Signal-front gravidynamics of vector fields in the ray gauge

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The dynamical evolution of a gravitating vector field (massive or massless) is analyzed along a signal-front coordinate (or null coordinate). Moreover, we specialize our frame of reference by adding the ray-gauge condition, which essentially consists of taking the affine parameter of the null geodesics contained in the signal-front hypersurfaces as the second kind of nonspatial coordinate. Then we extend previous results of Aragone and Chela-Flores to this matter-source case. We exhibit the algebraic and differential constraints for the present system, and the reduction process is accomplished. It is shown how the gravidynamics of the vector fields can be studied in terms of the minimum number of independent physical fields, half the usual timelike canonical first-order approach. It is found that the signal-front energy density is always non-negative. Moreover, the connection between vanishing signal energy fields and the Robinson-Trautman and Kundt plane radiative solutions is also shown. Finally, the limit of the matter signal energy, when the mass tends to zero, is shown to be (even in an arbitrary nonflat background) the sum of the related Maxwell signal energy plus the signal energy of a scalar massless field.

I. INTRODUCTION

In two previous papers[1,2] the dynamics of linearized gravitation, free Maxwell fields, and matterless general relativity were analyzed by taking as the evolving coordinate u a signal-front (or null) family of hypersurfaces, i.e., $g^{uu}=0$. Besides many interesting aspects of this inequivalent picture of relativistic dynamics[3] (which incidentally is the singular case of the Arnowitt, Deser, and Misner[4] canonical formulation), we are going to look in the present article at an illustrative interacting system, the gravitating vector field. This system is a good example to show how far the signal-front approach could be considered as a more transparent frame for relativistic interactions. We are going to see that the reduction process is simpler than in the timelike formulation. The differential constraints are going to be either two-dimensional differential operators on a two-dimensional Riemannian variety or ordinary differential equations in the associated quasinull variable v.

Furthermore, we give a reduced action showing explicitly the non-negative character of the signal-front energy density $\mathcal{G}^u$, the generator of the u evolution of the system. This quantity turns out to be a functional of the minimum number of independent physical variables needed to represent each field: three for the neutral massive vector field plus two for the helicity-two massless Einstein field.

Also in connection with the properties of $\mathcal{G}^u$, we discuss in Sec. V vacuum gravitational fields of vanishing signal-front energy. We show how one is led in a natural way, in the frame of the present dynamical approach, to the Robinson-Trautman and Kundt shear-free radiative solutions.

In the next section we are going to briefly describe the tensor-vector interacting system in a covariant way. We also define the field variables intrinsically associated to a $2+2$ point of view, what can be called the signal-front kinematics. The third section shall be devoted to the first stage of the reduction process. The fourth shall be dedicated to the last stage of the reduction process which leads to the signal-energy density $\mathcal{G}^u$ in terms of the physical variables. Finally, in the last section, the results obtained are discussed.[5]

II. THE SELF-INTERACTING TENSOR–LINEAR-VECTOR SYSTEM

A gravitating linear vector field can be described by the first-order action

$$A=\kappa_1^{-2}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)(-g)^{1/2}d^4x+\kappa_2^{-2}\int d^4x[A_{\mu\nu}F^{\mu\nu}+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}(-g)^{-1/2}-\tfrac{1}{2}m^2A_\mu A^\mu(-g)^{1/2}]\equiv A^G+A^M,\quad(1)$$

which is the sum of the Palatini Lagrangian A^G for the self-interacting Einstein tensor field,[6] plus the first-order vector action A^M. Here the symmetric quantities $g_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ represent the tensor field, A_μ and $F^{\mu\nu}$ the vector field, and κ_1 and κ_2 are their characteristic strengths, while m is the mass of the spin-one particle. Regarding their transformation laws, $\{g_{\mu\nu}\}$ is a tensor, $\Gamma^\alpha_{\mu\nu}$ an affinity, A_μ a vector, and $F^{\mu\nu}$ an antisymmetric density.

Independent variations of $g_{\mu\nu}$, Γ, A, and F yield the covariant field equations governing the evolution of this interacting system:

$$F_{\mu\nu}=(-g)^{1/2}(\partial_\mu A_\nu-\partial_\nu A_\mu), \tag{2a}$$

$$\partial_\nu F^{\mu\nu}+m^2(-g)^{1/2}A^\mu=0, \tag{2b}$$

$$2g_{\alpha\beta}\Gamma^\beta_{\mu\nu}=\partial_\mu g_{\nu\alpha}+\partial_\nu g_{\mu\alpha}-\partial_\alpha g_{\mu\nu}, \tag{2c}$$

$$\begin{aligned}\kappa^{-2}G_{\mu\nu}\equiv\kappa^{-2}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\\ =\tfrac{1}{2}(-g)^{-1}F_{\mu\alpha}F_\nu{}^\alpha-\tfrac{1}{8}(-g)^{-1}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\\ +m^2(A_\mu A_\nu-\tfrac{1}{2}A_\alpha A^\alpha g_{\mu\nu})\equiv T_{\mu\nu}.\end{aligned} \tag{2d}$$

Equations (2) imply the divergenceless character of the energy-momentum tensor $T_{\mu\nu}$ as well as the antisymmetry of $F_{\mu\nu}$, which also constrains A_μ to be either divergenceless or massless:

$$m^2\partial_\mu(-g)^{1/2}A^\mu=0. \tag{3}$$

In both cases Eq. (3) ensures the pure spin-one content of the vector field. Now we are going to study the kinematics of this system when, at least, one of the four coordinates u is null (or signal-like). That means that the hypersurfaces $u=\text{const}$ are characteristic hypersurfaces, i.e., $g^{uu}=0$. Besides this choice, we shall work in the ray gauge,[7] which physically means that the second "timelike" coordinate v is going to denote an affine parameter along the congruence of rays $v=\text{variable}$ contained in $\Sigma_u\equiv\{u=\text{const}\}$. This condition restrains the family of frames of reference to those satisfying $g_{iv}=0$. Moreover, an additional gauge condition shall be used: $g_{uv}=-1$, already mentioned in Ref. 2, where its nondynamical character was shown.[8]

After this choice was made, the four-dimensional contravariant and covariant components of the metric tensor can be given in terms of the six independent field variables g_{ij}, N^l, and n. They look like (where a presubscript indicates the dimensions of the space where the quantity is meaningful)[9]

$$\begin{aligned}{}_4g_{ij}\equiv g_{ij},\quad {}_4g_{ui}\equiv g_{ij}N^j\equiv N_i,\quad {}_4g_{vi}=0,\\ {}_4g_{uu}\equiv 2n+N^iN_i,\quad {}_4g_{uv}=-1,\quad {}_4g_{vv}=0,\end{aligned} \tag{4a}$$

$$\begin{aligned}{}_4g^{ij}={}_2g^{ij},\quad {}_4g^{ui}=0,\quad {}_4g^{iv}=N^i,\\ {}_4g^{uu}=0,\quad {}_4g^{uv}=-1,\quad {}_4g^{vv}=-2n.\end{aligned} \tag{4b}$$

The four-volume element has the value $(-{}_4g)^{1/2}d^4x={}_2g^{1/2}d^2\vec{x}\,du\,dv$.[10] It is worth remembering that the 2+2 geometrical structure is determined by the two good tangent vectors $e_i=\partial_i(\cdots)$ plus the two null vectors $e_{(+)}$ and $e_{(-)}$, "unitary" in the sense that $e_{(+)}\cdot e_{(-)}=-1$, $e_{(\pm)}\cdot e_i=0$ with respective values

$$e_{(+)}=-N^i\partial_i+\partial_u+n\partial_v,\quad e_{(-)}=\partial_v. \tag{5}$$

At this point, trying to keep the analogy with the weak-linearized case one is thinking of the Minkowski space as having the structure of a two-dimensional noncompact (NC) spacelike variety $\Sigma_{uv}\equiv\{u=\text{const},\ v=\text{const},\ x^i=\text{variable}\}$ times the two-dimensional $\{(u,v)\}$ noncompact 0-signature variety. However, we are also going to see that the two-dimensional spacelike, compact (C) variety times the noncompact (NC) $\{(u,v)\}$ 0-signature variety structure enters into the present dynamical approach, in a very natural fashion, on the same footing with the "noncompact" case. We shall see, especially in connection with the signal-like energy density $\mathcal{J}^u$, that at a certain point mathematical problems make us consider two separate alternatives, one of them leading to the two-dimensional NC × two-dimensional NC structure and the other bringing to our attention the two-dimensional C × two-dimensional NC case.

Once we have broken the four-covariance taking g_{ij}, N^l, and n as the independent variables representing the Einstein field, we have to be very careful in the independent variables, which shall be taken to represent the vector field. It turns out that $A_i, A_v=A_\mu e^\mu_{(-)}, A_u$ (even if $A_+\equiv A_\mu e^\mu_{(+)}=A_u+nA_v-N^lA_l$) and

$$B\equiv(2\,{}_2g)^{-1}\epsilon^{lm}\,{}_4F_{lm},$$

$$E\equiv{}_4F^{uv}={}_4F^{\mu\nu}e_{(-)\mu}e_{(+)\nu},$$

$$E_{iv}\equiv{}_4F_{iv},$$

and

$$\begin{aligned}E_{i+}\equiv{}_4F_{i\mu}e^\mu_{(+)}\\ ={}_4F_{iu}+n\,{}_4F_{iv}-N^j\,{}_4F_{ij}\ \text{instead of}\ {}_4F^{\mu\nu}\end{aligned}$$

constitute a good set of independent variables in order to analyze the signal dynamics of the vector field.

In a more precise way, when the loss of the four-general covariance is mentioned in the present context, we mean that one is restricting the group of covariance to a smaller group. This is equivalent to picking out a subclass of frames of reference and looking at the physics on these frames where phenomena can be better understood.

It was shown in Ref. 2 that the group G^l which leaves invariant both Σ_u, $\Sigma_{uv}=\{u=\text{const},\ v=\text{const}\}$, and the property that the lines $v=\text{variable}$

are light rays has its elements γ of the form $x^i \to \bar{x}^i = \gamma^i(x^j, u)$, $u \to \bar{u} = \gamma^u(u)$, $v \to \bar{v} = \gamma^v(u, v)$. However, as we also assumed $g_{uv} = -1$, G^l has to be pared off, leaving us the final group G^l_m which we shall employ throughout this article,[11]

$$G^l_m = \{x^i \to \bar{x}^i = \gamma^i(x^j, u), u \to \bar{u} = a_1(u), \quad v \to \bar{v} = \dot{a}_1^{-1}(u) v + a_2(u)\}. \tag{6}$$

Now, let us see what the dynamics looks like in terms of these intrinsically defined 2+2 variables.

III. THE REDUCTION PROCESS: SETTING UP THE SIGNAL (2 + 2) DYNAMICS

In terms of the independent variables g_{ij}, N^l, n, A_i, A_v, A_u, B, E, E_{iv}, and E_{i+} the differential vector action dA^M achieves the form

$$\begin{aligned}\kappa_2^2 dA^M \equiv \{ & -E\dot{A}_v - E^i_v \dot{A}_i + B\epsilon^{ij} A_{i,j} - A_u [C_Q - m^2 g^{1/2} A_v] \\ & + (A_{j,i} - A_{i,j}) N^i E^j_v + n m^2 g^{1/2} A_v^2 - m^2 g^{1/2} A_v A_i N^i + (A_{v,i} - A'_i)(E^i_+ + nE^i_v + N^i E) \\ & - g^{-1/2} E^i_+ E_{iv} + \tfrac{1}{2} g^{1/2} B^2 - \tfrac{1}{2} g^{-1/2} E^2 - \tfrac{1}{2} m^2 g^{1/2} A_i A^i - \partial_i (A_u E^i_v) - \partial_v (A_u E)\} \, d^2\vec{x}\, du\, dv,\end{aligned} \tag{7a}$$

where C_Q means[12] $[\partial_v(\cdots) \equiv (\cdots)'$ and $\partial_u(\cdots) \equiv (\cdots)^{\cdot}]$,

$$C_Q \equiv E^i_{v,i} + \partial_v E \equiv E^i_v + E', \tag{7b}$$

and Latin indices have been raised with the two-dimensional contravariant tensor ${}_2g^{ij}$, the inverse of g_{ij}. For instance, $E^i_v = g^{il} E_{lv}$, $A^i = g^{il} A_l$, etc., in the above expressions (7). The last two terms of (7a) being either spacelike or v-exact derivatives shall be discarded from now on; we are going to discuss what happens after making independent variations of the vector variables in the action (7a).

Variations of E^i_+ and B yield two algebraic constraints

$$g^{1/2} B = \epsilon^{ij} \partial_i A_j, \quad E_{iv} = g^{1/2}(\partial_i A_v - A'_i), \tag{8}$$

which from now on shall be taken as definitions of the quantities B, E_{iv} each time they shall be written. Therefore, after introducing the values of B and E_{iv} given at (8), we are left with five independent variables A_i, A_v, A_u, and E (see Ref. 13) and the reduced vector action:

$$\begin{aligned}\kappa_2^2 dA^M_* = \{ & -E\dot{A}_v - E^i_v \dot{A}_i - A_u [C_Q - m^2 g^{1/2} A_v] + \epsilon_{ij} N^i g^{1/2} B E^j_v + n m^2 g^{1/2} A_v^2 \\ & + n g^{-1/2} E_{iv} E^i_v + g^{-1/2} E_{iv} E N^i - m^2 g^{1/2} A_v A_i N^i - \tfrac{1}{2} g^{-1/2} E^2 - \tfrac{1}{2} g^{1/2} B^2 - \tfrac{1}{2} m^2 g^{1/2} A_i A^i\} \, d^2\vec{x}\, du\, dv \\ & + \text{exact divergence } d^2\vec{x}\, du\, dv,\end{aligned} \tag{9}$$

where E_{iv} and B are given by Eqs. (8). The vector field equations coming from independent variations of the five variables are

$$\begin{aligned}-(g^{1/2} g^{ij} A'_j)^{\cdot} - (g^{1/2} g^{ij} \dot{A}_j)' + g^{1/2} \epsilon^{li} B_{,l} + (g^{1/2} A_v^{|i})^{\cdot} + (g^{1/2} A_u^{|i})' + 2(nE^i_v)' + (N^i E)' + (N^l \epsilon_l{}^i g B)' + \epsilon^{il} \epsilon_{mj} (N^m E^j_v)_{,l} \\ = m^2 g^{1/2} A^i + m^2 g^{1/2} A_v N^i,\end{aligned} \tag{10a}$$

$$(m^2 - \Delta_g) A_v = g^{-1/2} E' - D^j A'_j, \tag{10b}$$

$$\dot{E} + g^{1/2} D^i \dot{A}_i - g^{1/2} \Delta A_u + m^2 g^{1/2} A_u + 2n m^2 g^{1/2} A_v - m^2 g^{1/2} A_l N^l = \{2nE^i_v + EN^i + gB\epsilon^i_l N^l\}_{,i}, \tag{10c}$$

and

$$A'_u - \dot{A}_v + g^{-1/2} E_{iv} N^i - g^{-1/2} E = 0, \tag{10d}$$

which are easily analyzed going to the flat case and using the 2-dimensional $T+L$ decomposition for A_i $(\rho_i(\cdots) \equiv (-\Delta_2)^{-1/2} \partial_i(\cdots))$, i.e.,

$$A_i \equiv \epsilon_{il} \rho_l A^T + \rho_i A^L. \tag{11}$$

It turns out that the $3 = 2\times 1 + 1$ physical dynamical variables (corresponding to the $2j+1$ variables that a j-spin neutral massive field has) are A^T, A^L, and E while A_u and A_v can be obtained in terms of them through the constraint equations. Eliminating A_v from dA^M_* we arrive at the final form of the vector action on a flat space:

$$\kappa_2^2 dA^M_{**}(g = \eta) = \left\{ A^{T\prime} \dot{A}^T + E' \frac{1}{m^2 - \Delta} \dot{E} + \dot{A}^L \frac{m^2}{m^2 - \Delta} A^{L\prime} - \tfrac{1}{2} E^2 - \tfrac{1}{2} A^T{}_{,i}{}^2 - \tfrac{1}{2} m^2 (A^T)^2 - \tfrac{1}{2} m^2 (A^L)^2 \right\} d^2\vec{x}\, du\, dv, \tag{12a}$$

where the typical $\dot{q}\,q' - \frac{1}{2} q_{,i}{}^2$ structure of the signal-front dynamics appears transparently for the transverse mode. Moreover, introducing the new variables ϵ and a^L

$$\epsilon \equiv (m^2 - \Delta)^{-1/2} E, \quad a^L \equiv m(m^2 - \Delta)^{-1/2} A^L, \tag{12b}$$

in this action we obtain the null-canonical form

$$\kappa_2{}^2 dA^M_{**}(g=\eta) = \{A^{T\prime}\dot{A}^T + \dot{\epsilon}\,\epsilon' + a^{L\prime}\dot{a}^L - \tfrac{1}{2}\epsilon(m^2-\Delta)\epsilon - \tfrac{1}{2}A^T{}_{,i}{}^2 - \tfrac{1}{2}m^2(A^T)^2 - \tfrac{1}{2}a^L(m^2-\Delta)a^L\} d^2\vec{x}\,du\,dv. \tag{12c}$$

Here we can take the limit for $m\to 0$,[14] obtaining the addition of the Maxwell action in the null-canonical form $\kappa_2{}^2 dA^{(1)}_{**} = \{A^{T\prime}A^{T\cdot} + \epsilon'\epsilon^{\cdot} - \frac{1}{2}A^T{}_{,i}{}^2 - \frac{1}{2}\epsilon_{,i}{}^2\} d^2\vec{x}\,du\,dv$ plus the action due to the scalar massless field a^L:

$$\kappa_2{}^2 dA^{(2)}_{**} = \{a^{L\prime}a^{L\cdot} - \tfrac{1}{2}a^L{}_{,i}\,a^L{}_{,i}\} d^2\vec{x}\,du\,dv.$$

At the level of the gravitating action A^M_* given in Eq. (9) a similar structure can be reached by solving the Coulomb constraint (10b). Doing this we have for A_v the value

$$A_v = \frac{1}{m^2 - \Delta_g}\{g^{-1/2}E' - D^j A_j'\}. \tag{13}$$

Introduction of this value of A_v into A^M_* also makes A_u disappear, since A_u is precisely the Lagrangian multiplier associated to the Coulomb constraint, as can be seen from Eq. (9).

That gives for the final reduced action A^M_{**} which is a functional of the two-covariant vector field A_i and the two-density E

$$\kappa_2{}^2 dA^M_{**} = \{-E^i_v\dot{A}_i - E\dot{A}_v - \tfrac{1}{2}g^{1/2}B^2 - \tfrac{1}{2}g^{-1/2}E^2 - \tfrac{1}{2}m^2 g^{1/2}A_iA^i + n\,g^{1/2}C^M_v + N^i g^{1/2}C^M_i\} d^2\vec{x}\,du\,dv, \tag{14}$$

where B, $E^i_v = g^{ij}E_{jv}$, A_v have been defined by Eqs. (8) and (13), respectively, and C^M_v, C^M_i are the coefficients of $n\,g^{1/2}$ and $N^i g^{1/2}$ in Eq. (9).

Now we can recall some results already presented in Ref. 2 concerning the structure of the Palatini action A^G after the first stage of the reduction process has been done. This part of the reduction does not depend upon the matter source one is dealing with; it merely consists of the elimination of the signal algebraic constraints. That led us to A^G_* given by[15]

$$\kappa_1{}^2 dA^G_* = \{\tfrac{1}{2}g^{1/2}g'^{ij}\dot{g}_{ij} - \tfrac{1}{2}g^{-3/2}g'\dot{g} + g^{1/2}{}_2R(g_{ij}) + n\,g^{1/2}C^G_v + N^i g^{1/2}C^G_i + \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}\} d^2\vec{x}\,du\,dv, \tag{15a}$$

where C^G_v, C^G_i the scalar constraint and the vector quantity originating the vector constraint are respectively given by

$$C^G_v \equiv g^{ij}g''_{ij} - \tfrac{1}{2}g'^{ij}g'_{ij} \tag{15b}$$

and

$$C^G_i \equiv D_j(g'_i{}^j - \delta_i{}^j g'_l{}^l). \tag{15c}$$

Therefore, the total action of the gravitating vector field, at an intermediate stage of the reduction process characterized by the absence of algebraic constraints either vectorial or gravitational, being the addition of $A^G_* + A^M_*$, reaches the form $(\kappa^2 \equiv \kappa_1{}^2\kappa_2{}^{-2})$

$$\begin{aligned}\kappa_1{}^2 dA_* &= \kappa_1{}^2 dA^G_* + \kappa_1^2 dA^M_*\\ &= \{\tfrac{1}{2}g^{1/2}g'^{ij}\dot{g}_{ij} - \tfrac{1}{2}g^{-3/2}g'\dot{g} - \kappa^2 E^i_v\dot{A}_i - \kappa^2 E\dot{A}_v\\ &\quad + \kappa^2 A_u[m^2 g^{1/2}A_v - C_Q] + n\,g^{1/2}(C^G_v + \kappa^2 C^M_v) + g^{1/2}N^i(C^G_i + \kappa^2 C^M_i)\\ &\quad + g^{1/2}{}_2R(g_{lm}) + \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime} - \tfrac{1}{2}\kappa^2 g^{1/2}B^2 - \tfrac{1}{2}\kappa^2 g^{-1/2}E^2\\ &\quad - \tfrac{1}{2}\kappa^2 m^2 g^{1/2}A_iA^i\} d^2\vec{x}\,du\,dv.\end{aligned} \tag{16}$$

Variations of the vector field variables in A_* yield the set of Eqs. (10), which determines the evolution of A_μ, E on an external gravitational field, while the Einstein field evolves according to the equations obtained from A_* by making independent variations (in decreasing complexity) of g_{ij}, N^l, and n. They are given by the set of Eqs. (17) and (18):

$$
\begin{aligned}
E^{ij} \equiv{} & \tfrac{1}{2}(g^{1/2}g^{ij}\dot{g}^{\,l}_{l})' + \tfrac{1}{2}(g^{1/2}g^{ij}g'^{\,l}_{l}) - \tfrac{1}{2}(g^{1/2}g'^{ij})^{\cdot} - \tfrac{1}{2}(g^{1/2}\dot{g}^{ij})' + \tfrac{1}{2}g^{1/2}g'^{k}_{k}\dot{g}^{ij} \\
& + \tfrac{1}{2}g^{1/2}\dot{g}^{k}_{k}g'^{ij} - \tfrac{1}{2}g^{1/2}g'^{i}_{k}\dot{g}^{kj} - \tfrac{1}{2}g^{1/2}\dot{g}^{i}_{k}g'^{kj} + \tfrac{1}{4}g^{1/2}g^{ij}(\dot{g}_{lm}g'^{lm} - g'^{l}_{l}\dot{g}^{m}_{m}) \\
& + \tfrac{1}{2}ng^{1/2}g^{ij}(g''^{l}_{l} - \tfrac{1}{2}g'^{lm}g'_{lm}) + ng^{1/2}(g'^{i}_{k}g'^{kj} - g''^{ij}) + (ng^{1/2}g^{ij})'' + (ng^{1/2}g'^{ij})' \\
& + \tfrac{1}{2}g^{1/2}N^{i\prime}N^{j\prime} + \tfrac{1}{4}g^{1/2}g^{ij}g_{lm}N^{l\prime}N^{m\prime} + \tfrac{1}{2}g^{1/2}N^{l}(D_{m}g'^{m}_{l} - D_{l}g'^{m}_{m})g^{ij} - g^{ij}(g^{1/2}N^{l})'_{,l} \\
& + \tfrac{1}{2}(g^{1/2}(N^{i|j} + N^{j|i}))' + \tfrac{1}{2}g^{1/2}N^{l}(D_{l}g'^{ij} - D^{i}g'^{j}_{l} - D^{j}g'^{i}_{l}) + \tfrac{1}{2}g^{1/2}(D_{l}N^{l})g'^{ij} \\
& + \tfrac{1}{2}g^{1/2}D^{j}(g'^{i}_{l}N^{l}) + \tfrac{1}{2}g^{1/2}D^{i}(g'^{j}_{l}N^{l}) - \tfrac{1}{2}g^{1/2}g^{ij}D_{l}(g'^{l}_{m}N^{m}) \\
& \qquad = \kappa^{2}\{ng^{-1/2}E^{i}_{v}E^{j}_{v} - \tfrac{1}{2}ng^{-1/2}g^{ij}E^{l}_{v}E_{lv} + \tfrac{1}{2}g^{1/2}BN^{l}(\epsilon^{i}{}_{l}E^{j}_{v} + \epsilon^{j}{}_{l}E^{i}_{v} - \epsilon_{lm}E^{m}_{v}g^{ij}) + \tfrac{1}{2}E^{i}_{v}D^{j}A_{u} + \tfrac{1}{2}E^{j}_{v}D^{i}A_{u} \\
& \qquad\quad - \tfrac{1}{2}g^{ij}E^{l}_{v}D_{l}A_{u} - \tfrac{1}{4}g^{ij}(g^{1/2}B^{2} + g^{-1/2}E^{2}) + \tfrac{1}{2}g^{ij}\mathring{A}_{l}E^{l}_{v} - \tfrac{1}{2}E^{i}_{v}\mathring{A}^{j} - \tfrac{1}{2}E^{j}_{v}\mathring{A}^{i} \\
& \qquad\quad + \tfrac{1}{2}m^{2}g^{1/2}g^{ij}(\tfrac{1}{2}A_{l}A^{l} - A_{u}A_{v} - nA_{v}^{2} + N^{l}A_{l}A_{v}) - \tfrac{1}{2}m^{2}g^{1/2}A^{i}A^{j}\},
\end{aligned}
\tag{17}
$$

$$
(g^{1/2}g_{ij}N^{j\prime})' = g^{1/2}C_{i} \equiv g^{1/2}C^{G}_{i} + \kappa^{2}g^{1/2}C^{M}_{i} \equiv g^{1/2}(D_{j}g'^{j}_{i} - D_{i}g'^{l}_{l}) + \kappa^{2}g^{1/2}\epsilon_{il}E^{l}_{v}B + \kappa^{2}g^{-1/2}E_{iv}E - \kappa^{2}m^{2}g^{1/2}A_{v}A_{i}, \tag{18a}
$$

and finally, by independent variations of n:

$$
C_{v} \equiv C^{G}_{v} + \kappa^{2}C^{M}_{v} \equiv g^{ij}g''_{ij} - \tfrac{1}{2}g'_{ij}g'^{ij} + \kappa^{2}g^{-1}E_{iv}E^{iv} + \kappa^{2}m^{2}A_{v}^{2} = 0. \tag{18b}
$$

These equations keep up the same structure already discussed in the vacuum case: The first three equations contain the dynamics, how the tensor field evolves along the signal-front coordinate u, while the two-vector equation (18a) and the two-scalar equation (18a) have a very different character from a dynamical point of view. They are both differential constraints restricting the independence of the gravity field variables, in the same way as the Coulomb constraint determines the value of A_v in terms of the physical variables A_i, E.

Nevertheless, even if they are both differential constraints, they are not playing the same role in the present gauge conditions. In fact, while the scalar constraint $C_v = 0$ is associated to an ordinary Lagrange multiplier, the vector constraint (18a) does not have an associated multiplier which could be discarded after solving it. Going one step further in dealing with the difficult nonlinear Einstein constraints (18) shall be our next goal.

IV. THE REDUCTION PROCESS: DEALING WITH THE DIFFERENTIAL CONSTRAINTS

One of the technical advantages of signal dynamics lies in the simplification which introduces in the structure of the arrived-differential constraints. Instead of having, as in the case of the 3+1 Arnowitt, Deser, Misner (ADM) formulation, differential three-dimensional constraints mixing up the six coordinates ${}_{3}g_{ij}$ and the six associated momentum variables ${}_{3}\pi_{ij}$, we now have ordinary differential equations in the variable v, the kinematical complications have been decreased, and the variables look like something less coupled through the field equations.

It is very easy to find the solution of the vector constraint (18a). After integrating twice we get for the value of N^i in terms of $g_{ij}(x, u, v_0) \equiv g_{0ij}$, $N^i(\vec{x}, u, v_0) \equiv N^i_0$, $N^{i\prime}(\vec{x}, u, v_0) \equiv N^{i\prime}_0$, and g_{ij}

$$
N^i(\vec{x}, u, v) = N^i_0 + g_0{}^{1/2}g_{0lm}N^{m\prime}_0 \int_{v_0}^{v} g^{-1/2}g^{il}(\vec{x}, u, \nu_1)\,d\nu_1 + \int_{v_0}^{v} g^{-1/2}g^{il}(\vec{x}, u, \nu_2)\,d\nu_2 \int_{v_0}^{\nu_2} g^{1/2}C_l(\vec{x}, u, \nu_1)\,d\nu_1. \tag{19a}
$$

So, having explicitly solved the vector constraint, we are allowed to introduce this value (19a) into the action A_* (which depends quadratically upon N^i). More precisely, using the constraint (18a) the two terms in dA_* containing N^i become, on the field equations (FE),

$$
\begin{aligned}
\{g^{1/2}N^{i}C_{v} + \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}\}_{\mathrm{FE}}\,d^{2}\vec{x}\,du\,dv &= \{(g^{1/2}g_{ij}N^{j\prime})'N^{i} + \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}\}_{\mathrm{FE}}\,d^{2}\vec{x}\,du\,dv \\
&= \{(g^{1/2}g_{ij}N^{j\prime}N^{i})' - \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}\}_{\mathrm{FE}}\,d^{2}\vec{x}\,du\,dv \\
&= -\tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}\,d^{2}\vec{x}\,du\,dv + \text{e.d. } d^{2}\vec{x}\,du\,dv,
\end{aligned}
\tag{19b}
$$

where $N^{i\prime}$ can be calculated from (19a),[16] in terms of g_{ij}, E, A_i, and A_v.[17] But we have also commented upon the dynamical meaning of the term $A_u(m^2 g^{1/2}A_v - C_Q)$ in the preceding section; it turned out that such a term provides the value of A_v as a function of g_{ij}, A_i, and E. Consequently, feeding back Eq. (19b) with this information one can now say that N^i is a function of g_{ij}, A_i, and E, understanding that N^i is given by

Eq. (19a) and with A_v satisfying the Coulomb constraint as shown in Eq. (13).

Moreover, as we are enforcing A_v to verify the Coulomb constraint, A_u is not needed anymore, and this term disappears from the action A_*. In the same way we get rid of the variable n. It is the Lagrange multiplier associated to the scalar constraint (18b), and once we take for granted that along its evolution our system satisfies $C_v=0$, we have to rewrite the action A_* without this linear constraint term $\sim ng^{1/2}C_v$.

Summing up, the unconstrained action A_{**} can be cast in the form

$$\kappa_1^{\,2} dA_{**} = \{\tfrac{1}{2}g^{1/2}(g^{il}g^{jm}-g^{ij}g^{lm})g'_{lm}\dot{g}_{ij} - \kappa^2 E\dot{A}_v - \kappa^2 E^i_v \dot{A}_i + g^{1/2}\,{}_2R(g_{lm}) - [\tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime} + \tfrac{1}{2}\kappa^2 g^{1/2}B^2 + \tfrac{1}{2}\kappa^2 g^{-1/2}E^2 + \tfrac{1}{2}\kappa^2 m^2 g^{1/2}A_iA^i]\}\, d^2\vec{x}\,du\,dv, \tag{20a}$$

with the following restrictive equations holding:

$$E_{iv} = g^{1/2}(A_{v,i} - A'_i), \tag{21a}$$

$$A_v = \frac{1}{(m^2-\Delta_g)}(g^{-1/2}E' - D^jA'_j), \tag{21b}$$

$$(g^{1/2}g_{ij}N^{j\prime})' = g^{1/2}D_l(g'^{\,l}_{\,i} - \delta^l_i g'^{\,m}_{\,m}) + \kappa^2 g^{1/2}(\epsilon_{il}E^l_v B + g^{-1}E_{iv}E - m^2 A_v A_i), \tag{21c}$$

and

$$C_v \equiv g^{ij}g''_{ij} - \tfrac{1}{2}g'_{ij}g'^{\,ij} + \kappa^2 g^{-1}E^i_v E_{iv} + \kappa^2 m^2 A_v^{\,2} = 0. \tag{21d}$$

In order to reach the shortest expression in the present signal dynamics formulation we can take into account a property which shall be discussed in the Appendix. The property states that[18] for a two-dimensional Riemannian variety one has

$$\int g^{1/2}\,{}_2R(g_{lm})\,d^2\vec{x}\,du\,dv = \int \partial_i R^i\, d^2\vec{x}\,du\,dv, \tag{22}$$

for some R^i. Consequently we are allowed to discard the two-dimensional Ricci-scalar density in the action (20a), which achieves the final value

$$\kappa_1^{\,2} dA_{**} = \{\tfrac{1}{2}g^{1/2}(g^{il}g^{jm} - g^{ij}g^{lm})g'_{lm}\dot{g}_{ij} - \kappa^2 E\dot{A}_v - \kappa^2 E^i_v\dot{A}_i - \mathcal{J}^u\}\, d^2\vec{x}\,du\,dv, \tag{23a}$$

where the signal-front energy density $\mathcal{J}^u$ is given by

$$\mathcal{J}^u = \tfrac{1}{2}g_{ij}N^{i\prime}N^{j\prime}g^{1/2} + \tfrac{1}{2}\kappa^2 g^{1/2}B^2 + \tfrac{1}{2}\kappa^2 g^{-1/2}E^2 + \tfrac{1}{2}\kappa^2 m^2 g^{1/2}A_iA^i. \tag{23b}$$

From the explicit structure of $\mathcal{J}^u$ shown in Eq. (23b) it is almost trivial to see its non-negative character. Furthermore, looking at Eq. (23b) we realize that $\mathcal{J}^u$ is the sum of the gravity signal-front energy density $\mathcal{J}^u_G$ and the matter signal-front energy density $\mathcal{J}^u_M$, both separately definite non-negative quantities:

$$\mathcal{J}^u = \mathcal{J}^u_G + \mathcal{J}^u_M \geq 0, \tag{23c}$$

$$\mathcal{J}^u_G \equiv \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}, \qquad \mathcal{J}^u_M \equiv \tfrac{1}{2}\kappa^2 B^2 g^{1/2} + \tfrac{1}{2}\kappa^2 g^{-1/2}E^2 + \tfrac{1}{2}\kappa^2 m^2 g^{1/2}A_iA^i. \tag{23d}$$

The matter signal energy density $\mathcal{J}^u_M$ consists of two parts, the first two terms being always present either in the massive or in the massless vector theory. More precisely, if we define new variables ϵ and a^L such that

$$E \equiv g^{1/2}(m^2-\Delta_g)^{1/2}\epsilon$$

and

$$mg^{1/2}A^i \equiv g^{il}g^{1/2}\partial_l[(m^2-\Delta_g)^{1/2}(-\Delta_g)^{-1/2}a^L] + m\epsilon^{il}\partial_l A^T,$$

it is straightforward to check that the null matter energy $E^u_M(m) = \int \mathcal{J}^u_M d^2\vec{x}\,dv$ tends to the sum of the null Maxwell energy $E^u(\text{Maxwell})$ plus the null energy of the scalar massless field represented by a^L:

$$E^u_M(m) \xrightarrow[m\to 0]{} \int \tfrac{1}{2}\kappa^2\{B^2 + [(-\Delta_g)^{1/2}\epsilon]^2\}d^2\vec{x}\,dv + \int \tfrac{1}{2}\kappa^2 g^{1/2}g^{ij}\partial_i a^L \partial_j a^L d^2\vec{x}\,dv. \tag{23e}$$

We leave the analysis of the limit when $m\to 0$ of the dynamical germ of the matter action $\{-\kappa^2 E\dot{A}_v - \kappa^2 E^i_v \dot{A}_i\}\, d^2\vec{x}\,du\,dv$ for later work. We now proceed with the dynamics of the gravitating massive vector.

Making variations of A_{**} on the field equations we get the u generator of the present system:

$$G^u(A_{**}) = \int_{\Sigma_u} \{\pi^{ij}\delta g_{ij} - \kappa^2 E\delta A_v - \kappa^2 E^i_v \delta A_i\}\, d^2\vec{x}\,dv, \tag{24}$$

where the nonindependent quantities π^{ij} (see Ref. 19) have been introduced in order to simplify the notation and δA_i, δA_v, and δg_{ij} obey all the constraints. These quantities π^{ij} are defined by

$$\pi^{ij} \equiv \tfrac{1}{2}g^{1/2}(g^{il}g^{jm} - g^{ij}g^{lm})g'_{lm}. \tag{25a}$$

They are also useful in writing down the Einstein constraints (21c) and (21d) in a briefer form,

$$[g^{1/2}g_{ij}N^{j\prime}]' = 2\pi^{ij}{}_{|j} + \kappa^2 g^{1/2}[\epsilon_{il}E^l_v B + g^{-1}E_{iv}E - m^2 A_v A_i], \tag{25b}$$

$$g^{1/2}g_{ij}\pi'^{ij} + \pi^l_l \pi^m_m - \pi^{ij}\pi_{ij} + \tfrac{1}{2}\kappa^2 E^i_v E_{iv} + \tfrac{1}{2}\kappa^2 m^2 A_v{}^2 = 0, \tag{25c}$$

strongly resembling the structure of the constraints in the ADM[20] canonical formulation.

At this point we want to clarify the physical consequences of Einstein's constraint equations [(21c) and (21d)] because the structure of the vector field part does not need additional decompositions to be understood. In fact, even in the massless (Maxwell) case, after being aware that the longitudinal part A^L of the two-dimensional coordinate-dependent decomposition given in Eq. (11) can be arbitrarily chosen the only thing one has to do is to pick out some specific gauge. For instance, $A^L = 0$ could be taken. Gravity constraints are more difficult to understand, but the analysis performed for the linearized theory and for the vacuum case[21] shall help us in extending these ideas to the present situation.

Let us recall that besides the $T+L$ two-dimensional decomposition for two-dimensional vectors V_i,

$$V_i \equiv \epsilon_{il}\rho_l V^T + \rho_i V^L, \quad \rho_l \equiv (-\Delta)^{-1/2}\partial_l(\cdots), \tag{26a}$$

$$\Delta(\cdots) \equiv \partial_{11}(\cdots) + \partial_{22}(\cdots), \tag{26b}$$

there can also be introduced a $T+L$ two-dimensional decomposition for symmetric second-order tensors t_{ij}. In brief, each symmetric two-dimensional tensor can be split into three orthogonal[22] pieces t^T_{ij}, $\hat{t}^T_{ij}$, and t^L_{ij} such that $t^T{}_{ij}$ is transverse, i.e.,

$$t_{ij} = t^T_{ij} + \hat{t}^T_{ij} + t^L_{ij} = t^T_{ij} + t_{i,j} + t_{j,i}, \qquad t^T_{ij,j} = 0. \tag{27}$$

Each of these three pieces can be calculated by using the flat Riesz operators ρ_i already defined in Eqs. (26) yielding $(\rho_i\rho_i = -1)$

$$t^T_{ij} = \epsilon_{il}\epsilon_{jm}\rho_l\rho_m t^T, \quad t^T = t_{ii} - \rho_i\rho_j t_{ij},$$
$$\hat{t}_{ij} = (\epsilon_{il}\rho_l\rho_j + \epsilon_{jl}\rho_l\rho_i)\hat{t}^T, \quad \hat{t}^T = \epsilon_{ij}\rho_l\rho_j t_{il}, \tag{28}$$
$$t^L_{ij} = \rho_i\rho_j t^L, \quad t^L = \rho_i\rho_j t_{ij}.$$

When one is dealing with tensors on a differential variety, this decomposition has the drawback of not being covariant, i.e., is coordinate-dependent. On the other hand, it has the advantage of not depending upon the elements (metric g_{ij}, affinity Γ^l_{ij}) which enrich the structure of the variety, allowing us to use it for analyzing the physical content of precisely these elements.[23]

In the following we are going to apply decomposition (28) to the two-metric g_{ij}, and we shall refer to g^T, $\hat{g}^T$, and g^L in the sense of Eqs. (28). There are also two two-vector quantities Γ^u_{iu} and $\nu_i \equiv g^{1/2}g_{ij}N^{j\prime}$ which shall be projected according to (26) giving rise to Γ^T, Γ^L, and to ν^T and ν^L, respectively:

$$\Gamma^u_{iu} = \tfrac{1}{2}(g_{ij}N^j)' \equiv \epsilon_{il}\rho_l\Gamma^T + \rho_i\Gamma^L,$$
$$\nu_i = g^{1/2}g_{ij}N^{j\prime} = \epsilon_{il}\rho_l\nu^T + \rho_i\nu^L. \tag{29}$$

The Einstein scalar constraint $C_v = 0$, which in the absence of matter has already been analyzed, keeps its significance: It determines g^L as a functional of g^T, $\hat{g}^T$, A_i, and E. So we have to process the information carried by the Einstein vector constraint Eq. (21c), which is essentially a two-covariant equation.[24] Taking its T and L projections we shall obtain respectively

$$\nu^{T\prime} = \epsilon_{li}\rho_l\nu'_i = \epsilon_{li}\rho_l[g^{1/2}C_i](g^T, \hat{g}^T, g^L, A_i, A_v, E), \tag{30a}$$

$$\nu^{L\prime} = -\rho_i\nu'_i = -\rho_i[g^{1/2}C_i](g^T, \hat{g}^T, g^L, A_i, A_v, E), \tag{30b}$$

where by A_v and g^L is meant their corresponding values as given by the Coulomb constraint (21b) and $C_v = 0$.

Equation (30a) provides the value of the nongauge scalar function $\hat{g}^T$ in terms of ν^T, g^T, A_i, and E, while Eq. (30b) determines the value of $\nu^{L\prime}$ as a functional of g^T, ν^T, A_i, E if we imagine that g^T, g^L, and A_v have been substituted by their respective values obtained from Eq. (30a), $C_v = 0$, and the Coulomb constraint. That way we shall end up with an unconstrained action A_{**} only depending upon the minimum number of independent physical variables the vector-tensor system one might expect to have: three for the vector field plus two for the gravity forces, where the two gravitational variables should be g^T and ν^T.

This interpretation is supported by what happens if we make a linearized approximation to our system. In that case the signal-energy density $\mathcal{J}^u_G$ tends to

$$\mathcal{J}^u_G \to \tfrac{1}{2}\delta_{ij}N^{i\prime}N^{j\prime} = \tfrac{1}{2}N'_i N'_i, \tag{31a}$$

and consequently one has for the global signal energy E^u_G (recalling that ${}_2\Gamma^u_{iu} = N'_i = \nu_i$ in the ray gauge)

$$E^u_G \to E^u_{G0} = \int_{\Sigma_u} \tfrac{1}{2} N'_i N'_i \, d^2\vec{x}\, dv$$

$$= \int_{\Sigma_u} \left\{\tfrac{1}{2}\nu^{T2} + \tfrac{1}{2}\nu^{L2}\right\} d^2\vec{x}\, dv. \qquad (31b)$$

But, in the rigid ray gauge $\Gamma^u_{uv}=0$ (see Ref. 25) and that implies, in the language of the present variables, that $\Gamma^L = 2\nu^L$ represents the dynamical quantity g^T through

$$\tfrac{1}{2}\nu^L = \Gamma^L = -\tfrac{1}{2} g^T. \qquad (31c)$$

Therefore, E^u_G (linearized) turns out to be

$$E^u_{G0} = \int \left\{\tfrac{1}{2}\nu^{T2} + \tfrac{1}{2} g^{T2}\right\} d^2\vec{x}\, dv, \qquad (31d)$$

which is exactly the conserved null Hamiltonian found for the free linearized gravitational field in Ref. 1.

More physical insight about the non-negative signal energy E^u_G shall be achieved after studying its vanishing. This is our next aim.

V. VANISHING SIGNAL-ENERGY SOLUTIONS

If one assumes enough smoothness in the components of the metric tensor, then to investigate the fields for which $E^u = 0$ is equivalent to looking for the fields for which $\mathcal{G}^u = 0$ is verified throughout the four-dimensional variety.

Because of the complexity of the problem, we are going to restrict ourselves in this section to dealing with the pure gravitational case ($\kappa^2 = 0$). We shall see that there is a natural connection between the family of vanishing signal-energy solutions and two well-known radiative solutions: the Robinson-Trautman[26] (of spherical type) and a broad class of the plane-fronted Kundt[27] waves.

From $\mathcal{G}^u_G = \frac{1}{2} g_{ij} N^{i\prime} N^{j\prime} = 0$ we obtain for N^i

$$N^i = N^i(\vec{x}, u). \qquad (32)$$

This is an invariant structure for N^i under the elements of $G^l_m = \{g^l, a_1, a_2\}$,[28] as one can easily check. In order to solve the 10 Einstein equations $R_{\mu\nu} = 0$ [equivalent to the set of the seven equations (17) and (18) for $\kappa^2 = 0$ plus the three subsidiary[29] conditions $R_{ui} = R_{uu} = 0$] we should have to introduce $N^i(\vec{x}, u)$ $g_{ij}(\vec{x}, u, v)$ and $n(\vec{x}, u, v)$ into these field equations and try to obtain their exact solution. However, even at the cost of losing generality and in order to obtain some results one could investigate whether some more specific ansatz for the structure of N^i, g_{ij}, and n might consistently lead to exact solutions. For instance, let us look for solutions of the type $g_{ij}(\vec{x}, u, v) = h(\vec{x}, u, v)\delta_{ij}$.[30]

Substitution of this ansatz into the scalar gravitational constraint $C_v = C^G_v = 0$ gives

$$-1/4 C^G_v \equiv (h^{-1}h')' + \tfrac{1}{2} h^{-2} h'^2 = 0. \qquad (33)$$

This ordinary differential equation determines the v structure of h giving rise in a unified way both to the expanding spherical solutions and to the nonexpanding plane waves, as has been pointed out in the article of Robinson and Trautman.[26] It contains simultaneously solutions of the C×NC type and of the NC×NC structure, as we shall see.

The first situation corresponds to the nonvanishing expansion $\Theta \equiv D_\mu e^\mu_{(-)}$ while the plane waves come out when $\Theta = h^{-1}h' = 0$ holds. It is worth mentioning that, because we are working in the ray gauge, the vector $e_{(-)} \equiv \partial_v$ is a null hypersurface orthogonal vector and consequently we always have vanishing twist. Moreover, due to the form shown in Eq. (33) of the gravity scalar constraint, both kinds of solutions are shear-free.[31]

Just for completeness, let us recall that the Robinson-Trautman expanding solution can be written in a frame of a reference system where $N^i(x, u)$ vanishes. That gives us the metric

$$ds_{RT}^2 \equiv v^2 a(\vec{x}, u)\delta_{ij}\, dx^i dx^j - 2du\left\{dv - \left[b(u)v^{-1} + \tfrac{1}{4} a^{-1}\Delta_2 \ln a - \tfrac{1}{2}\ln a v\right] du\right\}, \qquad (33a)$$

where $b(u)$ is arbitrary and $a(\vec{x}, u)$ has to satisfy the equation

$$a^{-1}\Delta a^{-1}\Delta \ln a = 6b \ln a + 4b. \qquad (33b)$$

On the other hand, the vanishing signal-energy plane waves found by Kundt can be written in a more compact form by going to a system of coordinates in which the two-space variety looks Euclidean. Making the best use of the gauge group G^l_m one obtains for the metric

$$ds_K^2 \equiv \delta_{ij}(dx^i + N\delta^i_1 du)(dx^j + N\delta^j_1 du) - 2du\left\{dv - \left[\tfrac{1}{2} vN_{,1} + a(\vec{x}, u)\right] du\right\}, \qquad (34a)$$

where the function $N = N(\vec{x}, u)$ is harmonic ($\Delta N = 0$) and "a" obeys the equation

$$\Delta a = N_{,1} - \tfrac{3}{2} N_{,1}{}^2 - \tfrac{1}{2} N_{,2}{}^2 - N N_{,1,1}. \qquad (34b)$$

These solutions constitute the whole class of the expansion free radiation fields of type III, N, or 0 with vanishing rotation, as has been shown in the article of Kundt.

At this point one might wonder whether the class of vanishing signal-energy fields only contains twist-free solutions of zero shear or if there are gravitational fields of vanishing signal energy with non-null $e_{(-)}$ shear. We shall see that the last alternative holds. There exist exact solutions of the vacuum gravity equations which have nonzero $e_{(-)}$ shear. In this connection it is very useful to parametrize the two-dimensional metric g_{ij} in the form $g_{ij}=g^{1/2}u_{ij}$ splitting up the two-volume and the unimodular two-dimensional deformation u_{ij}. More precisely in the following we are going to represent g_{ij} in the form

$$g_{ij}\equiv h\begin{pmatrix} e^{\alpha}\cosh\beta & \sinh\beta \\ \sinh\beta & e^{-\alpha}\cosh\beta \end{pmatrix} \tag{35}$$

(which already ensures the two-volume times the unimodular tensorial capacity structure of g_{ij} mentioned above), and thereafter use this representation to compute the Einstein scalar constraint and the shear σ of the congruence $e_{(-)}$. That gives for C_v^G

$$C_v^G\equiv 2h^{-2}h'^2-4h^{-1}h''-4\sigma(\alpha',\beta,\beta'), \tag{36a}$$

with σ having the value

$$\sigma(e_{(-)})=\tfrac{1}{2}\alpha'^2\cosh^2\beta+\tfrac{1}{2}\beta'^2. \tag{36b}$$

Equations (36a) shed some light on the problem because they show that even if $\sigma=0$ implies on the field equations either Kundt plane waves for $h'=0$ or Robinson-Trautman solution if $h^{-1}h'=\Theta(e_{(-)})\neq 0$, not necessarily every solution of the Einstein equations (in particular satisfying $C_v^G=0$) with vanishing null-energy, must have vanishing shear. Even more, it is possible to exhibit an exact gravitational field with vanishing signal energy (of course twist-free) with shear.

For instance, from the well-known class of plane-wave[32] solutions with $h=h(v)=\alpha=\alpha(v)$, $\beta=0$ one could select the metric (Θ being a real parameter)

$$ds^2=v^{1+\sin\theta}[\,v^{\cos\theta}dx^2+v^{-\cos\theta}dy^2]-2du\,dv \tag{37a}$$

and taking into account Eq. (36b) calculate its $e_{(-)}$ shear. It turns out to be

$$\sigma(e_{(-)})=\tfrac{1}{2}v^{-2}\cos^2\Theta>0, \tag{37b}$$

showing that the metric (37a) has simultaneously $\mathcal{J}_G^u=0$ and non-null $\sigma(e_{(-)})$, in contrast to what happens in the case of the RT and Kundt metrics.

VI. CONCLUSIONS

We have been able to define good physical variables to represent the gravitating massive vector field. By means of these variables we have analyzed its dynamical evolution along a signal-front coordinate reaching the unconstrained action A_{**}, which is functionally dependent upon the minimum number of independent and physically relevant variables. In the present system it corresponds to 5, 2 to represent the two helicity components of the Einstein field and 3 for each of the three components of the spin of the massive vector field. Moreover, the reduced action achieves a structure close to the typical signal canonical flat structure $\sim q'\dot{q}-\mathcal{J}^u$, with the total signal energy $\mathcal{J}^u$ explicitly showing its non-negative character. An interesting fact is the different structure shown by each of the three parts of the massive vector field; the electric part E contributes to the null energy $\mathcal{J}^u$ independently of its mass, the longitudinal part A^L only given an m^2 proportional term while the transverse component A^T gives the two kinds of contributions, a term independent of its mass plus other term, proportional to m^2.

Thereafter, making use of the two-dimensional coordinate-dependent $T+L$ decomposition of the different fields we clarified the gauge structure of this system and the consequences of having chosen the rigid-ray gauge to simplify the calculations. The role of the three differential gravitational constraints of the present system is carefully discussed. It is worth mentioning that one of the advantages of the signal-front dynamics seems to be the simplification of the differential constraints which became either an ordinary differential equation (the scalar one) or a differential operator on a two-dimensional Riemannian variety. We conjecture here that by employing conformal two-covariant techniques they shall be completely solved.[35]

In the limit of zero mass one obtains the Einstein-Maxwell system in a very clear way, and A^L becomes a new gauge function of the theory.

Finally we have seen how the Robinson-Trautman (spherical, C×NC) and the Kundt (plane, NC×NC) solutions fit in a natural way in the present formulation and we presented an example indicating that vanishing signal energy does not imply that the congruence generated by $e_{(-)}$ is necessarily shear-free.

APPENDIX: THE TWO-DIMENSIONAL SCALAR CURVATURE IS AN EXACT DIVERGENCE

In the reduction process, at a certain point we stated without proof that in the case of a two-dimensional Riemannian variety, the scalar curvature density $g^{1/2}\,{}_2R(g_{lm})$ is an exact divergence.

Let us recall[33] that an old result of differential geometry states that every 2-dimensional Riemannian variety is conformally Euclidean. Therefore, there always exists a transformation $x^i\to\bar{x}^j=f_c^j(x^i,u,v)$ such that the two-dimensional metric g_{ij} becomes

$$\bar{g}_{ij}=\bar{h}(\bar{x},u,v)\delta_{ij}, \tag{A1}$$

where we are thinking of u and v as parameters which describe a two-dimensional family of metrics g_{ij}, one for each value assigned to both variables. Even if the transformation induced by f^j_c on the general transformation of coordinates group $\{f^j_c, a_1(\cdots)=1, a_2(\cdots)=0\}$ in general does not belong to the rigid gauge covariance group G^l_m, it is very useful in order to calculate the scalar curvature density. In this conformally Euclidean frame,

$$g^{1/2}{}_2R(g_{ij}) = \bar{\Delta}_2 \ln \hbar \equiv (\ln \hbar)_{,\bar{i}\bar{i}}, \tag{A2}$$

and integrating in a four-volume we have after applying Gauss's theorem

$$\int_{V_4} g^{1/2}{}_2R\, d^2x\, du\, dv = \int_{U\times V} du\, dv \int_{\Sigma_{uv}} \partial_{\bar{i}} R^{\bar{i}}\, d^2\bar{\mathbf{x}}$$
$$= \int_{U\times V} du\, dv \int_{\partial\Sigma_{uv}} R^{\bar{i}} n_{\bar{i}}\, dl. \tag{A3}$$

The four-volume integral can be evaluated by calculating the flux of the vector $R^{\bar{i}}$ across the boundary $\partial\Sigma_{uv}$ of Σ_{uv} (see Ref. 34) and then integrating on u and v.

Then, if we want to calculate the four-volume integral of $g^{1/2}{}_2R$, we can do that by calculating the surface integral (dl is the differential length of the curve $\partial\Sigma_{uv}$, n_i the unit normal to $\partial\Sigma$):

$$\int_{U\times V} du\, dv \int_{\partial\Sigma} R^i n_i\, dl = \int \partial_i (g^{1/2} R^i)\, d^2x\, du\, dv \tag{A4}$$

in the original system of coordinates or as Eq. (A4) shows, applying Gauss's theorem again, in the "old" coordinates, where it is an exact two-divergence.

This is the reason explaining why the two-dimensional Einstein tensor $G_{ij} \equiv {}_2R_{ij} - \frac{1}{2} g_{ij}\, {}_2R$ vanishes identically, another well-known property of two-dimensional Riemannian varieties. So, we are allowed to discard this term from the reduced action A_{**} to obtain the field equations.

Nevertheless, the possible contribution of this exact divergence to the signal-front energy deserves a more detailed analysis because this term could be the origin of a dissipation mechanism, especially in the case where the two-dimensional variety Σ_{uv} is a noncompact one.

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[1]C. Aragone and R. Gambini, Nuovo Cimento 18B, 311 (1973);

[2]C. Aragone and J. Chela-Flores, Nuovo Cimento 25B, 225 (1975).

[3]This kind of dynamics was pioneered by P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[4]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 120, 313 (1960).

[5]Some previous results of this paper have been presented at the 24th Convención Anual de la Asovac, Maracaibo, 1974 (unpublished).

[6]The metric $g_{\mu\nu}$ has been chosen with signature +2 and $R_{\mu\nu} \sim \partial_\sigma \Gamma^\sigma_{\mu\nu}$.

[7]This kind of coordinates has been employed by I. Robinson and A. Trautman, Proc. R. Soc. London A265, 463 (1962).

[8]This gauge condition was introduced by R. G. Root, Phys. Rev. D 8, 3382 (1973).

[9]Latin indices run from 1 to 2 while Greek indices range over $(1,2,u,v)$.

[10]${}_2g$ means $\det\{g_{ij}\} = g_{11}g_{22} - g_{12}{}^2$.

[11]This group is very similar to the group considered by Robinson and Trautman in Ref. 7, a little bit more general than ours. The difference lies in the fact that they did not take into account Hamiltonian considerations.

[12]With respect to the group G^l_m, E behaves as a two-density, B as a two-scalar, E_{iv} up to an $e_{(-)}$ dilatation as a 2-vector density, E_{i+} up to the inverse dilatation as a 2-vector density.

[13]Besides B and E_{iv}, E_{i+} disappears because it is the Lagrange multiplier associated to Eq. (8) defining E_{iv}.

[14]In connection with this problem see the article of S. Deser, Ann. Inst. H. Poincaré 16, 79 (1972), where it is treated in the 3+1 canonical formalism. Observe that in the null formalism the correct canonical physical variable a^L is uniquely defined. On the contrary, in the 3+1 way, as p and q are independent variables, there is an arbitrary factor in the possible definition of p (and its inverse in q).

[15]We are using the following tensorial notation: $g'^{ij} \equiv g^{il} g^{jm} g'_{lm}$, $g'^l_l \equiv g^{ij} g'_{ij}$, $D_j(\cdots)$ is the two-dimensional covariant derivative.

[16]Incidentally, it is immediate to check that $N^{i\prime}$ is a two-contravariant vector under the group G^l_m, i.e., $\partial_{\bar{v}} N^{\bar{i}} = \theta^{\bar{i}}_j \partial_v N^j$.

[17]This is one way of looking at the constraint Eq. (18a). But it is also reasonable to use it to obtain the values of $\hat{g}^T$ and N'^L, keeping g^T and N'^T as final dynamical variables.

[18]The importance of dealing with $g^{1/2}{}_2R(g_{lm})$ was emphasized in a private communication by Professor S. Deser, in connection with the possibility of showing the non-negative character of the signal energy.

[19]$\pi^{ij} d\bar{u}$ behaves like a 2-contravariant density.

[20]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation, An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[21]Look at Refs. 1 and 2.

[22]In the global natural way, i.e., $\int t^T_{ij} \hat{t}^T_{ij}\, d^2\bar{\mathbf{x}} = \cdots = 0$.

[23]This kind of problems also appears in the canonical 3+1 ADM formulation. Recent work of Y. Choquet-Bruhat and S. Deser [Ann. Phys. (N.Y.) 81, 165 (1973)] and J. W. York, Jr. has substantially clarified the

problem of the 3 +1 constraints in a fully covariant way. See, for instance, the recent article of J. W. York, J. Math. Phys. 14, 456 (1973).

[24]Up to a factor $(d\bar{u}/du)$ with respect to an element of G^l_m.

[25]Look at Eq. (34a) of Ref. 1.

[26]I. Robinson and A. Trautman, Proc. R. Soc. London A265, 463 (1962).

[27]W. Kundt, Z. Phys. 163, 77 (1961).

[28]Under an element $(g^{\bar{l}}, a_1, a_2)$ g_{ij} transforms like a two-covariant tensor, while $N^j \to N^{\bar{j}} = a_1^{-1}(\theta^{\bar{j}}_i N^i - \theta^{\bar{j}})$, $n \to \bar{n} = \dot{a}_1^{-2} n + \dot{a}_2 \dot{a}_1^{-1} - v \ddot{a}_1 a_1^{-3}$.

[29]In the terminology of Sachs. See especially R. K. Sachs, J. Math. Phys. 3, 908 (1962).

[30]As in principle $g_{ij} = g_{ij}(x,u,v)$ and the elements of the covariance group $\{g^j(x,u), a_1(u), a_2(u)\}$ in the present treatment do not contain the affine parameter v, we are not allowed to go from a certain $g_{ij}(x,u,v)$ to the conformal structure $h(x,u,v)\delta_{ij} = \bar{g}_{ij}$ through an element of G^l_m.

[31]The shear of distortion of the congruence $\{e_{(-)}\}$ is defined by $D_\alpha e(-)_\beta D^\alpha e(-)^\beta - \frac{1}{2}(D_\alpha e(-)^\alpha)^2$.

[32]H. Bondi, F. A. Pirani, and I. Robinson, Proc. R. Soc. London A251, 519 (1959), and the article of J. Ehlers and W. Kundt, in *Gravitation, An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1967). The only flat solution of this type corresponds to $\theta = 0$, since the only nontrivially vanishing component of the Riemann tensor is $R^1{}_{v1v} = v^{-2} \times \sin\theta$.

[33]J. Schouten, *Ricci Calculus* (Springer, Heidelberg, 1954).

[34]Whenever there exists the boundary of the variety Σ_{uv}, the case of Σ_{uv} without boundary has to be discussed separately.

[35]After this paper was completed, we received a report by M. Kaku [Nucl. Phys. B91, 99 (1975)] in which the null constraint $C_v = 0$ has been solved.