states.[12] The existence of spacelike solutions of the infinite-component wave equations caused considerable disenchantment with this approach.[13] Again the relativistic H atom provides an interpretation for these solutions, because the relativistic bound electron has also negative-energy states which give to the whole atom spacelike solutions.[14] In the calculation of the polarizabilities in this paper, we shall in fact show the physical role and contributions of these, naively labeled as "unphysical," states, which play a role in second- and higher-order calculations.

We present a general theory of external interactions and, in particular, of static external electromagnetic interactions, and apply the results to the relativistic H atom, to the proton, pion, and other hadrons. The actual calculations of the matrix elements are discussed in the appendixes.

## II. WAVE EQUATION AND EXTERNAL EM FIELD

The wave equation for the proton is

$$\Omega\tilde{\psi} \equiv (J^{\mu}P_{\mu} + \beta S + \gamma)\psi(p) = 0, \qquad (1)$$

where the conserved current is given by

$$J_{\mu} = \alpha_1\Gamma_{\mu} + \alpha_2 P_{\mu} + \alpha_3 P_{\mu} S + i\alpha_4 L_{\mu\nu}g^{\nu} + i\alpha_5 C L^{*}_{\mu\nu}g^{\nu}. \qquad (2)$$

The constant parameters $\alpha_i$, $\beta$, and $\gamma$ have been determined previously. In Eq. (2) $\Gamma_{\mu}$, $S$, and $L_{\mu\nu}$, $L^{*}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}L^{\sigma\rho}$ are the O(4, 2) generators acting on the infinite-component states, while $P_{\mu} = (p' + p)_{\mu}$, and $q_{\mu} = (p' - p)_{\mu}$ occur when taking the matrix elements of $J_{\mu}$ between two states of momenta $p'$ and $p$. $C$ is a parity changing operator.

The solution of (1) gives for the mass spectrum the equation

$$\Omega(n, M) \equiv \{[\alpha_1{}^2 M^2 - (\alpha_3 M^2 + \beta)^2]^{1/2} n + \alpha_2 M^2 + \gamma\} = 0, \qquad (3)$$

from which we solve $M$ as a function of the principal quantum number $n$. Note that in the symmetric form of Eq. (1) the $j$ levels are degenerate for a given $n$. At first we introduce the external field by mimimal coupling, $P_{\mu} \to P_{\mu} - eA_{\mu}$, so that the interaction terms for Eq. (1) are

$$I = I_{\mathrm{I}} + I_{\mathrm{II}} \equiv eJ^{\mathrm{em}}_{\mu}A^{\mu} + (\alpha_2 + \alpha_3 S)A_{\mu}A^{\mu}. \qquad (4)$$

Here $I_{\mathrm{II}} \equiv \rho_{\mu\nu}A^{\mu}A^{\nu}$ and is recognized as a "seagull" term, with $\rho_{\mu\nu} = (\alpha_2 + \alpha_3 S)g_{\mu\nu}$.

However, in quantum electrodynamics the seagull term must satisfy the condition $\rho_{0\nu} = 0$ in order for the theory to be canonical.[15] Indeed if one makes the minimal coupling to an equation in Hamiltonian form, the terms proportional to $A_0 A^0$

are not present. In our problem we wish to calculate the response of the system to a stationary electromagnetic field. The field fixes the rest frame of the problem, and since we wish to know the change in energy, we are forced to use a canonical formulation. Thus we use as the second interaction term

$$I_{II} = -(\alpha_2 + \alpha_3 S)\,\vec{A}\cdot\vec{A}. \tag{5}$$

Physically, the absence of the $A^0A_0$ term makes sense, because it would lead, when present, to a contribution to the electric polarizability called "dielectric polarizability."[16] Classically this would arise from a charge cloud of magnetic monopoles. The experimental absence of free magnetic monopoles points to the absence of $A_0A^0$ terms in the interaction. Furthermore, in the nonrelativistic limit the interaction (5) leads to the correct coupling to the external field where the $A_0A^0$ term is again absent.

$I_{II}$ in Eq. (5) does not contribute to the electromagnetic form factors, but to processes with two photons. A similar term $\vec{A}^2$ occurs in Schrödinger theory, and can be shown in that case to be the low-energy limit of virtual transitions to the negative-energy states of Dirac theory.[17] Now the negative states of the Dirac hydrogen atom are analogous to the spacelike solutions to infinite-component wave equations.[14] In this case also, the appearance of the interaction term $-(\alpha_2+\alpha_3 S)\vec{A}^2$ is intimately related to the appearance of spacelike solutions to Eq. (1). Indeed for $\alpha_2=\alpha_3=0$ the spacelike states disappear and the mass spectrum becomes [from (3)]

$$M^2 = \alpha_1^{-2}(\beta^2 + \gamma^2/n^2). \tag{6}$$

The introduction of the $\alpha_2$ and $\alpha_3$ terms changes the mass spectrum to a realistic one, increasing with $n$ and brings in spacelike states at the same time.

## III. ELECTROMAGNETIC POLARIZABILITIES

As in the case of H atom the calculation of the polarizabilities simplifies considerably if parabolic coordinates are used.[18] In general, the infinite sum over the discrete and continuous states can be avoided by solving a differential equation.[19] In a similar fashion we have developed a new perturbation theory of infinite-component wave equations which allows us the calculation of polarizabilities in closed form, already summed over all intermediate states.[20]

Up to second order, the perturbation theory on the new equation

$$[J^\mu P'_\mu + \beta S + \gamma + I]\psi'(p') = 0 \tag{7}$$

gives

$$\Omega(n, M') + \lambda\langle\psi|I_{II}|\psi\rangle - \lambda^2\langle\psi|I_I\Omega^{-1}I_I|\psi\rangle = 0, \tag{8}$$

where $\psi$ is the unperturbed wave function, e.g., the ground-state wave function.

The solution of Eq. (8) gives the perturbed mass spectrum $M'_n$. If for constant external electric and magnetic fields, we write this in the forms, respectively,

$$\begin{aligned} M' &= M - \tfrac{1}{2}\alpha E^2, \\ M' &= M - \tfrac{1}{2}\beta B^2, \end{aligned} \tag{9}$$

we can read off immediately the polarizabilities $\alpha$ and $\beta$.

### A. Electric Polarizabilities

For a uniform electric field

$$A^\mu = (zE, 0, 0, 0). \tag{10}$$

The coordinate $z$ in the space of group states must be represented by the operator $(1/M')M_3$, where $M_3 = L_{35}$ is the third component of the boost operator (Appendix B). This is the essence of the Epstein-Waller method.

Thus the first interaction term becomes

$$\begin{aligned} eJ_\mu A^\mu = \frac{eE}{M}[(\alpha_1\Gamma_0 + 2\alpha_2 M' + 2\alpha_3 SM')M_3 \\ + \alpha_4 L_{35}M' - \alpha_5 M' L_{12} C], \end{aligned} \tag{11}$$

and contributes in second order to the polarizabilities. The second interaction term Eq. (5) is zero for a pure electric field.

The evaluation of (8) and (9) is shown in Appendix B. The result for the ground state of a tower with arbitrary ground-state spin $|\mu|$, is

$$\begin{aligned} \alpha = \frac{e^2}{2M^3}\frac{\cosh^4\theta}{(1+\mu)\eta}[\tfrac{1}{2}(\mu+1)\delta^2 + 2\mu^2 b\tanh\theta\,\delta \\ + \tfrac{1}{2}b^2(2\mu^2+3\mu+2) \\ + 2\mu^2(\mu+1)b^2\tanh^2\theta], \end{aligned} \tag{12}$$

where

$|\mu|$ = spin of ground state of tower,

$\theta$ = "tilting angle" $= \tanh^{-1}[-(\alpha_3 M^2 + \beta)/\alpha_1 M]$,

$a = \alpha_1 + 2\alpha_3 M\tanh\theta$,

$b = \alpha_1\tanh\theta + 2\alpha_3 M$,

$\delta \equiv (3+2\mu)a + b\tanh\theta + \dfrac{4\alpha_2 M}{\cosh\theta} + \dfrac{2\alpha_4 M}{\cosh\theta}$,

$$\eta \equiv [(\alpha_1^2 - 2\beta\alpha_3 - 2\gamma\alpha_2/n^2)^2 - 4(\beta^2 + \gamma^2/n^2)(\alpha_3{}^2 + \alpha_2{}^2/n^2)]^{1/2}.$$

### B. Magnetic Polarizabilities

For a uniform magnetic field

$$A^\mu = (0, \vec{A}), \quad \vec{A} = \tfrac{1}{2}\vec{B}\times\vec{r} = \tfrac{1}{2}\vec{B}(-y, x, 0). \tag{13}$$

Again the coordinates $x$ and $y$ are represented in the group space by the operators $(1/M')M_1$ and $(1/M')M_2$. Thus the two interaction terms are

$$eJ_\mu A^\mu = \frac{eB}{2M'}[\alpha_1(\Gamma_1 M_2 - \Gamma_2 M_1) - 2M'\alpha_4 L_{12} + \alpha_5 2M' C M_3], \tag{14}$$

and

$$-e^2(\alpha_2 + \alpha_3 S)\vec{A}^2 = -\tfrac{1}{4}e^2 B^2(M_1{}^2 + M_2{}^2). \tag{15}$$

The evaluation of (8) and (9) yields for $I_{\text{I}}$ and $I_{\text{II}}$ separately

$$\begin{aligned}\beta_{\text{I}} = {} & \frac{e^2}{M^3}\frac{1}{(1+\mu)\eta}\left[\tfrac{1}{2}(\mu\alpha_1\sinh\theta)^2 - M\alpha_5\cosh\theta\mu\alpha_1\sinh\theta + (\mu+1)M^2\alpha_5{}^2\cosh^2\theta\right] \\ & + \frac{e^2}{M^3}(\tfrac{1}{2}\alpha_1\cosh\theta + \alpha_4 M + \alpha_5 M\sinh\theta)^2\frac{\mu^2}{N^4}\left(1 + \frac{\alpha_1{}^2(\alpha_1{}^2 - 4\beta\alpha_3)(1+\mu)^2M^2}{(\alpha_1 M^2 + \gamma)^2\eta}\right) \\ & + \frac{2e^2}{M^3}\alpha_1{}^{-1}N^{-4}\left(\frac{\alpha_1}{2M}\tanh\theta + \alpha_5\right)\cosh^3\theta(\alpha_3 M^2 - \beta)\mu^2(\tfrac{1}{2}\alpha_1\cosh\theta + \alpha_4 M + \alpha_5 M\sinh\theta) \\ & + \frac{2e^2}{M^3}N^{-4}\alpha_1\cosh\theta\mu^2(\tfrac{1}{2}\alpha_1\cosh\theta + \alpha_4 M + \alpha_5 M\sinh\theta),\end{aligned} \tag{16}$$

with the same notation as in (12), with the addition

$$N^2 = \alpha_1 n\cosh\theta + 2\alpha_2 M + 2n\alpha_3 M\sinh\theta, \tag{17}$$

$n$ is the principal quantum number, and

$$\begin{aligned}\beta_{\text{II}} = {} & -\frac{e^2}{M^3}\frac{M}{N^2}\{\alpha_2\cosh^2\theta\,\tfrac{1}{2}(\mu+1) + \alpha_2\sinh^2\theta\,\mu^2 + \alpha_3\sinh\theta\cosh^2\theta\,\tfrac{1}{2}(\mu+1)^2 + \alpha_3\sinh^3\theta\,\mu^2(1+\mu) \\ & + \alpha_3\sinh\theta\cosh^2\theta[\mu^2 + \tfrac{1}{2}(\mu+1)]\}.\end{aligned} \tag{18}$$

### C. Examples

#### 1. $\mu = 0$.

*(a) Relativistic H atom.*

$$\alpha_1 = 1, \quad \alpha_2 = -\frac{\alpha}{2m_2}, \quad \alpha_3 = \frac{1}{2m_2},$$

$$\alpha_4 = \alpha_5 = 0, \quad \beta = \frac{m_2{}^2 - m_1{}^2}{2m_2}, \quad \gamma = \alpha\frac{m_2{}^2 + m_1{}^2}{2m_2},$$

we obtain from (12)–(13), (17), and (18)

$$\begin{aligned}&\alpha \cong \tfrac{9}{2}a^3(1 + \tfrac{25}{18}e^4), \\ &\beta_{\text{I}} = 0, \\ &\beta_{\text{II}} = -\tfrac{1}{2}e^4 a^3,\end{aligned} \tag{19}$$

where $a = 1/me^2$ is the Bohr radius.

This result differs for $\alpha$ in second order from calculations using the Dirac equation.[21] This is understandable, because the parameters in (19) do not take electron spin into account, but only the recoil.

*(b) Pions.* We chose for the simplest possible case[22]

$$\alpha_1 = -1, \quad \alpha_2 = 7, \quad \alpha_3 \cong 0, \quad \alpha_4 = \alpha_5 = \beta = \gamma = 0.$$

This gives a linearly rising spectrum with $m_\pi = 144$ MeV. The polarizabilities are

$$\begin{aligned}&\alpha = \tfrac{1}{4}e^2/m^3, \\ &\beta_{\text{I}} = 0, \\ &\beta_{\text{II}} = -\tfrac{1}{2}e^2/m^3.\end{aligned} \tag{20}$$

These results are in agreement with the calculations in chiral quantum field theory.[23]

#### 2. $\mu = \frac{1}{2}$.

*(a) Proton.* Using Kleinert's parameters for the proton[4]

$$\begin{aligned}&\alpha_1 = -0.909, \quad \alpha_2 = 1.66, \quad \alpha_3 = 0.166, \\ &\alpha_4 = 0.618, \quad \alpha_5 = -1.265, \quad \beta = 0.618, \quad \gamma = -0.891,\end{aligned} \tag{21}$$

we use Eqs. (12), (16), and (18) with $\mu = \frac{1}{2}$ to obtain, in units of $10^{-4}\,\text{fm}^3$,

$$\alpha = 16, \quad \beta = 30. \tag{22}$$

However, the mass spectrum calculated with Eq. 3 gives a first excited state of 1.3 GeV, which is lower than the Roper resonance at 1.47 GeV.

We have used the same input data, but required the first excited state to remain close by 1.47 GeV. There remains some freedom in the value of the saturation mass, which we retain as a free parameter. The most important input is the coefficient of $t$ in the dipole fit to the magnetic form factor. Assuming an experimental uncertainty in this parameter of 10% we can give an uncertainty in our calculated values.

The results are

$$\alpha = 7.7 \pm 3, \quad \beta = 4.5 \pm 2. \tag{23}$$

To get these results we have had to lower the saturation mass to 2.2 GeV. For a saturation mass of around 3 GeV the results are

$$\alpha \cong 12, \quad \beta \sim 12. \tag{24}$$

*(b) Neutron.* If we write the conserved electromagnetic current as

$$J_\mu^{\text{em}} = Q(\alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S) + i\alpha_4 L_{\mu\nu} q^\nu + i\alpha_5 C L_{\mu\nu}^* q^\nu,$$

then for the neutron $\langle n | J_\mu^{\text{em}} | n \rangle = Q = 0$, and we have

$$J_\mu^{\text{em}} = i\alpha_4^{\text{neutron}} L_{\mu\nu} q^\nu + i\alpha_5^{\text{neutron}} L_{\mu\nu}^* q^\nu \tag{25}$$

[see Eq. (2) for comparison]. The anomalous magnetic moment is given by

$$\mu^{(n)} = -\alpha_4^{(n)} M^{(n)} - \alpha_5^{(n)} M^{(n)} \sinh\theta.$$

Measurement of the slope of the neutron form factor at the origin gives[4]

$$\left.\frac{dG_E^n(t)}{dt}\right|_{t=0} = -0.564$$
$$= \frac{-\mu^{(n)}}{4M_{(n)}^2}(3 + 4\sinh^2\theta) - \frac{1}{M_{(n)}} \sinh\theta \cosh^2\theta \, \alpha_5^{(n)}. \tag{26}$$

So $\alpha_4^{(n)}$ and $\alpha_5^{(n)}$ are determined uniquely by these two measurements. We assume the same values of $\theta$ as for the proton, and this gives good fits to the neutron form factors. The results for the polarizabilities are

$$\alpha \cong 0, \quad \beta = 14 \pm 1, \tag{27}$$

where the units are $10^{-4}\,\text{fm}^3$, and the indicated uncertainty is the sensitivity of the calculation to a 10% experimental uncertainty of the $\theta$ parameter.

### D. Comparison with Experiment

Our results for the proton give $\beta < \alpha$ mainly because of the negative contribution of $I_{\text{II}}$. This diamagnetic contribution has until recently[24] been overlooked. Even with this negative contribution it is difficult to make $\beta$ so small as the recently measured value.[25] Our prediction for the magnetic polarizability lies somewhere in between the experimental value and the dispersion-theory result.[26,24] On the other hand the electric polarizability seems to agree very well with the experimental result.

We give a prediction for the neutron that $\alpha \cong 0$ and $\beta \cong 14$. This may be measurable.[27]


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## APPENDIX A: POSITION OPERATOR IN GROUP SPACE

We would like to determine the position operator in group space, in order to compute the matrix elements of the perturbation. Assume we have $x_{\text{op}} = -i\partial/\partial p$ in an infinite-component wave equation in momentum space:

$$[W(p) + f(\vec{x}_{\text{op}})]\psi(p) = 0. \tag{A1}$$

Here $\psi(p)$ is the infinite-component wave function and $W(p)$ is a Lorentz-invariant operator. Now

$$\psi(p) = e^{-i\vec{\xi}_p \cdot \vec{M}}\psi(0), \tag{A2}$$

where $e^{-i\vec{\xi}_p\cdot\vec{M}}$ is the "boost" operator. Operating on the left with $e^{i\vec{\xi}_p\cdot\vec{M}}$ and using the Lorentz-invariant properties of $W$, we obtain

$$[W(0) + f(e^{i\vec{\xi}\cdot\vec{M}}\vec{x}e^{-i\vec{\xi}\cdot\vec{M}})]\psi(0) = 0. \tag{A3}$$

It has been shown[28] that

$$\lim_{p\to 0} e^{i\vec{\xi}_p\cdot\vec{M}}\vec{x}_{\text{op}}e^{-i\vec{\xi}_p\cdot\vec{M}} = \vec{x}_{\text{op}} + \frac{\vec{M}}{M'} + \cdots, \tag{A4}$$

where $\vec{M}$ is the Lorentz "boost" operator and $M'$ the mass of the wave function. In our calculation we are interested in the low-energy limit, so the wave equation becomes

$$[W(0) + f(\vec{x}_{\text{op}} + \vec{M}/M')]\psi(0) = 0. \tag{A5}$$

$\vec{x}_{\text{op}}$ is diagonal in group space and can be thought of as an average position operator. It gives no contribution to the polarizability. Thus it is sufficient to replace $\vec{x}$ in the perturbation function by the group space operator $\vec{M}/M'$.

An analogous effect occurs in the Dirac equation when we make a Foldy-Wouthuysen transformation to a basis in which the energy operator is diagonal. In that case the position operator becomes nondiagonal, e.g., nonlocal.

## APPENDIX B: CALCULATION OF ELECTRIC POLARIZABILITY $\alpha$

The wave equation is given entirely in terms of Lie-group elements of the dynamical group O(4,2). The wave function is then a member of a specific representation of O(4,2). It is given by

$$\psi(p) = e^{-i\vec{\xi}_p\cdot\vec{M}} e^{i\theta T} |\mu n_1 n_2 m\rangle\,, \tag{B1}$$

where $\vec{\xi}_p$ is the boost parameter, given by

$\hat{\xi}_p \sinh\xi = \vec{q}/M$,

$M$ = mass of the state,

$\vec{M}$ is the Lorentz boost operator $= L_{i5}$,

$T$ is the tilt operator $= L_{45}$,

$\theta$ is the tilting angle, depending on $m'$ and other parameters of the wave equation,

$\mu$ characterizes the representation and is the lowest spin in the tower of states,

$n_1, n_2$ are the parabolic quantum numbers, familiar from the hydrogen atom,

$m$ is the $z$ component of spin.

For simplicity we will calculate the polarizability only for the lowest member of each tower. We choose the positive-parity state, to correspond to the baryons. A useful representation for the spinor $|\mu n_1 n_2 m\rangle$ is given in terms of creation and destruction operators as

$$\begin{aligned} |\mu n_1 n_2 m\rangle &= C_{n_1 n_2 m} (a_1^\dagger)^{n_2+(|m+\mu|+m+\mu)/2} (a_2^\dagger)^{n_1+(|m-\mu|-m+\mu)/2} (b_1^\dagger)^{n_1+(|m-\mu|+m-\mu)/2} (b_2^\dagger)^{n_2+(|m+\mu|-m-\mu)/2} |0\rangle\,,\\ C_{n_1 n_2 m}{}^{-2} &= \left(n_2+\frac{|m+\mu|+m+\mu}{2}\right)!\left(n_1+\frac{|m-\mu|-m+\mu}{2}\right)!\left(n_1+\frac{|m-\mu|+m-\mu}{2}\right)!\left(n_2+\frac{|m+\mu|-m-\mu}{2}\right)!\,. \end{aligned} \tag{B2}$$

The positive-parity ground state of a tower is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|\mu 00\mu\rangle + |-\mu 00\mu\rangle\}. \tag{B3}$$

The generators of O(4,2) can be easily expressed in terms of creation and destruction operators acting on the space of representation specified by (B2).

It must be emphasized that the choice of the above representation is only for calculational convenience, it does not mean that the particle is composed of four quarks. Once O(4,2) is specified as the dynamical symmetry, the details of the composition of the particle become irrelevant to an observer outside the particle.

The perturbation theory for infinite-component wave equations is covered elsewhere. We begin with a perturbed wave equation

$$[\Omega(p') + H]\psi'(p') = 0. \tag{B4}$$

Because of the interaction the particle has a new rest mass $M'$. We have shown that

$$M' = M - \frac{\gamma_1}{N^2} - \frac{\gamma_2+\gamma_3}{N^2} - \frac{\gamma_1^{\,2}}{2N^4}\left[1+\frac{\alpha_1^{\,2}(\alpha_1^{\,2}-4\beta\alpha_3)n^2M^2}{(\alpha_2 M^2+\gamma)^2\eta}\right], \tag{B5}$$

with

$$\begin{aligned} N^2 &\equiv \langle\psi|J_0|\psi\rangle\\ &= (N\alpha_1\cosh\theta_n + 2\alpha_2 M_n + 2n\alpha_3 M_n \sinh\theta_n) = 1\,, \quad \text{for the ground state,} \end{aligned}$$

$$\eta \equiv [(\alpha_1^{\,2} - 2\beta\alpha_3 - 2\gamma\alpha_2/n^2)^2 - 4(\beta^2+\gamma^2/n^2)(\alpha_3^{\,2}+\alpha_2^{\,2}/n^2]^{1/2},$$

$\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are parameters,

$n$ = principal quantum number $= n_1 + n_2 + m + 1$.

Here

$$\begin{aligned} \gamma_1 &= \langle\psi|e^{-i\theta T} H e^{i\theta T}|\psi\rangle\,,\\ \gamma_3 &= -\langle\psi|e^{-i\theta T} H e^{i\theta T}\Omega^{-1}(0) P e^{-i\theta T} H e^{i\theta T}|\psi\rangle. \end{aligned} \tag{B6}$$

$P$ projects out $\psi$. Here $\gamma_1$ is first order in $H$, $\gamma_3$ second order. Actually $\theta$ and $H$ are functions of $M'$, but are evaluated at $M$. For $\gamma_3$ this has no effect because the difference is of third order. However, when $\gamma_1$ is evaluated at $M$ there is an error of of second order in the perturbation, and it has to be considered; $\gamma_2$ is just this contribution, and is easily evaluated once one knows $\gamma_1(M)$.

For an external static electric field in the $z$ direction, the electromagnetic potential can be represented by

$$A^\mu = E(z000). \tag{B7}$$

Thus from Eqs. (4) and (5), $H$ is given by

$$\begin{aligned} H &= eJ_\mu^{\text{em}} A^\mu - (\alpha_2 + \alpha_3 S)\vec{A}^2\\ &= eJ_0^{\text{em}} A^0\\ &= eE[J_0 z + (\alpha_4, \alpha_5)\text{ terms}]. \end{aligned} \tag{B8}$$

The $(\alpha_4, \alpha_5)$ terms are found by using the fact that

$$i\alpha_4 L_{\mu\nu} q^\nu A^\mu + i\alpha_5 C L^*_{\mu\nu} q^\nu A^\mu = \tfrac{1}{2}\alpha_4 L_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\alpha_5 C L^*_{\mu\nu} F^{\mu\nu}, \quad \text{(B9)}$$

using the antisymmetry of $L_{\mu\nu}$, $L^*_{\mu\nu}$, and the fact that $\partial_\mu$ in momentum space is just $-iq_\mu$.

Now for a pure $E$ field in the $z$ direction,

$$F^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \end{bmatrix}. \quad \text{(B10)}$$

It is easy to show that $L^*_{\mu\nu}F^{\mu\nu} = -L_{\mu\nu}F^{*\mu\nu}$, where the asterisk represents the dual

$$L^*_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}L^{\sigma\rho}.$$

Using

$$F^{*\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(B11)}$$

and $J_0 = \alpha_1\Gamma_0 + \alpha_2 P_0 + \alpha_3 SP_0$, we find

$$H = \frac{eE}{M'}\{[\alpha_1\Gamma_0 + 2\alpha_2 M' + 2\alpha_3 SM']M_3 + \alpha_4 L_{30}M' - \alpha_5 M' L_{12} C\}. \quad \text{(B12)}$$

But since $L_{30} \equiv L_{35} = M_3$,

$$H = \frac{eE}{M'}\{[\alpha_1\Gamma_0 + 2\alpha_2 M' + 2\alpha_3 SM' + \alpha_4 M']M_3 - \alpha_5 M' L_{12} C\}. \quad \text{(B13)}$$

Here $M'$ is actually the interacting mass which we are trying to calculate. The dependence on $M'$ is used to determine $\gamma_2$, but for $\gamma_1$ and $\gamma_3$ we simply set $M' = M$.

Performing the tilt operation in (B6) we find that

$$H' \equiv e^{-i\theta T} H e^{i\theta T} = \frac{eE}{M'}\tfrac{1}{2}\cosh^2\theta'\Big( a'\Gamma_0 M_3 + a' M_3 \Gamma_0 + \frac{4\alpha_2 M'}{\cosh\theta'} M_3 + \frac{4\alpha_2 M' \tanh\theta'}{\cosh\theta'} A_3 + a'\Gamma_0 A_3 \tanh\theta'$$
$$+ a' A_3 \Gamma_0 \tanh\theta' + b' S M_3 + b' M_3 S + b' S A_3 \tanh\theta' + b' A_3 S \tanh\theta'\Big)$$
$$+ \frac{eE}{M'}(\alpha_4 M' \cosh\theta' M_3 + \alpha_4 M' \sinh\theta A_3 - \alpha_5 M' C L_{12}), \quad \text{(B14)}$$

where

$$a' = \alpha_1 + 2\alpha_3 M' \tanh\theta',$$
$$b' = \alpha_1 \tanh\theta' + 2\alpha_3 M'.$$

Dependence on $M'$ is indicated by a prime.

In (B14) the dependence on $M'$ is kept so that $\gamma_2$ can be evaluated, for $\gamma_1$ and $\gamma_3$ again we replace $M' \to M$.

The operators we need are given by $(\mu > 0)$

$$\begin{aligned}
M_3|\mu 00\mu\rangle &= -\tfrac{1}{2}\{|\mu 01\mu\rangle(2\mu+1)^{1/2} + |\mu 10\mu\rangle\},\\
L_{12}|\mu 00\mu\rangle &= \mu|\mu 00\mu\rangle,\\
A_3|\mu 00\mu\rangle &= -\mu|\mu 00\mu\rangle,\\
S|\mu 00\mu\rangle &= \tfrac{1}{2}|\mu 01\mu\rangle(2\mu+1)^{1/2} - |\mu 10\mu\rangle,\\
SA_3|\mu 00\mu\rangle &= -\tfrac{1}{2}\mu[|\mu 01\mu\rangle(2\mu+1)^{1/2} - |\mu 10\mu\rangle],\\
SM_3|\mu 00\mu\rangle &= -\tfrac{1}{4}\{|\mu 02\mu\rangle[2(2\mu+2)(2\mu+1)]^{1/2} - 2|\mu 20\mu\rangle + 2\mu|\mu 00\mu\rangle\},
\end{aligned} \quad \text{(B15)}$$

plus others for $|-\mu 00\mu\rangle$. The operators $CL_{12}$, $SM_3$, and $A_3$ give a nonzero result for the matrix element

$$\langle \mu 00\mu | O^p | \mu 00\mu \rangle.$$

but when evaluated between parity eigenstates they all vanish because they change the parity. Thus $\gamma_1 = 0$, which is to be expected, because a state with definite parity must have zero electric dipole moment.

If $\gamma_1 = 0$, then also $\gamma_2 = 0$, because $\gamma_2$ arises solely from the effect of replacing $M'$ by $M$ in $\gamma_1$.

Using (B15), $\gamma_3$ can be evaluated. We insert into (B6) a complete set $\sum_n |n\rangle\langle n| = 1$ (this is complete only for group space wave functions). Then

$$\exp(i\theta T)\Omega^{-1}(0)\exp(-i\theta T)|n\rangle = \Omega_n{}^{-1}|n\rangle,$$

with $\Omega_n{}^{-1}$ just a $c$ number. The reason for this is that $\theta$ was chosen specifically to diagonalize $\Omega$. If we did not use the present method of perturbation theory, we would not get such a $c$ number and, in general, infinitely many states would contribute to the sum over intermediate states.

Remembering that $P$ projects out of the ground state, we find that only states with $n=n_0+1$ and $n=n_0+2$ contribute to the sum. The result for $\gamma_3$ is, after considerable algebra,

$$\gamma_3=\frac{e^2E^2}{M^2}(\tfrac{1}{2}\cosh^2\theta)^2\frac{\operatorname{sgn}(\alpha_2M^2+\gamma)}{[\alpha_1{}^2M^2+(\alpha_3M^2+\beta)^2]^{1/2}}$$
$$\times\left\{\tfrac{1}{2}(\mu+1)\left[a(3+2\mu)+6\tanh\theta+\frac{4\alpha_2M}{\cosh\theta}+\frac{2\alpha_4M}{\cosh\theta}\right]^2+2\mu^2b\left[a(3+2\mu)+b\tanh\theta+\frac{4\alpha_2M}{\cosh\theta}+\frac{2\alpha_4M}{\cosh\theta}\right]\right.$$
$$\left.+\tfrac{1}{2}b^2[(\mu+1)(2\mu+1)+1]+2(\mu+1)\mu^2b^2\tanh^2\theta\right\}.$$

Putting the result into (B5) and comparing with the formula

$$M'=M-\tfrac{1}{2}\alpha E^2$$

we easily obtain Eq. (12).

## APPENDIX C: CALCULATION OF MAGNETIC POLARIZABILITY

For a uniform $B$ field in the $z$ direction we use

$$A^\mu=(0,\vec{A}),\quad \vec{A}=\tfrac{1}{2}B(-y,x,0). \qquad \text{(C1)}$$

Then

$$H=eJ_\mu A^\mu-(\alpha_2+\alpha_3S)\vec{A}^2$$
$$=\frac{eB\alpha_1}{2M'}(\Gamma_1M_2-M_1)-eB\alpha_4L_{12}$$
$$+eB\alpha_5CM_3-\frac{1}{4}\frac{e^2B^2}{M'^2}(M_1{}^2+M_2{}^2), \qquad \text{(C2)}$$

where we have replaced $\vec{x}$ by $\vec{M}$ and used Eqs. (2) and (B9), and also the analogs to (B10) and (B11),

$$F^{\mu\nu}=\begin{bmatrix}0&0&0&0\\0&0&-B&0\\0&B&0&0\\0&0&0&0\end{bmatrix},\qquad F^{*\mu\nu}=\begin{bmatrix}0&0&0&B\\0&0&0&0\\0&0&0&0\\-B&0&0&0\end{bmatrix}. \qquad \text{(C3)}$$

By rotation invariance, $M_1{}^2+M_2{}^2=2M_3{}^2$ for the ground state. Taking the tilt, we find

$$H'\equiv e^{-i\theta T}He^{i\theta T}=\frac{eB}{2M'}\cosh\theta'\,\alpha_1(\Gamma_1M_2+\Gamma_1A_2\tanh\theta-\Gamma_2M_1-\Gamma_2A_1\tanh\theta)$$
$$-e\alpha_4L_{12}B+e\alpha_5BC(\cosh\theta M_3+\sinh\theta A_3)+\text{seagull}. \qquad \text{(C4)}$$

The seagull interaction term is

$$-\frac{1}{2}\frac{e^2B^2}{M'^2}(\alpha_2+\alpha_3S\cosh\theta+\alpha_3\Gamma_0\sinh\theta)(M_3\cosh\theta+A_3\sinh\theta)^2.$$

Proceeding as before, we find that all these operators have diagonal components, and if we ignore the seagull term for the present we get

$$\gamma_1=-(e\mu B/M')(\tfrac{1}{2}\alpha_1\cosh\theta'+\alpha_4M'+\alpha_5M'\sinh\theta'). \qquad \text{(C5)}$$

From (B5), we see that this is just the interaction of a magnetic dipole with the field.

Since $\gamma_1\neq 0$ in this case $\gamma_2$ will also contribute. We simply expand (C5) in a Taylor series about $M$, and use $M'\cong M-\gamma_1M$ to first order in $B$. It is found after much algebra that

$$\gamma_2=\frac{e^2\mu^2B^2}{M^2}\cosh^3\theta(\tfrac{1}{2}\alpha_1\tanh\theta+M\alpha_5)(\tfrac{1}{2}\alpha_1\cosh\theta+M\alpha_4+\alpha_5M\sinh\theta)\left(\frac{\alpha_3M^2-B}{\alpha_1M^2}\right). \qquad \text{(C6)}$$

The calculation of $\gamma_3$ is more algebra and yields

$$\gamma_3=\frac{e^2B^2\,\mathrm{sgn}(\alpha_2M^2+\gamma)}{[\alpha_1{}^2M^2-(\alpha_3M^2+\beta)^2]^{1/2}}\left[\left(\frac{\mu\alpha_1\sinh\theta}{2M}\right)^2-\alpha_5\cosh\theta\left(\frac{\mu\alpha_1\sinh\theta}{2M}\right)+2(\mu+1)\left(\frac{\alpha_5-\cosh\theta}{2}\right)^2\right]. \tag{C7}$$

Now considering the seagull term, it is already of second order, and will give a contribution to $\gamma_1$ but only a third-order contribution to $\gamma_2$. We find

$$\gamma_1(\text{seagull})=-\frac{\alpha e^2B^2}{M^2}[\alpha_2\cosh^2\theta\,\tfrac{1}{2}(\mu+1)+\alpha_2\sinh^2\theta\,\mu^2+(\mu^2+\tfrac{1}{2}\mu+\tfrac{1}{2})\alpha_3\sinh\theta\cosh^2\theta$$
$$+\frac{\alpha_3}{2}(\mu+1)^2\sinh\theta\cosh\theta+\alpha_3\mu^2(1+\mu)\sinh^3\theta]. \tag{C8}$$

Putting (C5)–(C8) into (B5), and comparing with the relation

$$M'=M-(e\mu K/M)B-\tfrac{1}{2}\beta B^2, \tag{C9}$$

we find

$$K=\text{magnetic moment}$$
$$=-(\tfrac{1}{2}\alpha_1\cosh\theta+\alpha_4+\alpha_5\sinh\theta), \tag{C10}$$

and Eqs. (16) and (18).

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